Eberhard-Karls-Universität Tübingen
Mathematische Institut

# Harnack type inequalities for nonlinear PDE's involving critical Sobolev exponents

Diplomarbeit

*von*

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To my wife Amber, and our children Rania, Aziz and Rihem, whose love and patience make dreams come true.

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# DECLARATION

I have written this thesis independently, solely based on the literature and tools mentioned in the chapters and the appendix.

Hiermit erkläre ich, die vorliegende Arbeit selbstständig verfasst zu haben und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt zu haben.

Tübingen, den November 30, 2009
Ort, Datum

Dhia Eddine Mansour

# ZUSAMMENFASSUNG IN DEUTSCHER SPRACHE

In dieser Arbeit werden wir die semilineare partielle Differentialgleichung

$$\Delta^2 u = u^{\frac{n+4}{n-4}}$$

für $n \geq 5$ betrachten. Der auf der rechten Seite auftretende Exponent $\frac{n+4}{n-4}$ ist der bekannte kritische Sobolev-Exponent, wenn wir beachten, daß bei der Sobolev-Einbettung $\mathcal{H}^2(\Omega) \hookrightarrow L^{\frac{n+4}{n-4}+1}(\Omega)$ zwar noch stetig, aber nicht mehr kompakt eingebettet wird. Dies hat zur Folge, daß das Euler-Langrange-Funktional nicht die so genannte Palais-Smale-Bedingung erfüllt.

Wir werden diese Gleichung auf dem Ball $B_{3R}$ für einen vorgegeben Radius $R$ betrachten, wobei wir uns nur für positive Lösungen $u$ interessieren. Für eine solche positive Lösung $u$ ist es unser Hauptresultat, eine Abschätzung vom Harnack-Typ zu beweisen, d.h. für jeden Radius $R$ größer 0 gibt es eine von $R$ und $u$ unabhängige Konstante $C$ mit

$$\left(\max_{B_R} u\right) \cdot \left(\min_{B_{2R}} u\right) \leq CR^{n-4}.$$

Ein wichtiges – aber einfaches Korollar – aus dieser Abschätzung ist eine neue Energie-Abschätzung,

$$\int_{B_R} u^{\frac{2n}{n-4}} \leq C.$$

Als wichtige Vorarbeit und Motivation für unser Haupttheorem gibt es bereits einige bekannte Resultate für den Laplace-Operator – im Gegensatz zum Bi-Laplace-Operator, den wir betrachten. Seit etwa zwanzig Jahren gibt es Ergebnisse, die unser Problem für den Laplace-Operator betrachten – mit angepassten Koeffizienten. Eine (bis auf diese Koeffizienten) gleiche Abschätzung vom Harnack-Typ wurde hierfür bereits bewiesen.

Wir werden zunächst das eben beschriebene Problem mit angepassten Koeffizienten für den Laplace-Operator untersuchen und führen einen ausführlichen detaillierten Beweis von Y.Y. Li und L. Zhang vor. Zusätzlich werden bereits hier als Korollar eine ähnliche Energie-Abschätzung wie oben beweisen. Die hier benutzte

Technik werden wir im Kern bei unserem Hauptresultat wiedererkennen, jedoch stark variieren müssen. Für eine leicht verallgemeinerte Version dieses klassischen Theorems zeigen wir mit der gleichen Technik ebenfalls eine Abschätzung vom Harnack-Typ. Wir kehren dann wieder zum bekannten Resultat für den Laplace-Operator zurück und beweisen hierfür Abschätzungen vom Harnack-Typ auf einer Halbball mit Randdaten. Diese Abschätzung für das Randwertproblem werden wir im Zuge dieser Arbeit nicht verallgemeinern.

Die grundsätzliche Beweistechnik, die in allen besprochenen Theoreme vorkommt, ist die so genannte Technik der "Moving Spheres". Diese Technik tritt in verschiedenen Variationen auf, wobei wir uns im ersten Teil dieser Arbeit an der Technik von Y.Y. Li und L. Zhang aus dem Jahre 2003 orientieren und für unser Haupttheorem über den Bi-Laplace-Operator zusätzlich an der von X. Xu aus dem Jahre 2000. Die Technik der Moving Spheres basieren auf der bekannten Kelvin-Transformation und dem starken Maximumsprinzip.

Motiviert durch die Verallgemeinerungen des Problems für den Laplace-Operator im Bezug auf die rechte Seite und des Randwertproblems, scheint es möglich zu sein, auch diese für den Bi-Laplace-Operator mit dieser Technik zu zeigen. Dies sind aber noch offene Probleme.

Die bekannten Theoreme aus dem ersten Teil dieser Arbeit wurden als wichtiges Hilfsmittel für a priori Abschätzungen des Yamabe-Problems, welches in der konformen Geometrie auftritt, benutzt. Mit Hilfe unseres Resultats sollte möglich sein, a priori Abschätzungen für den Yamabe-Paneitz Operator zu beweisen.

CONTENTS

# 0

# INTRODUCTION

Harnack inequalities, named after *Carl Gustav Axel von Harnack* (1851-1888), were originally proved for harmonic functions in the plane, much later however, they became an important tool in the general theory of harmonic functions as well as in partial differential equations.

In this work, we study a special form of Harnack type inequality, namely, that the product of the maximum of a positive solution over some ball with radius $R$, and the minimum over some ball with radius $2R$, is bounded by some constant times $R^\alpha$ $(\alpha = 2 - n; \alpha = 4 - n)$.

First, we study the constant scalar curvature equation

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}} \qquad \text{in } B_{3R} \;,\; n \geq 3 \tag{0.1}$$

where $B_{3R}$ is a ball centered at the origin and has radius $3R$.

This model equation (0.1) arises in many physical contexts and has been under extensive study for decades, but its greatest interest in recent years lies in its relation to the well known Yamabe problem. From this geometric point of view: Let $g = u^{\frac{4}{n-2}}\delta$ be a Riemannian metric conformal to Euclidean metric $\delta$, then the scalar curvature with respect to $g$ is $4n(n-1)$. $\frac{n+2}{n-2}$ is the critical Sobolev exponent. Indeed $\frac{n+2}{n-2} + 1 = \frac{2n}{n-2}$ is the critical Sobolev embedding exponent of $\mathcal{H}^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$, where $\Omega \subset\subset \mathbb{R}^n$ is a bounded open domain of $\mathbb{R}^n, n \geq 3$. We recall that this embedding is continuous but not compact. It follows that the associated Euler-Lagrange functional does not satisfy the Palais-Smale condition. Such a lack of compactness presents us from using the Methods of Calculus of variation to prove existence of solution to the observe equation. An interesting feature of the above equation is its conformal invariance which induces blow up phenomena. Understanding blow up phenomena in conformally invariant equations has been a great challenge and a major topic in modern nonlinear Analysis. A major tool to understand blow up phenomena in Yamabe type equation is the following *Harnack type Inequality*:

**Theorem 1.**

*Let $u \in C^2(B_{3R})$ be a positive solution of (0.1), then*

$$\left(\max_{\overline{B}_R} u\right)\left(\min_{\overline{B}_{2R}} u\right) \leq C(n)R^{2-n} \tag{0.2}$$

This Harnack inequality was an important step in the derivation of some a priori estimate for the Yamabe equation in the works of L. Zhang and Y.Y. Li (e.g. [10], [11] and [12]). Also, for some more general conformally invariant equations, Y.Y. Li and A. Li used the Harnack type inequality to derive some a priori estimate and Liouville type results [7], [8] and [6]. The Harnack type inequality (0.2) was first obtained in 1989 by R. Schoen in [13]. He proved it by employing the Liouville type theorem of Caffarelli, Gidas and Spruck [5]. In 2003, Li and Zhang expanded the study of this theorem by giving a different proof in an elegant paper [9], based on the well known Moving Sphere Method, without using the Liouville type theorem. They do not only gave a different proof, they also simplified the Moving Spheres Method decidedly. Furthermore with the same method they gave a proof for such Harnack type inequality for more general right hand side $g(x,u)$ with several properties. In particular assuming that $g$ satisfies

$$g \text{ is continuous and positive in}(0,\infty), \text{and} \sup_{0<s\leq t} g(s) < \infty, \forall t < \infty, \tag{0.3}$$

$$s^{-\frac{n+2}{n-2}} g(s) \text{ is non-increasing in } (0,\infty), \tag{0.4}$$

and

$$\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) \text{exists and belongs to} (0,\infty). \tag{0.5}$$

**Theorem 2.**

*Let $g$ satisfy (0.3)-(0.5), and let $u$ be a positive solution of*

$$-\Delta u = g(u) \quad in \quad B_{3R} \tag{0.6}$$

*with*

$$\max_{\overline{B}_R} u \geq 1.$$

*Then*

$$\left(\max_{\overline{B}_R} u\right)\left(\min_{\overline{B}_{2R}} u\right) \leq CR^{2-n},$$

*where $C$ depends only on $n$ and $g$.*

*Remark* 1. Under a slightly stronger hypothesis that $g$ is locally Lipschitz in $(0, \infty)$, the above Theorem was established by C.C. Chen and C.S. Lin in [1] by a delicate moving plane method.

Also in [9] by using the Moving Sphere Method Li and Zhang established the following theorem, which is a boundary version of the Schoen's Harnack Inequality. Let $B_R^+ = \{x = (x', t) \in B_R \,|\, t > 0\}$ denote the Half ball, and $\partial' B_R^+ = \partial B_R^+ \cap \{t = 0\}$.

**Theorem 3.**

*For $n \geq 3$ and $c \in \mathbb{R}$, let $u \in C^1(\overline{B_{3R}^+}) \cap C^2(B_{3R}^+)$ be a positive solution of*

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } B_{3R}^+, \\ \frac{\partial u}{\partial t} = cu^{\frac{n}{n-2}} & \text{on } \partial' B_{3R}^+. \end{cases} \tag{0.7}$$

*Then, for some constant $C = C(n, c)$*

$$\left( \max_{\overline{B_R^+}} u \right) \left( \min_{\overline{\partial B_{2R}^+}} u \right) \leq C(n) R^{2-n}.$$

The main aim of this work is first to study the Method of Moving Sphere by giving a detailed and comprehensive proof of the above three theorems.
Although the arguments of our proofs are taken from the paper of Li and Zhang [9], but the structure and some proof steps are greatly different from the template.

The last chapter of this thesis is dedicated to the extension of the above arguments to some fourth order equation involving critical Sobolev exponent. Namely we prove:

**Theorem 4.**

*For $n \geq 5$, let $B_{3R}$ be a ball of radius $3R$ in $\mathbb{R}^n$, and let $u \in C^2(B_{3R})$ be a positive solution of*

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \;\; \Delta u < 0 \;\; \text{in } B_{3R}$$

*then*

$$\left( \max_{\overline{B}_R} u \right) \left( \min_{\overline{B}_{2R}} u \right) \leq C(n) R^{4-n} \tag{0.8}$$

The Harnack type inequality (0.8) provides very crucial information. One immediate consequence is that the following energy estimate hold true.

$$\int_{B_R} u^{\frac{2n}{n-4}} \leq C. \tag{0.9}$$

We point out that the proof of theorem 4 does not rely on the Liouville type theorem of C.S.Lin ([2]), but some technique arguments are taken from the work of X. Xu [14] and the references therein, who proved the Liouville type theorem of C.S.Lin by applying the Method of Moving Spheres.

As far as we know this is the first work concerning Harnack type inequality result for the Bilaplacian operator.

This document is split into four chapters and two appendix.

- In Chapter 1: We give a proof of the Harnack type inequality of R. Schoen (Theorem 1). As consequence, we establish an energy estimate.
- In Chapter 2: We establish Theorem 2 by essentially the same arguments in Chapter 1.
- In Chapter 3: We prove the Harnack type inequality on half balls (Theorem 3). Our proof is along the line of the proof of Li and Zhang.
- In Chapter 4: We give a proof of our result (Theorem 4), and we establish the energy estimate (0.9).
- In Appendix A: We present some calculus Lemma needed in Chapters 1-3.
- In Appendix B: We present some calculus Lemma used in Chapter 4.

# 1

## HARNACK TYPE INEQUALITY FOR A SECOND ORDER PDE INVOLVING SOBOLEV EXPONENT

In this chapter we present a proof of the following Harnack type inequality for a second order PDE involving sobolev exponent, which was first discovered by R. Schoen in [13].

**Theorem 1.** *For $n \geq 3$, let $B_{3R}$ be a ball of center $0$ and radius $3R$ in $\mathbb{R}^n$, and let $u \in C^2(B_{3R})$ be a positive solution of*

$$-\Delta u = n(n-2)u^{\frac{n+2}{n-2}} \quad in \;\; B_{3R} \tag{1.1}$$

*then*

$$\left(\max_{\overline{B}_R} u\right)\left(\min_{\overline{B}_{2R}} u\right) \leq C(n)R^{2-n} \tag{1.2}$$

A consequence of this Harnack inequality is the following energy estimate.

**Corollary 1.1.** *Let $u$ be as in Theorem 1. Then*

$$\int_{B_R} u^{\frac{2n}{n-2}} \leq C(n).$$

*Remark* 2. The above Theorem was established by Li and Zhang in [9], their proof was by the Method of Moving Spheres. In the following we will expand on Li and Zhang's proof as well as making it more detailed and comprehensive, furthermore laying the ground for our proofs in the chapters to come.

Before starting with the main proof, let us prove the following selection Lemma.

**Lemma 1.1.** *Let $u \in C^0(\overline{B}_1)$ be a positive function. Then for every $a > 0$, there exists $|x| < 1$ such that*

*(i)*

$$u(x) \geq \frac{1}{2^a} \max_{B_\sigma(x)} u$$

*(ii)*

$$\sigma^a u(x) \geq \frac{1}{2^a} u(0)$$

*where* $\sigma = \frac{1-|x|}{2}$

*Proof.* We consider

$$v(y) = (1-|y|)^a u(y)$$

Since $u$ is continuous and positive, it is clear that $v$ attains its maximum over $\overline{B_1}$ in some point $x \in B_1$. Therefore let $x \in B_1$ with

$$v(x) := \max_{y \in \overline{B_1}} v(y) \qquad \text{and} \qquad \sigma = \frac{1-|x|}{2} > 0.$$

Then we have for all $y \in \overline{B_1}$

(i)

$$v(x) = (1-|x|)^a u(x) \geq (1-|y|)^a u(y)$$

which implies that

$$u(x) \geq \frac{(1-|y|)^a}{(1-|x|)^a} u(y) \qquad \text{for all } \; y \in B_1.$$

Next we show that $\overline{B_\sigma(x)} \subseteq B_1$, so let $z \in \overline{B_\sigma(x)}$, then we get

$$|z| = |z + x - x| \leq |z - x| + |x| \leq \sigma + |x| = \frac{1-|x|}{2} + |x| = \frac{1+|x|}{2} < 1.$$

Thus it follows

$$\begin{aligned} u(x) &\geq \max_{B_\sigma(x)} \left( \frac{1-|\cdot|}{1-|x|} \right)^a u \\ &\geq \max_{B_\sigma(x)} \left( \frac{1-|\cdot - x| - |x|}{1-|x|} \right)^a u \\ &\geq \max_{B_\sigma(x)} \left( 1 - \frac{\sigma}{1-|x|} \right)^a u \end{aligned}$$

also using $\frac{\sigma}{1-|x|} = \frac{1}{2}$, we deduce

$$u(x) \geq \frac{1}{2^a} \max_{B_\sigma(x)} u.$$

(ii) Likewise for all $y \in \overline{B_1}$

$$(1-|x|)^a u(x) \geq (1-|y|)^a u(y),$$

which implies that

$$\frac{(1-|x|)^a}{2^a} u(x) \geq \frac{(1-|y|)^a}{2^a} u(y).$$

In particular for $y = 0$, we get

$$\sigma^a u(x) \geq \frac{1}{2^a} u(0).$$

□

*Proof of Theorem 1.* It suffices to prove theorem 1.1 for $R = 1$. The general case follows by working with $v(\cdot) = R^{\frac{n-2}{2}} u(R\cdot)$. We argue by contradiction. If (1.2) were not true, there would exist for each integer $j = 1, 2, \ldots$ solutions $\{u_j\}$ of (1.1), such that

$$u_j(\overline{x_j}) \min_{\overline{B}_2} u_j > j, \tag{1.3}$$

where

$$u_j(\overline{x_j}) = \max_{\overline{B}_1} u_j.$$

Applying Lemma 1.1 to $u = u_j\left(\frac{1}{4}\cdot + \overline{x}_j\right)$ and $a = \frac{n-2}{2}$, we find

$$x_j \in B_{\frac{1}{4}}(\overline{x}_j) \Rightarrow 4\left|x_j - \overline{x}_j\right| < 1$$

such that

$$u_j\left(\frac{1}{4}\left(4\left(x_j - \overline{x}_j\right)\right) + x_j\right) \geq 2^{\frac{2-n}{2}} u_j\left(\frac{1}{4}x + \overline{x}_j\right) \quad \text{for } x \in B_{\sigma_j}\left(4(x_j - \overline{x}_j)\right)$$

thus

$$u_j(x_j) \geq 2^{\frac{2-n}{2}} \max_{B_{\frac{\sigma_j}{4}}(x_j)} u_j,$$

and

$$(\sigma_j)^{\frac{n-2}{2}} u_j(x_j) \geq 2^{\frac{2-n}{2}} u_j(\overline{x}_j),$$

where

$$\sigma_j = \frac{1}{2}\left(1 - 4\left|x_j - \overline{x}_j\right|\right) \leq \frac{1}{2}.$$

Therefore we have

$$u_j(x_j) \geq 2^{\frac{2-n}{2}} (\sigma_j)^{\frac{2-n}{2}} u_j(\overline{x}_j) \geq u_j(\overline{x_j}), \tag{1.4}$$

and, using (1.3) and $\max_{\overline{B}_1} u_j \geq \min_{\overline{B}_2} u_j$, we define

$$\gamma_j := \frac{1}{4} u_j(x_j)^{\frac{2}{n-2}} \sigma_j \geq \frac{1}{8} u_j(\overline{x}_j)^{\frac{2}{n-2}} \geq \frac{1}{8}\left[u_j(\overline{x}_j) \min_{\overline{B}_2} u_j\right]^{\frac{1}{n-2}} \geq \frac{1}{8} j^{\frac{1}{n-2}} \to \infty \tag{1.5}$$

and

$$\Gamma_j := \frac{1}{2} u_j(x_j)^{\frac{2}{n-2}} \geq 4\gamma_j \to \infty.$$

Next we set

$$w_j(y) = \frac{1}{u_j(x_j)} u_j\left(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}}\right), \qquad y \in B_{3\Gamma_j}$$

Since for all $|y| \leq 3\Gamma_j$ we have by the triangle inequality

$$\left|x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}}\right| \leq |x_j| + \frac{|y|}{u_j(x_j)^{\frac{2}{n-2}}} \leq \frac{5}{4} + \frac{3}{2} < 3,$$

then we get by a simple calculation

$$\begin{aligned}
-\Delta w_j &= -\Delta\left(\frac{1}{u_j(x_j)} u_j\left(x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}}\right)\right) \\
&= \frac{1}{u_j(x_j)}\left[-\Delta\left(u_j\left(x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}}\right)\right)\right] \\
&= \frac{1}{u_j(x_j)}\left[\frac{1}{u_j(x_j)^{\frac{2}{n-2}}} \frac{1}{u_j(x_j)^{\frac{2}{n-2}}}\left(-\Delta u_j\left(x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}}\right)\right)\right] \\
&= \frac{1}{u_j(x_j)^{\frac{n+2}{n-2}}}\left[n\,(n-2)\, u_j^{\frac{n+2}{n-2}}\left(x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}}\right)\right] \\
&= n\,(n-2)\, w_j^{\frac{n+2}{n-2}}, \qquad w_j > 0 \text{ on } B_{3\Gamma_j}
\end{aligned} \tag{1.6}$$

We also have

$$\begin{aligned}
u_j(x_j) &\geq 2^{\frac{2-n}{2}} \max\left\{u_j(y) \;\middle|\; |y - x_j| < \frac{1}{4}\sigma_j\right\} \\
&\geq 2^{\frac{2-n}{2}} \max\left\{u_j\left(\frac{y}{u_j(x_j)^{\frac{2}{n-2}}} + x_j\right) \;\middle|\; |y| < \frac{1}{4}\sigma_j u_j(x_j)^{\frac{2}{n-2}} = \gamma_j\right\}
\end{aligned}$$

Thus by the definition of $w_j$

$$w_j(0) = 1 \geq 2^{\frac{2-n}{2}} \max_{B_{\gamma_j}} w_j. \tag{1.7}$$

Again by the triangle inequality follows for all $|y| \leq \frac{3}{2}\Gamma_j$

$$\left| x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \right| \leq \frac{5}{4} + \frac{3}{4} = 2.$$

Therefore by the superharmonicity of $w_j$ and the strong maximum principle,

$$\begin{aligned} \min_{\overline{B_{\frac{3}{2}\Gamma_j}}} w_j &= \min_{\partial B_{\frac{3}{2}\Gamma_j}} w_j \\ &\geq \frac{\min_{\overline{B}_2} u_j}{u_j(x_j)} \end{aligned}$$

by (1.3) and (1.4)

$$\begin{aligned} &> \frac{j}{u_j(x_j)^2} \\ &= \frac{j}{2^{n-2}}\Gamma_j^{2-n} \end{aligned}$$

All together we have

$$\begin{cases} -\Delta w_j = n(n-2)w_j^{\frac{n+2}{n-2}} & \text{in } B_{3\Gamma_j}, \\ w_j(0) = 1, \\ w_j \leq 2^{\frac{n-2}{2}} & \text{in } B_{\gamma_j} \\ w_j > \frac{j}{2^{n-2}}\Gamma_j^{2-n} & \text{in } B_{\frac{3}{2}\Gamma_j}. \end{cases} \tag{1.8}$$

So far, we have just constructed solutions $\{w_j\}$ of the equation (1.1) that are defined on very large balls. The next step is to show that this sequence has a converging subsequence in $C^2$ norm on any compact subset of $\mathbb{R}^n$. This will be deduced by the following Lemma.

**Lemma 1.2.** *Let $w_j$ be defined as above. Then*

$$\|w_j\|_{C^{2,\alpha}(B_{\frac{\gamma_j}{4}})} \leq D$$

*where $D$ a constant independent of $j$.*

*Proof.* On $B_{\gamma_j}$ we have

$$\begin{aligned} |\Delta w_j| &= n(n-2)w_j^{\frac{n+2}{n-2}} \\ &\leq n(n-2)2^{\frac{n+2}{2}} \\ &\leq C \end{aligned}$$

where $C$ independent of $j$.
Consequently for all $p$

$$\Delta w_j \in L^p\left(B_{\gamma_j}\right).$$

By the standard $L^p$-regularity, we get for each open subset $\Omega \subset\subset B_{\gamma_j}$

$$w_j \in w^{2,p}(\Omega).$$

From Morrey's Theorem and (1.6) follows

$$w_j \in C^{0,\alpha}\left(B_{\gamma_j/2}\right) \Longrightarrow \Delta w_j \in C^{0,\alpha}\left(B_{\gamma_j/2}\right) \quad \text{where } \ 0<\alpha<1.$$

Therefore by $C^\alpha$-regularity, we get $w_j \in C^{2,\alpha}$, and

$$\begin{aligned}\|w_j\|_{C^{2,\alpha}\left(B_{\gamma_j/4}\right)} &\le C\left(\|\Delta w_j\|_{C^{0,\alpha}\left(B_{\gamma_j/2}\right)} + \|w_j\|_{L^2\left(B_{\gamma_j/2}\right)}\right)\\ &\le D.\end{aligned}$$

□

From Lemma 1.2 we see that $\|w_j\|_{C^{2,\alpha}\left(B_{\gamma_j/4}\right)}$ is *uniformly* bounded, which means that the functions $w_j$ and their first and second derivatives are equicontinuous. Thus by the *Arzela-Ascoli* compactness criterion, we can find a subsequence -still denoted by $w_j$- which converges uniformly to some $w \in C^2(\Omega)$, where $\Omega \subset\subset \mathbb{R}^n$ and $w$ is a positive solution of (1.1)
Therewith the first step has been performed, we start now discussing the Kelvin's transformation for Laplace operator.

For $x \in \mathbb{R}^n$, let $w_{j,x,\lambda}$ denote the Kelvin transform of $w_j$ with respect to the $B_\lambda(x)$, i.e

$$w_{j,x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2} w_j\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right), \quad y \in B_{3\Gamma_j}(x)\backslash\overline{B_\lambda(x)}$$

By Lemma A.1 and the fact that $w_j$ is a solution of (1.1), we obtain with the following direct calculation that $w_{j,x,\lambda}$ is also a solution of the problem (1.1).

$$\begin{aligned}-\Delta w_{j,x,\lambda} &= \left(\frac{\lambda}{|y-x|}\right)^{n+2}\left[-\Delta w_j\left(x + \frac{\lambda^2(\cdot - x)}{|\cdot - x|^2}\right)\right]\\ &= \left(\frac{\lambda}{|y-x|}\right)^{n+2} n(n-2) w_j^{\frac{n+2}{n-2}}\left(x + \frac{\lambda^2(\cdot - x)}{|\cdot - x|^2}\right)\\ &= n(n-2) w_{j,x,\lambda}^{\frac{n+2}{n-2}}. \qquad (1.9)\end{aligned}$$

Next we want to compare, for any fixed $x \in \mathbb{R}^n$, $w_j$ with $w_{j,x,\lambda}$ and we shall always take without loss of generality $x = 0$. For $x \neq 0$ the arguments follow similarly.
Set for $0 < \lambda < \frac{1}{2}\Gamma_j$

$$\Sigma_{j,x,\lambda} := B_{\Gamma_j}(x) \backslash \overline{B_\lambda}(x)$$

We note that for $x \neq 0$, there holds $B_{\Gamma_j}(x) \subset B_{\frac{3}{2}\Gamma_j}$ for $j$ large enough since $\frac{1}{2}\Gamma_j \to \infty$.
We use $w_{j,\lambda}$ to denote $w_{j,0,\lambda}$ i.e

$$w_{j,\lambda}(y) = \left(\frac{\lambda}{|y|}\right)^{n-2} w_j\left(\frac{\lambda^2 y}{|y|^2}\right)$$

We define for $y \in \Sigma_{j,x,\lambda}$

$$f_{j,x,\lambda}(y) = w_j(y) - w_{j,x,\lambda}(y)$$

Then by (1.8) and (1.9), we obtain

$$-\Delta f_{j,x,\lambda} = w_j^{\frac{n+2}{n-2}} - w_{j,x,\lambda}^{\frac{n+2}{n-2}} \quad \text{in} \quad \Sigma_{j,x,\lambda}. \tag{1.10}$$

We use $f_{j,\lambda}, w_{j,\lambda}$ and $\Sigma_{j,\lambda}$ to denote $f_{j,0,\lambda}, w_{j,0,\lambda}$ and $\Sigma_{j,0,\lambda}$.
The next lemma says that we can start the Method of Moving Spheres.

**Lemma 1.3.** *For every $x \in \mathbb{R}^n$, there exists $\lambda_{j,x} > 0$ small enough such that*

$$f_{j,x,\lambda} \geq 0 \quad \textit{for all } 0 < \lambda < \lambda_{j,x} \textit{ and } \quad \lambda \leq |y - x| \leq \Gamma_j.$$

*Proof.* As explained above, without loss of generality, we may take $x = 0$. Then we will prove it by two steps.

- *Step 1.* There exists $R_j < \Gamma_j$ such that for $\lambda < |y| < R_j$ we have

$$f_{j,\lambda}(y) \geq 0.$$

  For $\lambda < |y| < R_j$ and $R_j$ small enough, we have by lemma 1.2 and the fact that $w_j$ is positive

$$d_j := \inf w_j(y) > 0. \tag{1.11}$$

$$c_j := \sup \left|\nabla w_j\right| < \infty. \tag{1.12}$$

  Let in polar coordinates,

$$g_j(r, \theta) := r^{\frac{n-2}{2}} w_j(r, \theta).$$

Then for $0 < r < R_j \leq \frac{n-2}{2} \cdot \frac{d_j}{c_j}$ we get

$$\begin{aligned} \frac{\partial}{\partial r}\left(g_j(r,\theta)\right) &= \frac{n-2}{2} r^{\frac{n}{2}-2} w_j(r,\theta) + r^{\frac{n}{2}-1} \frac{\partial}{\partial r} w_j(r,\theta) \\ &= r^{\frac{n}{2}-2}\left(\frac{n-2}{2} w_j(r,\theta) + r\frac{\partial}{\partial r} w_j(r,\theta)\right) \\ &\geq r^{\frac{n}{2}-2}\left(\frac{n-2}{2} w_j(r,\theta) - r\left|\frac{\partial}{\partial r} w_j(r,\theta)\right|\right) \\ &\geq r^{\frac{n}{2}-2}\left(\frac{n-2}{2} d_j - \underbrace{r \cdot c_j}_{<\frac{n-2}{2} d_j}\right) \\ &> 0. \end{aligned}$$

Therefore with $r_1 \cdot \theta = \frac{\lambda^2 y}{|y|^2}$ and $r_2 \cdot \theta = y$, we have for all $0 < \lambda < |y| < R_j$

$$f_{j,\lambda}(y) \geq 0.$$

- *Step 2.* There exists $0 < \lambda_j \leq R_j$ such that for all $0 < \lambda \leq \lambda_j$

$$f_{j,\lambda} \geq 0 \quad \text{for all} \quad R_j \leq |y| \leq \Gamma_j.$$

Let

$$\lambda_j := R_j \left(\frac{\min_{B_{\Gamma_j} \setminus B_{R_j}} w_j}{\max_{B_{R_j}} w_j}\right)^{\frac{1}{n-2}} \leq R_j.$$

Then

$$\left(\frac{\lambda_j}{R_j}\right)^{n-2} \max_{\overline{B_{R_j}}} w_j = \min_{B_{\Gamma_j} \setminus B_{R_j}} w_j$$

Therefore for every $0 < \lambda \leq \lambda_j$ and $|y| \geq R_j$, we have

$$\begin{aligned} w_{j,\lambda}(y) &= \left(\frac{\lambda}{|y|}\right)^{n-2} w_j\left(\frac{\lambda^2 y}{|y|^2}\right) \\ &\leq \left(\frac{\lambda_j}{R_j}\right)^{n-2} \max_{|y| \geq R_j} w_j\left(\frac{\lambda^2 y}{|y|^2}\right) \end{aligned}$$

since $\frac{\lambda^2}{|y|} \leq \frac{\lambda^2}{R_j} \leq R_j$, follows

$$\begin{aligned}
&\leq \left(\frac{\lambda_j}{R_j}\right)^{n-2} \max_{\overline{B_{R_j}}} w_j \\
&= \min_{B_{\Gamma_j} \setminus B_{R_j}} w_j \\
&\leq w_j(y).
\end{aligned}$$

Combining *Step 1* and *Step 2*, we get

$$f_{j,\lambda} \geq 0 \text{ for all } 0 < \lambda \leq \lambda_j \text{ and } \lambda \leq |y| \leq \Gamma_j.$$

□

Therefore we can define

$$\overline{\lambda}_j(x) := \sup\left\{0 < \mu \ ; \ w_{j,x,\lambda}(y) \leq w_j(y), \forall y \in \overline{\Sigma_{j,x,\lambda}}, 0 < \lambda \leq \mu\right\},$$

where $\Sigma_{j,x,\lambda} := B_{\Gamma_j}(x) \setminus \overline{B_\lambda(x)}$.
The purpose of the Moving Spheres Method is to show that $\overline{\lambda}_j(x)$ is unbounded for every $x \in \mathbb{R}^n$ for $j$ large enough. This will be attained by an application of the strong maximum principle and the Hopf Lemma.

**Lemma 1.4.** *For every* $x \in \mathbb{R}^n$,

$$\lim_{j \to \infty} \overline{\lambda}_j(x) = \infty.$$

*Proof.* Without loss of generality, we may take $x = 0$. This proof is through a contradiction argument. Suppose the contrary, then there exist

$$\overline{\lambda}_j \leq C < \gamma_j, \tag{1.13}$$

for some constant C independent of $j$. Here we have used the fact $\gamma_j \to \infty$(see (1.5)). To reach a contradiction we will only need to show that

$$f_{j,\overline{\lambda}_j}(y) > 0 \qquad \text{for } y \in \overline{\Sigma_{\overline{\lambda}_j}} \setminus \partial B_{\overline{\lambda}_j} \tag{1.14}$$

and

$$\frac{df_{j,\overline{\lambda}_j}}{dr}(y) > 0 \qquad \text{for } y \in \partial B_{\overline{\lambda}_j} \tag{1.15}$$

Indeed from (1.15) and the continuity of $\nabla f_{j,\overline{\lambda}_j}$, there exists $\overline{\lambda}_j < R_j < \Gamma_j$ such that

$$\frac{d}{dr} f_{j,\lambda} > 0, \qquad \text{for } \overline{\lambda}_j \leq \lambda \leq R_j, \ \lambda \leq r \leq R_j.$$

Consequently, since $f_{j,\lambda} = 0$ on $\partial B_\lambda$, we have

$$f_{j,\lambda}(y) > 0, \qquad \text{for } \overline{\lambda}_j \leq \lambda < R_j,\ \lambda < |y| \leq R_j. \tag{1.16}$$

By (1.14) we can find $c_j > 0$ such that

$$w_j(y) - w_{j,\overline{\lambda}_j}(y) \geq \frac{c_j R_j^{n-2}}{|y|^{n-2}} \qquad \forall |y| \geq R_j.$$

Therefore for $|y| \geq R_j$

$$w_j(y) - w_{j,\lambda}(y) \geq \frac{c_j R^{n-2}}{|y|^{n-2}} - \left(w_{j,\lambda}(y) - w_{j,\overline{\lambda}_j}(y)\right). \tag{1.17}$$

By the uniform continuity of $w_j$ on $\overline{B_{R_j}}$, there exists $0 < \epsilon_j < R_j - \overline{\lambda}_j$ such that for all $\overline{\lambda}_j \leq \lambda \leq \overline{\lambda}_j + \epsilon_j$ and $|y| \geq R_j$

$$\begin{aligned}
\left|\lambda^{n-2} w_j\left(\frac{\lambda^2 y}{|y|^2}\right) - \overline{\lambda}_j^{n-2} w_j\left(\frac{\overline{\lambda}_j^2 y}{|y|^2}\right)\right| &= \left|\lambda^{n-2}(w_j(\frac{\lambda^2 y}{|y|^2}) - w_j(\frac{\overline{\lambda}_j^2 y}{|y|^2})) + (\lambda^{n-2} - \overline{\lambda}_j^{n-2}) w_j(\frac{\overline{\lambda}_j^2 y}{|y|^2})\right| \\
&\leq R^{n-2}\epsilon_j + \left|(\lambda - \overline{\lambda}_j)\Sigma_{k=0}^{n-3}\lambda^k \overline{\lambda}_j^{n-3-k}\right| w_j(\frac{\overline{\lambda}_j^2 y}{|y|^2}) \\
&\leq R^{n-2}\epsilon_j + o(R^{n-2}\epsilon_j)
\end{aligned}$$

after choosing an appropriate $\epsilon_j$

$$< \frac{c_j R^{n-2}}{2}.$$

Therefore and by (1.17)

$$w_j(y) - w_{j,\lambda}(y) > 0, \qquad \text{for } \overline{\lambda}_j \leq \lambda \leq \overline{\lambda}_j + \epsilon_j,\ R_j \leq |y| \leq \Gamma_j.$$

Estimates (1.16) and the above violate the definition of $\overline{\lambda}_j$.

Now we go back and try to establish (1.14) and (1.15).

By the definition of $\overline{\lambda}_j$,

$$w_{j,\overline{\lambda}_j} \leq w_j \quad \text{in } \Sigma_{\overline{\lambda}_j}$$

By (1.8) and (1.9) we get

$$-\Delta\left(w_j - w_{j,\overline{\lambda}_j}\right) = n(n-2)\left(w_j^{\frac{n+2}{n-2}} - w_{j,\overline{\lambda}_j}^{\frac{n+2}{n-2}}\right) \geq 0, \qquad \text{in } \Sigma_{\overline{\lambda}_j}. \tag{1.18}$$

We also have

$$\max_{\partial B_{\Gamma_j}} w_{j,\overline{\lambda}_j} = \max_{\partial B_{\Gamma_j}} \left(\frac{\overline{\lambda}_j}{|y|}\right)^{n-2} w_j \left(\frac{\overline{\lambda}_j^2}{\Gamma_j^2} y\right)$$

so by (1.16), we get

$$\begin{aligned} &\leq \left(\frac{C}{\Gamma_j}\right)^{n-2} \max_{\partial B_{\Gamma_j}} w_j \left(\frac{\overline{\lambda}_j^2}{|y|^2} y\right) \\ &= C^{n-2}\Gamma_j^{2-n} \max_{\partial B_{\frac{\overline{\lambda}_j^2}{\Gamma_j}}} w_j \end{aligned}$$

and because of $\frac{\overline{\lambda}_j^2}{\Gamma_j} < \frac{\gamma_j^2}{\Gamma_j} = \frac{1}{4}\frac{\Gamma_j^2\sigma_j^2}{\Gamma_j} = \frac{1}{4}\Gamma_j\sigma_j^2 < \frac{1}{2}\Gamma_j\sigma_j = \gamma_j$, follows

$$\leq C^{n-2}\Gamma_j^{2-n} \max_{B_{\gamma_j}} w_j$$

also using (1.7), we deduce

$$\leq D\ \Gamma_j^{2-n} \tag{1.19}$$

for some $D$ independent of $j$.
Therefore, by (1.19) and (1.8), for large $j$, we get

$$\min_{\partial B_{\Gamma_j}} \left(w_j - w_{j,\overline{\lambda}_j}\right) \geq \min_{\partial B_{\Gamma_j}} w_j - \max_{\partial B_{\Gamma_j}} w_{j,\overline{\lambda}_j} > \frac{j}{2^{n-2}}\Gamma_j^{2-n} - D\ \Gamma_j^{2-n} > 0.$$

We recall that

$$w_j - w_{j,\overline{\lambda}_j} = 0 \qquad \text{on } \ \partial B_{\overline{\lambda}_j}.$$

Thus all together we have

$$\begin{cases} -\Delta f_{j,\overline{\lambda}_j} \geq 0 & \text{in } \ \Sigma_{\overline{\lambda}_j}, \\ f_{j,\overline{\lambda}_j} > 0 & \text{on } \ \partial B_{\Gamma_j} \\ f_{j,\overline{\lambda}_j} = 0 & \text{on } \ \partial B_{\overline{\lambda}_j}. \end{cases}$$

An application of *the strong maximum principle* and *the Hopf Lemma* yield estimates (1.14) and (1.15). □

By Lemma 1.2 we know that along a subsequence,

$$w_j \longrightarrow w \qquad \text{in } C^2_{loc}(\mathbb{R}^n)$$

for some solution $w$ of

$$\begin{aligned} &-\Delta w = n(n-2)w^{\frac{n+2}{n-2}} \qquad \text{on } \mathbb{R}^n \\ &w(0) = 1 \end{aligned} \tag{1.20}$$

By the convergence of $w_j$ to $w$ and the fact that $\overline{\lambda}_j(x) \to \infty$ for every $x \in \mathbb{R}^n$, we have

$$w_{x,\lambda}(y) \leq w(y), \qquad \forall\, |y-x| \geq \lambda > 0.$$

It follows, by Lemma A.2, $w \equiv w(0) \equiv 1$. Which is impossible because of (1.20). Theorem 1 is established.

□

Now thanks to *Theorem 1* we give a

*Proof of Corollary 1.1.* Clearly, we only need to establish it for $R = 1$. Let $G$ be the Green's function on $B_3$, i.e.

$$\begin{cases} -\Delta G(x, \cdot) = \delta_x & \text{in } B_3 \\ G(x, \cdot) = 0 & \text{on } \partial B_3. \end{cases}$$

Then it is clear by the maximum principle that there exists a constant $C \geq 1$ such that for all $y$ and $\eta \in B_2$ we have

$$G(y, \eta) \geq C^{-1}$$

and by the Hopf Lemma for every fixed $y \in B_3$ and $s \in \partial B_3$

$$\frac{\partial G(y, s)}{\partial \nu} < 0.$$

Now we let for $y \in \overline{B_2}$

$$u(y) := \min_{\overline{B_2}} u.$$

Then we get by the Green's representation formula,

$$\begin{aligned} u(y) &= \int_{B_3} G(y, \eta)\,(-\Delta u)\,\mathrm{d}\eta - \int_{\partial B_3} \frac{\partial G(y, s)}{\partial \nu} u(s)\,\mathrm{d}s \\ &\geq (n(n-2)) \int_{B_3} G(y, \eta) u^{\frac{n+2}{n-2}}(\eta)\,\mathrm{d}\eta \\ &\geq n(n-2)C^{-1} \int_{B_1} u^{\frac{n+2}{n-2}}(\eta)\,\mathrm{d}\eta \end{aligned}$$

Therefore by Theorem 1

$$\begin{aligned} \int_{B_1} u^{\frac{2n}{n-2}} &\leq \max_{\overline{B_1}} u \int_{B_1} u^{\frac{n+2}{n-2}} \\ &\leq C \left( \max_{\overline{B_1}} u \right) \left( \min_{\overline{B_2}} u \right) \\ &\leq C. \end{aligned}$$

□

# 2

## A HARNACK TYPE INEQUALITY FOR MORE GENERAL EQUATIONS IN $\mathbb{R}^N$

In this chapter we give a proof of a Harnack type inequality for more general right hand side $g(y,u)$.
The proof goes along with the proof of *Theorem 1*.
In particular we assume that $g$ satisfies

$$g \text{ is continuous and positive in} (0,\infty), \text{and} \sup_{0<s\leq t} g(s) < \infty, \forall t < \infty, \tag{2.1}$$

$$s^{-\frac{n+2}{n-2}} g(s) \text{is non-increasing in } (0,\infty), \tag{2.2}$$

and

$$\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) \text{exists and belongs to} (0,\infty). \tag{2.3}$$

**Theorem 2.** *Let $g$ satisfy conditions (2.1)- (2.3), and let $u$ be a positive solution of*

$$-\Delta u = g(u) \quad in \quad B_{3R} \tag{2.4}$$

*with*

$$\max_{\overline{B}_R} u \geq 1.$$

*Then*

$$\left(\max_{\overline{B}_R} u\right)\left(\min_{\overline{B}_{2R}} u\right) \leq C R^{2-n},$$

*where $C$ depends only on $n$ and $g$.*

*Remark* 3. The key argument to prove the above theorem, like in the proof of Theorem 1, is to apply the maximum principle and the Hopf Lemma. Because we are dealing, in this case, with a general right hand side $g(u)$, we would not be able to apply it directly. The reason therefore, being because, on one hand, the rescaling function $w_j$ will not be a solution of (2.4) and on the other hand neither would its Kelvin's transformation. Thanks to the condition made on $g$ we will be able to prove the above Theorem with the same arguments as in the proof of Theorem 1. We shall also mention that by allowing $\lim_{s\to\infty} s^{-\frac{n+2}{n-2}} g(s) = 0$ in (2.3) then the result is no longer valid (For more details see [9]).

*Proof of Theorem 2.* The proof is by contradiction argument. We suppose the contrary, then there exist for each integer $j = 1, 2, \ldots,$ solutions $\{u_j\}$ of (2.4), such that

$$u_j(\overline{x_j}) \min_{\overline{B}_{2R_j}} u_j > \frac{j}{R_j^{n-2}}, \tag{2.5}$$

where

$$u_j(\overline{x_j}) = \max_{\overline{B}_{R_j}} u_j \geq 1. \tag{2.6}$$

Our first step is finding a super harmonic sequence, that is related to the equation (2.4) and defined on very large balls. To do so we will follow the same arguments in the proof of *Theorem 1.* For convenience of the reader, we give all the details explicitly again.
Applying Lemma 1.1 to $u = u_j\left(\frac{R_j}{4}\cdot + \overline{x}_j\right)$ and $a = \frac{n-2}{2}$, we find

$$x_j \in B_{\frac{R_j}{4}}(\overline{x}_j) \;\Rightarrow\; \left|\frac{4}{R_j}(x_j - \overline{x}_j)\right| < 1$$

such that

$$u_j\left(\frac{R_j}{4}\left(\frac{4}{R_j}(x_j - \overline{x}_j)\right) + \overline{x}_j\right) \geq 2^{\frac{2-n}{2}} \max_{B_{\sigma_j}\left(\frac{4}{R_j}(x_j - \overline{x}_j)\right)} u_j\left(\frac{R_j}{4}\cdot + \overline{x}_j\right)$$

thus

$$u_j(x_j) \geq 2^{\frac{2-n}{2}} \max_{B_{\frac{R_j}{4}\sigma_j}(x_j)} u_j,$$

and

$$(\sigma_j)^{\frac{n-2}{2}} u_j(x_j) \geq 2^{\frac{2-n}{2}} u_j(\overline{x}_j),$$

where

$$\sigma_j = \frac{1}{2}\left(1 - \frac{4}{R_j}\left|x_j - \overline{x}_j\right|\right) \leq \frac{1}{2}.$$

Therefore we get

$$u_j(x_j) \geq u_j(\overline{x}_j) \geq 1 \tag{2.7}$$

and, also using (2.5) and $\max_{\overline{B}_{R_j}} u_j \geq \min_{\overline{B}_{2R_j}} u_j$, we define

$$\gamma_j := \frac{R_j}{4} u_j(x_j)^{\frac{2}{n-2}} \sigma_j \geq \frac{R_j}{8} u_j(\overline{x}_j)^{\frac{2}{n-2}} \geq \frac{R_j}{8} \left[ u_j(\overline{x}_j) \min_{\overline{B}_{2R_j}} u_j \right]^{\frac{1}{n-2}} \geq \frac{1}{8} j^{\frac{1}{n-2}} \to \infty \tag{2.8}$$

and

$$\Gamma_j := \frac{R_j}{2} u_j(x_j)^{\frac{2}{n-2}} \geq 4\gamma_j \to \infty.$$

Now we set

$$w_j(y) = \frac{1}{u_j(x_j)} u_j \left( x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \right) \qquad y \in B_{3\Gamma_j}.$$

Since for all $|y| \leq 3\Gamma_j$ we have by the triangle inequality

$$\left| x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \right| \leq |x_j| + \frac{|y|}{u_j(x_j)^{\frac{2}{n-2}}} \leq \frac{5}{4} R_j + \frac{3}{2} R_j < 3R_j$$

then we get with a simple calculation

$$\begin{aligned}
-\Delta w_j &= -\Delta \left( \frac{1}{u_j(x_j)} u_j \left( x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}} \right) \right) \\
&= \frac{1}{u_j(x_j)} \left[ -\Delta \left( u_j \left( x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}} \right) \right) \right] \\
&= \frac{1}{u_j(x_j)} \left[ \frac{1}{u_j(x_j)^{\frac{2}{n-2}}} \frac{1}{u_j(x_j)^{\frac{2}{n-2}}} \left( -\Delta u_j \left( x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}} \right) \right) \right] \\
&= \frac{1}{u_j(x_j)^{\frac{n+2}{n-2}}} \left[ g \left( u_j \left( x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}} \right) \right) \right] \\
&= u_j(x_j)^{-\frac{n+2}{n-2}} g \left( u_j(x_j) w_j \right) \geq 0 \quad , \quad \text{on} \quad B_{3\Gamma_j}
\end{aligned} \tag{2.9}$$

and because of

$$\begin{aligned}
u_j(x_j) &\geq 2^{\frac{2-n}{2}} \max \left\{ u_j(y) \;\middle|\; |y - x_j| < \frac{R_j}{4} \sigma_j \right\} \\
&\geq 2^{\frac{2-n}{2}} \max \left\{ u_j \left( \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} + x_j \right) \;\middle|\; |y| < \frac{R_j}{4} \sigma_j u_j(x_j)^{\frac{2}{n-2}} = \gamma_j \right\}
\end{aligned}$$

follows due to the definition of $w_j$

$$w_j(0) = 1 \geq 2^{\frac{2-n}{2}} \max_{B_{\gamma_j}} w_j \ . \tag{2.10}$$

Again by the triangle inequality follows for all $|y| \leq \frac{3}{2}\Gamma_j$

$$\left| x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \right| \leq \frac{5}{4}R_j + \frac{3}{4}R_j = 2R_j.$$

Therefore thanks to the superharmonicity of $w_j$ and the strong maximum principle

$$\begin{aligned} \min_{\overline{B}_{\frac{3}{2}\Gamma_j}} w_j &= \min_{\partial B_{\frac{3}{2}\Gamma_j}} w_j \\ &\geq \frac{\min_{\overline{B}_{2R_j}} u_j}{u_j(x_j)} \end{aligned}$$

by (2.5) and (2.7)

$$\begin{aligned} &> \frac{j}{u_j(x_j)^2 R_j^{n-2}} \\ &= \frac{j}{2^{n-2}}\Gamma_j^{2-n} \end{aligned}$$

All in all we have

$$\begin{cases} -\Delta w_j = u_j(x_j)^{-\frac{n+2}{n-2}} g\left(u_j(x_j) w_j\right) & \text{in } B_{3\Gamma_j}, \\ w_j(0) = 1, & \\ w_j \leq 2^{\frac{n-2}{2}} & \text{in } B_{\gamma_j} \\ w_j > \frac{j}{2^{n-2}}\Gamma_j^{2-n} & \text{in } B_{\frac{3}{2}\Gamma_j}. \end{cases} \tag{2.11}$$

Our first step is complete and we can start looking for a converging subsequence. Thanks to the condition on $g$, we will be able to show that $w_j$ have a converging subsequence in $C^1$ norm on any compact subset of $\mathbb{R}^n$. This will be deduced from the following lemma.

**Lemma 2.1.** *Let $w_j$ be defined as above. Then*

$$\left\| w_j \right\|_{C^{1,\alpha}(B_{\frac{\gamma_j}{2}})} \leq D$$

*where $D$ a constant independent of $j$.*

*Proof.* Because of (2.2) and (2.1) there exist a constant $C < \infty$ independent of $j$ such that

$$g(s) \leq C\left(1 + s^{\frac{n+2}{n-2}}\right), \quad \forall s > 0.$$

Therefore on $B_{\gamma_j}$

$$\begin{aligned}|\Delta w_j| &= u_j(x_j)^{-\frac{n+2}{n-2}} g\left(u_j(x_j) w_j\right) \\ &\leq u_j(x_j)^{-\frac{n+2}{n-2}} C\left(1 + u_j(x_j)^{\frac{n+2}{n-2}} w_j^{\frac{n+2}{n-2}}\right) \\ &= u_j(x_j)^{-\frac{n+2}{n-2}} C + C w_j^{\frac{n+2}{n-2}}\end{aligned}$$

using (2.7) and (2.10), we get

$$\begin{aligned}&\leq C + 2^{\frac{n+2}{2}} C \\ &\leq D\end{aligned}$$

where $D$ independent of $j$.
Consequently for all $p$

$$\Delta w_j \in L^p\left(B_{\gamma_j}\right).$$

By the standard $L^p$-regularity, we get for each open subset $\Omega \subset\subset B_{\gamma_j}$

$$w_j \in w^{2,p}(\Omega).$$

Therefore by $C^\alpha$-regularity, we have $w_j \in C^{1,\alpha}$ and

$$\begin{aligned}\|w_j\|_{C^{1,\alpha}\left(B_{\gamma_j/2}\right)} &\leq C\left(\|\Delta w_j\|_{L^p\left(B_{\gamma_j/2}\right)} + \|w_j\|_{L^2\left(B_{\gamma_j/2}\right)}\right) \\ &\leq D.\end{aligned}$$

□

Since $\gamma_j \to \infty$, we can easily by Lemma 2.1 and *Arzela-Ascoli* compactness criterion find a subsequence $w_j$ such that

$$w_j \longrightarrow w \qquad \text{in } C^1_{loc}(\mathbb{R}^n)$$

Now we are in the position to discuss the Kelvin's transformation for Laplace operator.
For $x \in \mathbb{R}^n$, let $w_{j,x,\lambda}$ denote the Kelvin transformation of $w_j$ with respect to the $B_\lambda(x)$, i.e

$$w_{j,x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2} w_j\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \quad \text{for } y \in B_{3\Gamma_j} \backslash \overline{B_\lambda(x)}$$

By Lemma A.1 and the fact that $w_j$ solution of (2.9), we obtain

$$
\begin{aligned}
-\Delta w_{j,x,\lambda} &= \left(\frac{\lambda}{|\cdot - x|}\right)^{n+2} \left[-\Delta w_j\left(x + \frac{\lambda^2(\cdot - x)}{|\cdot - x|^2}\right)\right] \\
&= \left(\frac{\overline{\lambda}_j}{|\cdot - x|}\right)^{n+2} u_j(x_j)^{-\frac{n+2}{n-2}} g\left(u_j(x_j) w_j\left(\frac{\overline{\lambda}_j^2}{|\cdot - x|^2}\cdot\right)\right) \\
&= \left(\frac{|\cdot - x|}{\overline{\lambda}_j}\right)^{-n-2} u_j(x_j)^{-\frac{n+2}{n-2}} g\left(u_j(x_j)\left(\frac{|\cdot - x|}{\overline{\lambda}_j}\right)^{n-2} w_{j,x,\lambda}\right) \qquad (2.12)
\end{aligned}
$$

Thanks to the condition on $g$ (2.2), we see that $-\Delta w_{j,x,\lambda} \geq 0$.
As in Theorem 1, we shall compare for any fixed $x \in \mathbb{R}^n, w_j$ with $w_{j,x,\lambda}$ and we shall always take without loss of generality $x = 0$. For $x \neq 0$, the argument follows similarly.
Set for $\lambda > 0$

$$\Sigma_{j,x,\lambda} := B_{\Gamma_j}(x) \backslash \overline{B_\lambda}(x)$$

The restriction of the domain to $B_{\Gamma_j}$ is needed for $x \neq 0$, seeing that we would still have $B_{\Gamma_j}(x) \subset B_{\frac{3}{2}\Gamma_j}$ for $j$ large enough since $\frac{1}{2}\Gamma_j \to \infty$.
We use $w_{j,\lambda}$ to denote $w_{j,0,\lambda}$ i.e

$$w_{j,\lambda}(y) = \left(\frac{\lambda}{|y|}\right)^{n-2} w_j\left(\frac{\lambda^2 y}{|y|^2}\right)$$

We define for $y \in \Sigma_{j,x,\lambda}$

$$f_{j,x,\lambda}(y) = w_j(y) - w_{j,x,\lambda}(y)$$

Since by (2.11) we have affirmed that $w_j$ is positive and in a small domain bounded, our next lemma can be proved with same argument as Lemma 1.3.

**Lemma 2.2.** *For every $x \in \mathbb{R}^n$, there exists $\lambda_{j,x} > 0$ small enough such that*

$$f_{j,x,\lambda} \geq 0 \quad \textit{for all } 0 < \lambda < \lambda_{j,x} \textit{ and } \lambda \leq |y - x| \leq \Gamma_j.$$

Therefore we can define

$$\overline{\lambda}_j(x) := \sup\left\{0 < \mu \ ; \ w_{j,x,\lambda}(y) \leq w_j(y), \forall y \in \overline{\Sigma_{j,x,\lambda}}, 0 < \lambda \leq \mu\right\}.$$

Next we show the same (Lemma 1.4) under the new condition:

**Lemma 2.3.** *For every $x \in \mathbb{R}^n$,*

$$\lim_{j\to\infty} \overline{\lambda}_j(x) = \infty.$$

*Proof.* Without loss of generality, we take $x = 0$. This proof is also with contradiction argument. Suppose the contrary, then there exist

$$\overline{\lambda_j} \leq C < \gamma_j, \tag{2.13}$$

for some constant $C$ independent of $j$. Here we have used the fact $\gamma_j \to \infty$(see (2.8)). We saw in the proof of Lemma 1.4 that we only need to show

$$f_{j,\overline{\lambda_j}}(y) > 0 \qquad \text{for} \quad y \in \overline{\Sigma_{\overline{\lambda_j}}} \backslash \partial B_{\overline{\lambda_j}} \tag{2.14}$$

and

$$\frac{df_{j,\overline{\lambda_j}}}{dr}(y) > 0 \qquad \text{for} \quad y \in \partial B_{\overline{\lambda_j}} \tag{2.15}$$

to reach a contradiction.
Once (2.14) and (2.15) are established, the rest of the proof of Lemma 2.2 is the same as the proof of Lemma 1.4.
Or alternatively we can prove it as follows. Assuming that (2.14) and (2.15) hold. Then we can easily find $a_j > 0$ such that

$$f_{j,\overline{\lambda_j}} \geq a_j \quad \text{for all} \quad y \in B_{\Gamma_j} \backslash B_{2\overline{\lambda_j}}. \tag{2.16}$$

By the definition of $\overline{\lambda_j}$ and (2.13) there exists for every fixed $j$ a sequence $\{\lambda_k\}$ such that

$$\lambda_k \longrightarrow \overline{\lambda_j} \text{ as } k \to \infty,$$

$$\overline{\lambda_j} < \lambda_k < \Gamma_j,$$

$$\inf_{\Sigma_{j,\lambda_k}} f_{j,\lambda_k} < 0.$$

It is not difficult to see from (2.16), that for $k$ large enough, we have

$$f_{j,\lambda_k}(y) \geq \frac{1}{2} a_j \quad \text{for } y \in B_{\Gamma_j} \backslash B_{2\overline{\lambda_j}}.$$

It follows that there exists $y_k \in B_{2\overline{\lambda_j}} \backslash B_{\lambda_k}$ such that

$$f_{j,\lambda_k}(y_k) = \min_{\Sigma_{j,\lambda_k}} f_{j,\lambda_k} < 0.$$

Since $f_{j,\lambda_k}(y) = 0$ for $|y| = \lambda_k$, we get

$$\lambda_k < |y_k| < 2\overline{\lambda_j}$$

$$\nabla f_{j,\lambda_k}(y_k) = 0$$

$$\Delta f_{j,\lambda_k}(y_k) \geq 0.$$

After passing to a subsequence (still denoted by $y_k$), $y_k \to y_0$. It follows

$$f_{j,\overline{\lambda_j}}(y_0) = 0 \quad \text{thus } |y_0| = \overline{\lambda_j}$$

and

$$\nabla_{f_{j,\overline{\lambda_j}}}(y_0) = 0 \ , \ \Delta f_{j,\overline{\lambda_j}}(y_0) \geq 0$$

a contradictory of (2.15). Thus (2.13) can not occur.

Now we are going to establish (2.14) and (2.15). To do so let us start with considering

$$\mathcal{M} = \left\{ y \ ; \ y \in \Sigma_{\overline{\lambda_j}} \ \wedge \ w_j(y) < \left( \frac{|y|}{\overline{\lambda_j}} \right)^{n-2} w_{j,\overline{\lambda_j}} \right\} \subseteq \Sigma_{\overline{\lambda_j}}.$$

First in view of (2.12) we have

$$-\Delta w_{j,\overline{\lambda_j}}(y) = \left( \frac{|y|}{\overline{\lambda_j}} \right)^{-n-2} u_j(x_j)^{-\frac{n+2}{n-2}} g\left( u_j(x_j) \left( \frac{|y|}{\overline{\lambda_j}} \right)^{n-2} w_{j,\overline{\lambda_j}}(y) \right) \quad \text{on} \quad \mathcal{M}$$

Knowing that $s^{-\frac{n+2}{n-2}} g(s)$ is non-increasing in$(0, \infty)$, we get on $\mathcal{M}$

$$\left(w_j u_j(x_j)\right)^{-\frac{n+2}{n-2}} g\left(u_j(x_j) w_j\right) \geq \left( w_{j,\overline{\lambda_j}} \left( \frac{|y|}{\overline{\lambda_j}} \right)^{n-2} u_j(x_j) \right)^{-\frac{n+2}{n-2}} g\left( u_j(x_j) \left( \frac{|y|}{\overline{\lambda_j}} \right)^{n-2} w_{j,\overline{\lambda_j}} \right)$$

therefore

$$-w_j^{-\frac{n+2}{n-2}} \Delta w_j \geq -w_{j,\overline{\lambda_j}}^{-\frac{n+2}{n-2}} \Delta w_{j,\overline{\lambda_j}}.$$

Since $-\Delta w_j$ and $-\Delta w_{j,\overline{\lambda_j}}$ are positive in $\mathcal{M}$ and $w_j \geq w_{j,\overline{\lambda_j}}$, we get

$$-\Delta w_j \geq -\Delta w_{j,\overline{\lambda_j}}, \qquad \text{in } \mathcal{M}.$$

thus

$$-\Delta f_{j,\overline{\lambda_j}} \geq 0 \tag{2.17}$$

Furthermore on $\partial\mathcal{M} \backslash \partial B_{\overline{\lambda_j}}$ we have $|y| > \overline{\lambda_j}$ and

$$w_j(y) - w_{j,\overline{\lambda_j}}(y) = w_{j,\overline{\lambda_j}}(y) \left( \frac{|y|^{n-2}}{\overline{\lambda_j}^{n-2}} - 1 \right) > 0.$$

We recall that

$$w_j - w_{j,\overline{\lambda_j}} = 0 \qquad \text{on} \;\; \partial B_{\overline{\lambda_j}}.$$

So it follows by the maximum principle that

$$f_{j,\overline{\lambda_j}} > 0 \qquad \text{on} \;\; \overline{\mathcal{M}} \backslash \partial B_{\overline{\lambda_j}}. \tag{2.18}$$

On the other hand, on $\Sigma_{\overline{\lambda_j}} \backslash \overline{\mathcal{M}}$ we have

$$w_j(y) > \frac{|y|^{n-2}}{\overline{\lambda_j}^{n-2}} w_{j,\overline{\lambda_j}}(y) > w_{j,\overline{\lambda_j}}(y)$$

thus

$$f_{j,\overline{\lambda_j}} > 0 \qquad \text{on} \;\; \Sigma_{\overline{\lambda_j}} \backslash \overline{\mathcal{M}}. \tag{2.19}$$

Moreover on $\partial B_{\Gamma_j}$

$$\max_{\partial B_{\Gamma_j}} w_{j,\overline{\lambda_j}} = \max_{\partial B_{\Gamma_j}} \left( \frac{\overline{\lambda_j}}{|y|} \right)^{n-2} w_j \left( \frac{\overline{\lambda_j}^2}{\Gamma_j^2} y \right)$$

so by (2.13), we get

$$\begin{aligned} &\leq \left( \frac{C}{\Gamma_j} \right)^{n-2} \max_{\partial B_{\Gamma_j}} w_j \left( \frac{\overline{\lambda_j}^2}{|y|^2} y \right) \\ &= C^{n-2} \Gamma_j^{2-n} \max_{\partial B_{\frac{\overline{\lambda_j}^2}{\Gamma_j}}} w_j \end{aligned}$$

and because of $\frac{\overline{\lambda_j}^2}{\Gamma_j} < \frac{\gamma_j^2}{\Gamma_j} = \frac{1}{4} \frac{\Gamma_j^2 \sigma_j^2}{\Gamma_j} = \frac{1}{4} \Gamma_j \sigma_j^2 < \frac{1}{2} \Gamma_j \sigma_j = \gamma_j$, follows

$$\leq C^{n-2} \Gamma_j^{2-n} \max_{B_{\gamma_j}} w_j$$

also using (2.11), we deduce

$$\leq D \; \Gamma_j^{2-n} \tag{2.20}$$

for some $D$ independent of $j$. Therefore, in view of (2.20) and (2.11), for large $j$, we get

$$\min_{\partial B_{\Gamma_j}} \left( w_j - w_{j,\overline{\lambda_j}} \right) \geq \min_{\partial B_{\Gamma_j}} w_j - \max_{\partial B_{\Gamma_j}} w_{j,\overline{\lambda_j}} > \frac{j}{2^{n-2}} \Gamma_j^{2-n} - D \; \Gamma_j^{2-n} > 0.$$

so

$$f_{j,\overline{\lambda_j}} > 0 \qquad \text{on} \;\; \partial B_{\Gamma_j}. \tag{2.21}$$

Consequently by (2.17), (2.18) and (2.20) we deduce estimate (2.14).
To show (2.15) we observe 2 cases:

For $y_0 \in \partial B_{\overline{\lambda_j}}$

- If $\frac{d}{dr}\left(w_j - w_{j,\overline{\lambda_j}}\right)(y_0) \geq (n-2)w_j(y_0)$,
then, because of $n > 2$ and $w_j > 0$, (2.15) follows immediately.

- Otherwise if $\frac{d}{dr}\left(w_j - w_{j,\overline{\lambda_j}}\right)(y_0) < (n-2)w_j(y_0)$, then

$$\frac{d}{dr}\left(\left(\frac{|y|}{\overline{\lambda_j}}\right)^{n-2} w_{j,\overline{\lambda_j}} - w_j(y)\right)_{|y=y_0} = (n-2)w_j(y_0) - \frac{d}{dr}\left(w_j - w_{j,\overline{\lambda_j}}\right)(y_0) > 0.$$

Therefore by the definition of $\mathcal{M}$, there exists some $\delta > 0$, such that

$$B_\delta(y_0) \cap \Sigma_{\overline{\lambda_j}} \subset \mathcal{M}.$$

Thanks to the Hopf Lemma(see(2.16)), we conclude

$$\frac{d}{dr}\left(w_j - w_{j,\overline{\lambda_j}}\right)(y_0) > 0.$$

Estimate (2.15) is established.

□

Thanks to Lemma 2.2, we know that along a subsequence,

$$w_j \longrightarrow w \qquad \text{in} \;\; C^1_{loc}(\mathbb{R}^n)$$

where $w$ satisfies $w \geq 0, w(0) = 1$.
By the convergence of $w_j$ to $w$ and the fact that $\overline{\lambda}_j(x) \to \infty$ for every $x \in \mathbb{R}^n$, we have

$$w_{x,\lambda}(y) \leq w(y), \qquad \forall\, |y - x| \geq \lambda > 0.$$

It follows, by Lemma A.2, $w \equiv w(0) \equiv 1$, which is impossible because of the following:
In view of (2.7) let

$$c = \limsup_{j\to\infty} u_j(x_j) \geq 1.$$

- If $c = \infty$, then (in the distributions sense)

$$
\begin{aligned}
-\Delta w &= \limsup_{j\to\infty} u_j(x_j)^{-\frac{n+2}{n-2}} g\left(u_j(x_j)w_j\right) \\
&= \limsup_{j\to\infty} u_j(x_j)^{-\frac{n+2}{n-2}} w_j^{-\frac{n+2}{n-2}} w_j^{\frac{n+2}{n-2}} g\left(u_j(x_j)w_j\right) \\
&= \lim_{s\to\infty} w_j^{\frac{n+2}{n-2}} s^{-\frac{n+2}{n-2}} g(s), \qquad\qquad s = u_j(x_j)w_j
\end{aligned}
$$

  by (2.3) and $w_j \to w$, we can find $a > 0$ such that

$$
= a w^{\frac{n+2}{n-2}}.
$$

- If $c < \infty$, then

$$
-\Delta w = c^{-\frac{n+2}{n-2}} g(cw).
$$

Neither of the above is possible since $w \equiv 1$.
Theorem 2 is established □

# 3

# A HARNACK TYPE INEQUALITY ON HALF EUCLIDEAN BALLS

In this chapter we give a proof of a Harnack type inequality on half Euclidean balls under geometrically natural boundary conditions.
For $x \in \mathbb{R}^n, n \geq 3$, we use the notation $x = (x', t)$ where $x' = (x_1, \ldots, x_{n-1})$. We will also use the following notations

$$B_R^T(x) = B_R(x) \cap \{t > T\}, \quad B_R^+(x) = B_R(x) \cap \{t > 0\}, \quad B_R^+ = B_R^+(0),$$
$$\partial' B_R^T(x) = \partial B_R^T(x) \cap \{t = T\}, \qquad \partial'' B_R^T(x) = \partial B_R^T(x) \cap \{t > T\},$$
$$\partial' B_R^+(x) = \partial B_R^+(x) \cap \partial \mathbb{R}_+^n, \qquad \partial'' B_R^+(x) = \partial B_R^+(x) \cap \mathbb{R}_+^n.$$

**Theorem 3.** *For $n \geq 3$ and $c \in \mathbb{R}$, let $u \in C^1(\overline{B_{3R}^+}) \cap C^2(B_{3R}^+)$ be a positive solution of*

$$\begin{cases} -\Delta u = n(n-2)u^{\frac{n+2}{n-2}} & \text{in } B_{3R}^+, \\ \frac{\partial u}{\partial t} = c u^{\frac{n}{n-2}} & \text{on } \partial' B_{3R}^+. \end{cases} \tag{3.1}$$

*Then, for some constant $C = C(n, c)$, there holds:*

$$\left( \max_{\overline{B_R^+}} u \right) \left( \min_{\overline{\partial B_{2R}^+}} u \right) \leq C(n) R^{2-n}.$$

Before we start the proof of Theorem 3, we show the following Lemma which is similar to Lemma 1.1:

**Lemma 3.1.** *Let $u \in C^0(\overline{B_1^{-T}})$ be a positive function, $T \geq 0$. Then for every $a > 0$, there exists $x \in B_1 \cap \{t \geq -T\}$ such that*

*(i)*

$$u(x) \geq \frac{1}{2^a} \max_{\overline{B_\sigma^{-T}(x)}} u$$

*(ii)*

$$\sigma^a u(x) \geq \frac{1}{2^a} u(0).$$

*where* $\sigma = \frac{1-|x|}{2}$.

*Proof.* we consider

$$v(y) = (1-|y|)^a u(y).$$

Since $u$ is continuous and positive, it is clear that $v$ attains its maximum over $\overline{B_1^{-T}}$ in some point $x \in B_1 \cap \{t \geq -T\}$. Hence let $x \in B_1 \cap \{t \geq -T\}$ with

$$v(x) := \max_{y \in B_1 \cap \{t \geq -T\}} v(y), \qquad \text{and} \qquad \sigma = \frac{1-|x|}{2}.$$

Then we have for all $y \in B_1 \cap \{t \geq -T\}$

(i)

$$v(x) = (1-|x|)^a u(x) \geq (1-|y|)^a u(y)$$

which implies that

$$u(x) \geq \frac{(1-|y|)^a}{(1-|x|)^a} u(y) \qquad \text{for all } \; y \in B_1 \cap \{t \geq -T\}\,.$$

Next we claim that $\overline{B_\sigma(x)} \subseteq B_1$ and prove it as follows: Let $z \in \overline{B_\sigma(x)}$,then

$$|z| = |z + x - x| \leq |z - x| + |x| \leq \sigma + |x| = \frac{1-|x|}{2} + |x| = \frac{1+|x|}{2} < 1$$

Therewith we get $\overline{B_\sigma^{-T}(x)} \subseteq B_1 \cap \{t \geq -T\}$.
Hence it follows

$$\begin{aligned} u(x) &\geq \max_{B_\sigma^{-T}(x)} \left( \frac{1-|\cdot|}{1-|x|} \right)^a u \\ &\geq \max_{B_\sigma^{-T}(x)} \left( \frac{1-|\cdot - x| - x}{1-|x|} \right)^a u \\ &\geq \max_{B_\sigma^{-T}(x)} \left( 1 - \frac{\sigma}{1-|x|} \right)^a u \end{aligned}$$

also using $\frac{\sigma}{1-|x|} = \frac{1}{2}$, we deduce

$$u(x) \geq \frac{1}{2^a} \max_{B_\sigma^{-T}(x)} u.$$

(ii) Likewise for all $y \in B_1 \cap \{t \geq -T\}$

$$(1-|x|)^a u(x) \geq (1-|y|)^a u(y)$$

which implies that

$$\frac{(1-|x|)^a}{2^a} u(x) \geq \frac{(1-|y|)^a}{2^a} u(y)$$

In particular for $y = 0$, we get

$$\sigma^a u(x) \geq \frac{1}{2^a} u(0).$$

□

*Proof of Theorem 3.* The proof is by contradiction argument. We suppose the contrary, then there exist solutions of (3.1) $u_j, j = 1, 2, \ldots,$ such that

$$u_j(\overline{x_j}) \min_{\partial B^+_{2R_j}} u_j > jR_j^{2-n}, \tag{3.2}$$

where

$$u_j(\overline{x_j}) = \max_{\overline{B^+_{R_j}}} u_j.$$

Applying Lemma 3.1 to $u = u_j\left(\frac{R_j}{4} \cdot + \overline{x_j}\right)$ with $a = \frac{n-2}{2}$ and $T = \frac{4\overline{x_{jn}}}{R_j}$ ($\overline{x_{jn}}$ denote the n-th component of $x_j$), we find

$$x_j \in B_{\frac{R_j}{4}}(\overline{x_j}) \cap \{x_{jn} \geq 0\} \Rightarrow \left(\frac{4}{R_j}\left|x_j - \overline{x_j}\right| < 1 \ \wedge \ \frac{4}{R_j}(x_{jn} - \overline{x_{jn}}) \geq -T\right)$$

such that

$$u_j\left(\frac{R_j}{4}\left(\frac{4}{R_j}(x_j - \overline{x_j})\right) + \overline{x_j}\right) \geq 2^{\frac{2-n}{2}} u_j\left(\frac{R_j}{4}x + \overline{x_j}\right) \quad \text{for} \quad x \in \overline{B^{-T}_{\sigma_j}}\left(\frac{4}{R_j}(x_j - \overline{x_j})\right).$$

Thus

$$u_j(x_j) \geq 2^{\frac{2-n}{2}} u_j(x) \quad \text{for} \quad x \in B_{\frac{R_j}{4}\sigma_j}(x_j) \cap \overline{\mathbb{R}^n_+}$$

and

$$(\sigma_j)^{\frac{n-2}{2}} u_j(x_j) \geq 2^{\frac{2-n}{2}} u_j(\overline{x}_j),$$

where

$$\sigma_j = \frac{1}{2}\left(1 - \frac{4}{R_j}\left|x_j - \overline{x}_j\right|\right) \leq \frac{1}{2}.$$

It follows that

$$u_j(x_j) \geq u_j(\overline{x_j}) \tag{3.3}$$

and, using (3.2) and $\max_{\overline{B^+_{R_j}}} u_j \geq \min_{\overline{\partial B^+_{2R_j}}} u_j$, we define

$$\gamma_j := u_j(x_j)^{\frac{2}{n-2}} \sigma_j \frac{R_j}{4} \geq \frac{R_j}{8} u_j(\overline{x}_j)^{\frac{2}{n-2}} \geq \frac{R_j}{8}\left[u_j(\overline{x}_j) \min_{\overline{B}_{3R_j}} u_j\right]^{\frac{1}{n-2}} \geq \frac{1}{8} j^{\frac{1}{n-2}} \to \infty \tag{3.4}$$

and

$$\Gamma_j := 2u_j(x_j)^{\frac{2}{n-2}} R_j \geq 16\gamma_j \to \infty. \tag{3.5}$$

In view of (3.3) and (3.2) we get

$$u_j(x_j) \inf_{\partial'' B^+_{2R_j}} u_j > jR_j^{2-n} \tag{3.6}$$

Let

$$T_j := u_j(x_j)^{\frac{2}{n-2}} x_{jn},$$

and set

$$w_j(y) = \frac{1}{u_j(x_j)} u_j\left(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}}\right), \quad y \in \Omega_j,$$

where

$$\Omega_j = \left\{y;\ x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \in B^+_{2R_j}\right\}.$$

Then we get exactly like by (1.6) in Theorem 1

$$-\Delta w_j = n(n-2) w_j^{\frac{n+2}{n-2}} \qquad \text{in } \Omega_j.$$

And on $t = -T_j$

$$\begin{aligned} \frac{\partial w_j}{\partial t} &= \frac{1}{u_j(x_j)} \frac{\partial}{\partial t}\left[u_j\left(x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}}\right)\right] \\ &= \frac{1}{u_j(x_j)} \frac{1}{u_j(x_j)^{\frac{2}{n-2}}} \frac{\partial u_j}{\partial t}\left(x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}}\right) \end{aligned}$$

using (3.1) we get

$$
\begin{aligned}
&= \frac{1}{u_j(x_j)^{\frac{n}{n-2}}} c\, u_j^{\frac{n}{n-2}} \left( x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}} \right) \\
&= c w_j^{\frac{n}{n-2}}
\end{aligned}
$$

and because of

$$
\begin{aligned}
u_j(x_j) &\geq 2^{\frac{2-n}{2}} \max \left\{ u_j(x) \;\mid\; |x - x_j| < \frac{R_j}{4}\sigma_j \;\;\wedge\;\; x_n \geq 0 \right\} \\
&\geq 2^{\frac{2-n}{2}} \max \left\{ u_j \left( \frac{x}{u_j(x_j)^{\frac{2}{n-2}}} + x_j \right) \;\mid\; |x| < \frac{R_j}{4}\sigma_j u_j(x_j)^{\frac{2}{n-2}} = \gamma_j \;\;\wedge\;\; x \in \omega_j \right\}
\end{aligned}
$$

follows

$$
1 = w_j(0) \geq 2^{\frac{2-n}{2}} \max_{\Omega_j \cap B_{\gamma_j}} w_j.
$$

So we have all together

$$
\begin{cases}
-\Delta w_j = n(n-2) w_j^{\frac{n+2}{n-2}} & \text{in } \Omega_j, \\
\frac{\partial w_j}{\partial t} = c w_j^{\frac{n}{n-2}} & \text{on } t = -T_j, \\
w_j(0) = 1 & \\
w_j(y) \leq 2^{\frac{n-2}{2}} & \text{for } y \in \Omega_j \cap B_{\gamma_j}
\end{cases}
\tag{3.7}
$$

Let

$$
\partial'' \Omega_j = \partial \Omega_j \cap \left\{ y; y_n > -T_j \right\}.
$$

Then for $y \in \partial'' \Omega_j$ we have

$$
\begin{aligned}
\left| x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \right| - |x_j| \leq \frac{|y|}{u_j(x_j)^{\frac{2}{n-2}}} &\leq \left| x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-2}}} \right| + |x_j| \\
2R_j - |x_j| \leq \frac{|y|}{u_j(x_j)^{\frac{2}{n-2}}} &\leq 2R_j + |x_j|
\end{aligned}
$$

and because of $|x_j| \leq |x_j - \overline{x_j}| + |\overline{x_j}| \leq \frac{R_j}{4} + R_j = \frac{5}{4} R_j$, we get

$$
\frac{3}{4} R_j \leq \frac{|y|}{u_j(x_j)^{\frac{2}{n-2}}} \leq \frac{13}{4} R_j
$$

So we have

$$
\frac{3}{8} \Gamma_j \leq |y| \leq \frac{13}{8} \Gamma_j \quad \forall y \in \partial'' \Omega_j. \tag{3.8}
$$

We know that

$$\inf_{\partial''\Omega_j} w_j = \frac{1}{u_j(x_j)} \inf_{\partial''\Omega_j} u_j \left( x_j + \frac{\cdot}{u_j(x_j)^{\frac{2}{n-2}}} \right)$$
$$= \frac{u_j(x_j) \inf_{\partial'' B^+_{2R_j}} u_j}{u_j(x_j)^2}$$

by (3.6)

$$> \frac{jR_j^{2-n}}{u_j(x_j)^2}$$
$$= j\Gamma_j^{2-n}$$

Hence we get

$$\inf_{\partial''\Omega_j} w_j > j\Gamma_j^{2-n} \tag{3.9}$$

Passing to a subsequence, we obtain

$$\lim_{j\to\infty} T_j = T \in [0, \infty].$$

We divide the remaining proof into two cases.
**Case 1:** $T = \infty$
**Case 2:** $T \in [0, \infty)$.

**Reaching a contradiction in Case 1.** Since $\min\{\gamma_j, T_j\} \to \infty$.
We can find, like in Theorem 1, a subsequence -still denoted by $w_j$- which converges uniformly to some $w \in C(\Omega)$, where $\Omega \subset\subset \mathbb{R}^n$ and $w$ is a positive solution of (3.1).

Next we discuss the Kelvin's transformation for Laplace operator.
For $x \in \mathbb{R}^n$ and $\lambda < \frac{T_j}{2}$, let $w_{j,x,\lambda}$ denote the Kelvin transformation of $w_j$ with respect to the $B_\lambda(x)$, i.e

$$w_{j,x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2} w_j\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right), \quad y \in \Sigma_{j,x,\lambda} := \Omega_j \backslash \overline{B_\lambda(x)}$$

In view of (1.9) we have

$$-\Delta w_{j,x,\lambda} = n(n-2) w_{j,x,\lambda}^{\frac{n+2}{n-2}} \tag{3.10}$$

As in the previous Theorem, we want compare for any fixed $x \in \mathbb{R}^n, w_j$ with $w_{j,x,\lambda}$ and we shall always take without loss of generality $x = 0$. This will be assured as

follows:
Since $x_j \in B_{\frac{R_j}{4}}(\overline{x_j}) \cap \overline{\mathbb{R}^n_+}$ and $\overline{x_j} \in \overline{B^+_{R_j}}$ we have

$$0 \le x_{jn} \le \frac{5}{4} R_j$$

and

$$0 \le T_j = u_j(x_j)^{\frac{2}{n-2}} x_{jn} \le \frac{5}{8}\Gamma_j$$

Then by the assumption $\lambda < \frac{T_j}{2} < \frac{5}{16}\Gamma_j$, we get for every $y \in B_\lambda(x)$

$$|y| \le |y-x| + |x| < \frac{5}{16}\Gamma_j + |x|,$$

and since $\Gamma_j \to \infty$, we have for $j$ large enough $|x| < \frac{1}{16}\Gamma_j$. Therefore in view of (3.8) follows $y \in \Omega_j$

We define for $y \in \Sigma_{j,x,\lambda}$

$$f_{j,x,\lambda}(y) = w_j(y) - w_{j,x,\lambda}(y)$$

and we use in the following $f_{j,\lambda}, w_{j,\lambda}$ and $\Sigma_{j,\lambda}$ to denote $f_{j,0,\lambda}, w_{j,0,\lambda}$ and $\Sigma_{j,0,\lambda}$.
Since $w_j$ positive and bounded on $B_{\Gamma_j}$, our next lemma can be established in the same way as lemma 1.3.

**Lemma 3.2.** *For every $x \in \mathbb{R}^n$, there exists $\lambda_{j,x} > 0$ small enough such that*

$$f_{j,x,\lambda} \ge 0 \quad \textit{for all } 0 < \lambda < \lambda_{j,x} \textit{ and } \; y \in \Sigma_{j,x,\lambda}.$$

Therefore we can define

$$\overline{\lambda}_j(x) := \sup\left\{0 < \mu \;;\; w_{j,x,\lambda}(y) \le w_j(y), \forall y \in \overline{\Sigma_{j,x,\lambda}}, 0 < \lambda \le \mu\right\}.$$

Next we show the following Lemma:

**Lemma 3.3.** *For every $x \in \mathbb{R}^n$,*

$$\lim_{j\to\infty} \overline{\lambda}_j(x) = \infty.$$

*Proof.* For simplicity, we take $x = 0$. This proof is as usual with contradiction argument. Suppose the contrary, then there exists a constant $C$ independent of $j$ such that

$$\overline{\lambda}_j \le C < \gamma_j. \tag{3.11}$$

Here we have used the fact $\gamma_j \to \infty$(see (3.4)).
As in the proof of Lemma 1.4 and Lemma 2.3, we only need to show

$$f_{j,\overline{\lambda}_j}(y) > 0 \qquad \text{for } \; y \in \overline{\Sigma_{j,\overline{\lambda}_j}} \backslash \partial B_{\overline{\lambda}_j} \tag{3.12}$$

and

$$\frac{df_{j,\overline{\lambda}_j}}{dr}(y) > 0 \qquad \text{for } \; y \in \partial B_{\overline{\lambda}_j} \tag{3.13}$$

to reach a contradiction.
Indeed we easily deduce from (3.12) and (3.13) that $f_{j,\lambda} \geq 0$ on $\overline{\Sigma_{j,\lambda}}$ for $\lambda$ close to $\overline{\lambda_j}$, violating the definition of $\overline{\lambda_j}$.
By the definition of $\overline{\lambda}_j$ we have

$$w_{j,\overline{\lambda}_j} \leq w_j \quad \text{in } \Sigma_{\overline{\lambda_j}},$$

and furthermore in view of (3.7) and (3.10)

$$-\Delta v_{j,\overline{\lambda}_j} = n(n-2)\left(w_j^{\frac{n+2}{n-2}} - w_{j,\overline{\lambda}_j}^{\frac{n+2}{n-2}}\right) \geq 0, \qquad \text{in } \Sigma_{\overline{\lambda_j}}. \tag{3.14}$$

Also we have

$$\begin{aligned}
\max_{\partial''\Omega_j} w_{j,\overline{\lambda_j}} &= \max_{\partial''\Omega_j} \left(\frac{\overline{\lambda}_j}{|\cdot|}\right)^{n-2} w_j\left(\frac{\overline{\lambda}_j^2}{|\cdot|^2}\,\cdot\right) \\
&\leq \max_{\partial''\Omega_j} \left(\frac{\overline{\lambda}_j}{|\cdot|}\right)^{n-2} \max_{\partial''\Omega_j} w_j\left(\frac{\overline{\lambda}_j^2}{|\cdot|^2}\,\cdot\right)
\end{aligned}$$

because of (3.8) $\frac{3}{8}\Gamma_j \leq |y| \leq \frac{13}{8}\Gamma_j$ and (3.11), we get

$$\leq \left(\frac{4}{3}\right)^{n-2} C^{n-2}\Gamma_j^{2-n} \max_{\partial''\Omega_j} w_j\left(\frac{\overline{\lambda}_j^2}{|\cdot|^2}\,\cdot\right)$$

and because of $\frac{\overline{\lambda}_j^2}{|y|} < \frac{8\gamma_j^2}{3\Gamma_j} = \frac{8}{3}\frac{\Gamma_j^2\sigma_j^2}{64\Gamma_j} = \frac{\Gamma_j\sigma_j^2}{24} < \frac{\Gamma_j\sigma_j}{8} = \gamma_j$, follows

$$\leq \left(\frac{4}{3}\right)^{n-2} C^{n-2}\Gamma_j^{2-n} \max_{B_{\gamma_j}} w_j$$

by using (3.7), we deduce

$$\max_{\partial''\Omega_j} w_{j,\overline{\lambda_j}} \leq D\Gamma_j^{2-n} \tag{3.15}$$

for some $D$ independent of $j$. Therefore, by (3.15) and (3.9), for large $j$, we get

$$\min_{\partial''\Omega_j}\left(w_j - w_{j,\overline{\lambda_j}}\right) \geq \min_{\partial''\Omega_j} w_j - \max_{\partial''\Omega_j} > j\Gamma_j^{2-n} - D\Gamma_j^{2-n} > 0.$$

Hence

$$f_{j,\overline{\lambda_j}} > 0 \qquad \text{on } \overline{\partial''\Omega_j}$$

An application of the Hopf Lemma and the strong maximum principle yields estimate (3.13) and

$$f_{j,\overline{\lambda_j}}(y) > 0 \qquad \text{for } y \in \Sigma_{j,\overline{\lambda_j}}.$$

To show (3.12), we still need to establish

$$f_{j,\overline{\lambda_j}}(y) > 0 \quad \text{on } \left\{t = -T_j\right\} \cap \partial\Omega_j.$$

This will follow from the following

**Lemma 3.4.** *Suppose $T_j \to \infty$ and $\left\{\overline{\lambda_j}\right\}$ are bounded. Then for any $N > 0$, there exists $j_0 > 1$ such that for $j > j_0$,*

$$\frac{\partial w_{j,\overline{\lambda_j}}}{\partial t}(z) > N w_{j,\overline{\lambda_j}}(z)^{\frac{n}{n-2}}, \quad \forall z \in \left\{t = -T_j\right\} \cap \partial\Omega_j.$$

Indeed, if for some $z$ with $z_n = -T_j$,

$$f_{j,\overline{\lambda_j}}(z) = 0.$$

Then $z$ is a minimum point and by Lemma 3.3 and for large $j$,

$$0 \le \frac{\partial f_{j,\overline{\lambda_j}}}{\partial t}(z) = c w_j(z)^{\frac{n}{n-2}} - \frac{\partial w_{j,\overline{\lambda_j}}}{\partial t}(z) = c w_{j,\overline{\lambda_j}}(z)^{\frac{n}{n-2}} - \frac{\partial w_{j,\overline{\lambda_j}}}{\partial t}(z) < 0.$$

A contradiction, (3.12) is established.

*Proof of Lemma 3.3.* Since $T_j \to \infty$ and $\left\{\overline{\lambda_j}\right\}$ are bounded from above by positive constants, we have, for large $j$,

$$\frac{1}{2}w(0) < w_j\left(\frac{\overline{\lambda_j}^2 z}{|z|^2}\right) < 2w(0) \text{ and } \left|\nabla w_j\left(\frac{\overline{\lambda_j}^2 z}{|z|^2}\right)\right| < |\nabla w(0)| + 1, \ \forall z \in \left\{t = -T_j\right\} \cap \partial\Omega_j.$$

By a direct computation

$$\begin{aligned} \frac{\partial w_{j,\overline{\lambda_j}}}{\partial t}(z) &\ge (n-2)\overline{\lambda_j}^{n-2} T_j |z|^{-n} w_j\left(\frac{\overline{\lambda_j}^2 z}{|z|^2}\right) - \overline{\lambda_j}^{n} |z|^{-n}\left|\nabla w_j\left(\frac{\overline{\lambda_j}^2 z}{|z|^2}\right)\right| \\ &\ge m\overline{\lambda_j}^{n-2} T_j |z|^{-n} \\ &> N w_{j,\overline{\lambda_j}}\left(\frac{\overline{\lambda_j}^2 z}{|z|^2}\right), \end{aligned}$$

where $m$ is a postive constant independent of $j$. Lemma 3.3 is established. So is Lemma 3.2. □

We know that along a subsequence,

$$w_j \longrightarrow w \qquad \text{in } C^2_{loc}(\mathbb{R}^n)$$

for some solution $w$ of

$$-\Delta w = n(n-2)w^{\frac{n+2}{n-2}}, \quad w > 0, \text{ on } \mathbb{R}^n \tag{3.16}$$

By the convergence of $w_j$ to $w$ and the fact that $\overline{\lambda}_j(x) \to \infty$ for every $x \in \mathbb{R}^n$, we have

$$w_{x,\lambda}(y) \le w(y), \qquad \forall\, |y-x| \ge \lambda > 0.$$

It follows, thanks to Lemma A.2, $w = constant$. Which is impossible because of (3.16).
The contradiction in Case 1 is reached. □

**Reaching a contradiction in Case 2.**
Since $T < \infty$ let $\hat{w}_j$ be a translation of $\hat{w}_j$ given by

$$\hat{w}_j(y) = w_j(y - T_j e_n), \quad y \in \hat{\Omega}_j,$$

where $e_n = (0', 1)$ and $\hat{\Omega}_j = \Omega_j + T_j e_n$.
Then we have in view of (3.7)

$$-\Delta\hat{w}_j = -\Delta w_j(\cdot - T_j e_n) = n(n-2)w_j(\cdot - T_j e_n) = n(n-2)\hat{w}_j \text{ in } \hat{\Omega}_j,$$

$$\frac{\partial \hat{w}_j}{\partial t} = \frac{\partial w_j(\cdot - T_j e_n)}{\partial t} = c w_j^{\frac{n}{n-2}}(\cdot - T_j e_n) = c\hat{w}_j^{\frac{n}{n-2}}, \quad \text{on } t = 0,$$

$$\hat{w}_j(T_j e_n) = 1$$

and because of $w_j(y) \le 2^{\frac{n-2}{2}}$ for $y \in \Omega_j$ and $|y| \le \gamma_j$

$$\hat{w}_j(y) \le 2^{\frac{n-2}{2}} \quad \text{for } y \in \hat{\Omega}_j \text{ and } |y| \le \gamma_j - T_j.$$

Since $\gamma_j \to \infty$ we have with the same arguments in theorem 1 that

$$\hat{w}_j \longrightarrow \hat{w} \qquad \text{in } C^2_{loc}(\overline{\mathbb{R}^n_+})$$

for some solution $\hat{w}$ of (3.1).
Let

$$\partial''\hat{\Omega}_j = \partial\hat{\Omega}_j \cap \{y; y_n > 0\}.$$

Then for $y \in \partial''\hat{\Omega}_j$ we have $y - T_j e_n \in \partial''\Omega_j$, which means by (3.8)that

$$\frac{3}{8}\Gamma_j \le |y - T_j e_n| \le \frac{13}{8}\Gamma_j,$$

so

$$\frac{3}{8}\Gamma_j - T_j \leq |y| \leq \frac{13}{8}\Gamma_j + T_j$$

and the fact that $T_j$ is bounded and $\Gamma_j \to \infty$, we can find $C$ such that

$$C^{-1}\Gamma_j \leq |y| \leq C\Gamma_j$$

For $x \in \partial\mathbb{R}^n_+$, let $\hat{w}_{j,x,\lambda}$ denote the Kelvin transformation of $\hat{w}_j$ with respect to the $B_\lambda(x)$, i.e

$$\hat{w}_{j,x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-2} \hat{w}_j\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right), \quad y \in \hat{\Sigma}_{j,x,\lambda} := \hat{\Omega}_j \backslash \overline{B_\lambda(x)}$$

By essentially the same arguments in the proof of Lemma 1.3, we can find $\lambda_{j,x} > 0$ such that

$$\hat{w}_{j,x,\lambda}(y) < \hat{w}_j(y) \quad \text{for } y \in \hat{\Sigma}_{j,x,\lambda} \text{ and } 0 < \lambda \leq \lambda_{j,x}.$$

Therefore we can define

$$\overline{\lambda}_j(x) := \sup\left\{0 < \mu \ ; \ \hat{w}_{j,x,\lambda}(y) \leq \hat{w}_j(y), \forall y \in \overline{\hat{\Sigma}_{j,x,\lambda}}, 0 < \lambda \leq \mu\right\}.$$

Like usual we show

**Lemma 3.5.** *For every $x \in \partial\mathbb{R}^n_+$,*

$$\lim_{j\to\infty} \overline{\lambda}_j(x) = \infty.$$

*Proof.* For simplicity, we take $x = 0$. This proof is also with contradiction argument. Suppose the contrary, then there exists a constant $C$ independent of $j$ such that

$$\overline{\lambda}_j \leq C < \gamma_j. \tag{3.17}$$

Here we have used the fact $\gamma_j \to \infty$(see (3.4)).
Let $v_\lambda = w_j - w_{j,\lambda}$. As in the proof of Lemma 1.3, we only need to show

$$v_{\overline{\lambda}_j}(y) > 0 \qquad \text{for } y \in \overline{\hat{\Sigma}_{\overline{\lambda}_j}} \backslash \partial B_{\overline{\lambda}_j} \tag{3.18}$$

and

$$\frac{dv_{\overline{\lambda}_j}}{dr}(y) > 0 \qquad \text{for } y \in \partial\hat{\Sigma}_{\overline{\lambda}_j} \cap \partial B^+_{\overline{\lambda}_j} \tag{3.19}$$

to reach a contradiction.
By the definition of $\overline{\lambda}_j$,

$$\hat{w}_{j,\overline{\lambda}_j} \leq \hat{w}_j \quad \text{in } \hat{\Sigma}_{\overline{\lambda}_j}.$$

Then with a direct computation we get

$$-\Delta\hat{w}_{j,\overline{\lambda}_j} = -\Delta\left(\frac{\overline{\lambda}_j^{n-2}}{|\cdot|^{n-2}}\hat{w}_j\left(\frac{\overline{\lambda}_j^2}{|\cdot|^2}\cdot\right)\right)$$
$$= \left(\frac{\overline{\lambda}_j}{|\cdot|}\right)^{n+2} n(n-2)\hat{w}_j^{\frac{n+2}{n-2}}\left(\frac{\overline{\lambda}_j^2}{|\cdot|^2}\cdot\right)$$
$$= n(n-2)\hat{w}_{j,\overline{\lambda}_j}^{\frac{n+2}{n-2}}.$$

Therefore it follows

$$-\Delta v_{\overline{\lambda_j}} = -\Delta\left(\hat{w}_j - \hat{w}_{j,\overline{\lambda_j}}\right) = n(n-2)\left(\hat{w}_j^{\frac{n+2}{n-2}} - \hat{w}_{j,\overline{\lambda}_j}^{\frac{n+2}{n-2}}\right) \geq 0 \qquad \text{in } \hat{\Sigma}_{\overline{\lambda_j}} \tag{3.20}$$

and

$$\frac{\partial v_{\overline{\lambda_j}}}{\partial t} = \frac{\partial\left(\hat{w}_j - \hat{w}_{j,\overline{\lambda_j}}\right)}{\partial t} = \frac{cn}{n-2}\xi^{\frac{2}{n-2}}v_{\overline{\lambda_j}} \quad \text{on } t = 0, \tag{3.21}$$

where $\xi(y)$is, given by the mean value theorem, between $\hat{w}_j(y)$ and $\hat{w}_{j,\overline{\lambda_j}}(y)$. We also have in view of (3.9)

$$\inf_{\partial''\hat{\Omega}_j}\hat{w}_j = \inf_{\partial''\Omega_j} w_j > j\Gamma_j^{2-n}$$

and like by (3.15), we can find $D$ independent of $j$ such that

$$\sup_{\partial''\hat{\Omega}_j}\hat{w}_{j,\overline{\lambda}} < D\Gamma_j^{2-n}$$

Therefore, for large $j$, we get

$$\inf_{\partial''\hat{\Omega}_j} v_{\overline{\lambda_j}} = \inf_{\partial''\hat{\Omega}_j}\left(\hat{w}_j - \hat{w}_{j,\overline{\lambda}}\right) > j\Gamma_j^{2-n} - D\Gamma_j^{2-n} > 0.$$

Thus, thanks to the strong maximum principle and the Hopf Lemma, we have

$$v_{\overline{\lambda_j}}(y) > 0 \qquad \text{for } y \in \hat{\Sigma}_{\overline{\lambda}_j} \cup \partial''\hat{\Omega}_j$$

and

$$\frac{dv_{\overline{\lambda}_j}}{dr}(y) > 0 \qquad \text{for } y \in \partial'' B^+_{\overline{\lambda}_j}$$

To show (3.18) and (3.19), we only need to establish

$$\begin{cases} v_{\overline{\lambda_j}}(y) > 0 & \text{for } y \in \partial' \hat{\Sigma}_{\overline{\lambda_j}} \backslash \overline{B_{\overline{\lambda_j}}} \\ \\ \frac{dv_{\overline{\lambda_j}}}{dr}(y) > 0 & \text{for } y \in \partial \mathbb{R}^n_+ \cap \partial B_{\overline{\lambda_j}} \end{cases} \tag{3.22}$$

where $\partial' \hat{\Sigma}_{\overline{\lambda_j}} = \partial \hat{\Sigma}_{\overline{\lambda_j}} \cap \{t = 0\}$.
The first equation is seen as follows:
Suppose that $v_{\overline{\lambda_j}} = 0$ at some point on $\partial' \hat{\Sigma}_{\overline{\lambda_j}} \backslash \overline{B_{\overline{\lambda_j}}}$. Then we get by (3.21)

$$\frac{\partial v_{\overline{\lambda_j}}}{\partial t} = 0$$

violating the Hopf Lemma. The second equation of (3.22) follows immediately from Lemma A.5.
Lemma 3.4 is established. □

By the convergence of $\hat{w}_j$ to $\hat{w}$ and the fact that $\overline{\lambda}_j(x) \to \infty$ for every $x \in \partial \mathbb{R}^n_+$, we have

$$\hat{w}_{x,\lambda}(y) \leq \hat{w}(y), \qquad \forall\, y \in \mathbb{R}^n_+ \text{ and } |y - x| \geq \lambda > 0.$$

It follows, thanks to Lemma A.3, $\hat{w}$ depends on $t$ only, a contradiction to Lemma A.4.
The contradiction in Case 2 is reached.
Theorem 3 is established. □

# 4

## HARNACK TYPE INEQUALITY FOR A FOURTH ORDER PDE INVOLVING SOBOLEV EXPONENT

In this chapter we give a proof of a Harnack type inequality for a fourth order PDE involving sobolev exponent.

**Theorem 4.** *For $n \geq 5$, let $B_{3R}$ be a ball of radius $3R$ in $\mathbb{R}^n$, and let $u \in C^2(B_{3R})$ be a positive solution of*

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \ \ \Delta u < 0 \ \ in \ \ B_{3R} \tag{4.1}$$

*then*

$$\left(\max_{\overline{B}_R} u\right)\left(\min_{\overline{B}_{2R}} u\right) \leq C(n)R^{4-n} \tag{4.2}$$

A consequence is the following energy estimate.

**Corollary 4.1.** *Let $u$ be as in Theorem 4. Then*

$$\int_{B_R} u^{\frac{2n}{n-4}} \leq C(n).$$

*Remark* 4. Our proof is by the Method of Moving Spheres, which consists of Kelvin's transformation and maximum principle. However, the biggest difficulty in dealing with the equation and its transformed is that they are of fourth order, therefore the principle does not hold in general.

*Proof of Theorem 4.* We only need to prove (4.2) for $R = 1$. The general case follows by applying the result to $v(\cdot) = R^{\frac{n-4}{2}} u(R\cdot)$. We prove it by contradiction argument. If (4.2) were not true, we would have positive solutions $\{u_j\}$ of (4.1) on $B_3$ such that

$$u_j(\overline{x_j}) \min_{\overline{B}_2} u_j > j, \tag{4.3}$$

where

$$u_j(\overline{x_j}) = \max_{\overline{B}_1} u_j.$$

As in the proof of Theorem 1, by applying Lemma 1.1 to $u(\frac{1}{4}\cdot+\overline{x_j})$ and $a=\frac{n-4}{2}$, we can find $x_j \in B_{\frac{1}{4}}(\overline{x_j})$ such that

$$u_j(x_j) \geq 2^{\frac{4-n}{2}} \max_{B_{\frac{1}{4}\sigma_j(x_j)}} u_j$$

and

$$(\sigma_j)^{\frac{n-4}{2}} u_j(x_j) \geq 2^{\frac{4-n}{2}} u_j(\overline{x_j})$$

where

$$\sigma_j = \frac{1}{2}\left(1 - \frac{1}{4}\left|x_j - \overline{x_j}\right|\right) \leq \frac{1}{2}.$$

Thus it follows

$$u_j(x_j) \geq u_j(\overline{x_j}), \tag{4.4}$$

and using (4.3), we get

$$\gamma_j := \frac{1}{4}u_j(x_j)^{\frac{2}{n-4}}\sigma_j \geq \frac{1}{8}u_j(\overline{x_j})^{\frac{2}{n-4}} \geq \frac{1}{8}\left[u_j(\overline{x_j})\right]^{\frac{1}{n-4}} \geq \frac{1}{8}j^{\frac{1}{n-4}} \longrightarrow \infty \tag{4.5}$$

and

$$\Gamma_j := \frac{1}{2}u_j(x_j)^{\frac{2}{n-4}} \geq 4\gamma_j \longrightarrow \infty. \tag{4.6}$$

Now we set

$$w_j(y) = \frac{1}{u_j(x_j)}u_j\left(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-4}}}\right), \qquad y \in B_{3\Gamma_j}.$$

By essentially the same calculation in the proof of Theorem 1, we see that $w_j$ satisfies

$$\begin{cases} \partial_i w_j(y) = \frac{1}{u_j(x_j)^{\frac{n-2}{n-4}}}\partial_i u_j\left(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-4}}}\right) & \text{in } B_{3\Gamma_j} \\ \Delta w_j = \frac{1}{u_j(x_j)^{\frac{n}{n-4}}}\Delta u_j\left(x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-4}}}\right) < 0 & \text{in } B_{3\Gamma_j} \\ \Delta^2 w_j = w_j^{\frac{n+4}{n-4}} \quad \text{and} \quad w_j > 0 & \text{in } B_{3\Gamma_j} \\ w_j(0) = 1 & \\ w_j \leq 2^{\frac{n-4}{2}} & \text{in } B_{\gamma_j} \end{cases} \tag{4.7}$$

By the triangle inequality, we have for all $y \in B_{\frac{3}{2}\Gamma_j}$

$$\left|x_j + \frac{y}{u_j(x_j)^{\frac{2}{n-4}}}\right| \leq |x_j| + \frac{|y|}{u_j(x_j)^{\frac{2}{n-4}}} \leq \frac{5}{4} + \frac{3}{4} = 2.$$

Therefore since $w_j$ superharmonic, it follows thanks to the strong maximum principle that

$$\min_{B_{\frac{3}{2}\Gamma_j}} w_j = \min_{\partial B_{\frac{3}{2}\Gamma_j}} w_j \geq \frac{\min_{\overline{B_2}} u_j}{u_j(x_j)}$$

and in view of (4.4) and (4.1)

$$\begin{aligned} &> \frac{j}{u_j(x_j)\cdot u_j(\overline{x_j})} \\ &\geq \frac{j}{u_j(x_j)^2} \\ &> j2^{4-n}\Gamma_j^{4-n} \end{aligned} \tag{4.8}$$

For later purposes we prove the following results.

**Lemma 4.1.** *$w_j$ and its Laplacian $-\Delta w_j$ satisfy the following estimates*

1. $-\Delta w_j(y) \geq c_j|y|^{2-n}$ *for* $1 \leq |y| \leq 2\Gamma_j$
2. $w_j(y) \geq c_j|y|^{4-n}$ *for* $1 \leq |y| \leq 2\Gamma_j$

*where*

$$c_j = \min\left\{\min_{\partial B_{2\Gamma_j}} w_j, \min_{\partial B_{2\Gamma_j}} \left(-\Delta w_j\right)\right\} > 0.$$

*Proof.* Since $w_j$ and its Laplacian $-\Delta w_j$ are *superharmonic* in $B_{2\Gamma_j}$, it follows by the maximum principle that for all $|y| \leq 2\Gamma_j$ we have

$$\begin{cases} w_j & \geq \min_{B_{2\Gamma_j}} w_j = \min_{\partial B_{2\Gamma_j}} w_j \geq c_j \\ -\Delta w_j & \geq \min_{B_{2\Gamma_j}} \left(-\Delta w_j\right) = \min_{\partial B_{2\Gamma_j}} \left(-\Delta w_j\right) \geq c_j \end{cases}$$

The fact that $c_j$ is positive, is seen from (4.8) and the non-negativity and the superharmonicity of $-\Delta w_j$ on $B_{3\Gamma_j}$. And since $|y| \geq 1$ and $n \geq 5$ we deduce

$$\begin{cases} w_j & \geq c_j \geq c_j|y|^{2-n} \\ -\Delta w_j & \geq c_j \geq c_j|y|^{4-n}. \end{cases}$$

Lemma 4.1 is established. □

Next we want to show that $w_j$ has a converging subsequence in $C^4$-norm on any compact subset of $\mathbb{R}^n$. This will be deduced from the following Lemma.

**Lemma 4.2.**
$$\|w_j\|_{C^{4,\alpha}\left(B_{\gamma_j/4}\right)} \le D$$
*where $D$ a constant independent of $j$ and $0 < \alpha < 1$.*

*Proof.* On $B_{\gamma_j}$ we have in view of (4.7)

$$|\Delta^2 w_j| = w_j^{\frac{n+4}{n-4}} \le 2^{\frac{n+4}{2}} \le C$$

where $C$ independent of $j$.
Consequently for all $p > 0$,

$$\Delta(\Delta w_j) \in L^p\left(B_{\gamma_j}\right).$$

Applying the standard $L^p-$regularity and Morrey's Theorem on $\Delta w_j$, we get for each open subset $\Omega \subset\subset B_{\gamma_j}$,

$$\Delta w_j \in w^{2,p}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega) \quad \text{for all } 0 < \alpha < 1.$$

Thus by $C^\alpha$-regularity and (4.4) follows

$$w_j \in C^{2,\alpha}\left(B_{\gamma_j/2}\right) \Rightarrow \Delta^2 w_j \in C^{2,\alpha}\left(B_{\gamma_j/2}\right).$$

Applying again $C^\alpha$- regularity on $\Delta^2 w_j$ as well as on $\Delta w_j$, we get $w_j \in C^{4,\alpha}\left(B_{\gamma_j/4}\right)$ and
$$\|w_j\|_{C^{4,\alpha}\left(B_{\gamma_j/4}\right)} \le C\left(\|\Delta w_j\|_{C^{2,\alpha}\left(B_{\gamma_j/2}\right)} + \|w_j\|_{L^2\left(B_{\gamma_j/2}\right)}\right) \le D.$$

□

From Lemma 4.2 we conclude that $\|w_j\|_{C^{4,\alpha}\left(B_{\gamma_j/4}\right)}$ is uniformly bounded. This means that the $w_j$, their first, second, third and fourth derivatives are equicontinuous. Thus by the *Arzela-Ascoli* Theorem, we can find a subsequence -still denoted by $w_j$- which converges uniformly to some $w \in C^4(\Omega)$, where $\Omega \subset\subset \mathbb{R}^n$ and $w$ is a positive solution of (4.1).

Now we are in the position to discuss the Kelvin's transformation for biharmonic operators. To do so, let us set for $x \in \mathbb{R}^n$

$$v_{j,x}(y) = |y-x|^{4-n} w_j\left(x + \frac{y-x}{|y-x|^2}\right), \text{ for } |y-x| \ge \frac{1}{3\Gamma_j}$$

and

$$v_{j,x,\lambda}(y) = \left(\frac{\lambda}{|y-x|}\right)^{n-4} v_{j,x}\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)$$

$$= \lambda^{4-n} w_j\left(x + \frac{y-x}{\lambda^2}\right), \qquad \text{for } y \in B_\lambda(x) \backslash B_{\frac{1}{3\Gamma_j}}(x)$$

First of all, thanks to the Kelvin's transformation rule (Lemma A.1) and the fact that $w_j$ is a solution of (4.1), we will be able to show that $v_{j,x}$ and $v_{j,x,\lambda}$ are also solutions of the problem (4.1).

**Lemma 4.3.** *$v_{j,x}$ and $v_{j,x,\lambda}$ satisfy the equations*

$$\Delta^2 v_{j,x} = v_{j,x}^{\frac{n+4}{n-4}} \tag{4.9}$$

$$\Delta^2 v_{j,x,\lambda} = v_{j,x,\lambda}^{\frac{n+4}{n-4}} \tag{4.10}$$

*Proof.* First we show (4.10). With a direct computation and in view of (4.7) we have

$$\begin{aligned}\Delta^2 v_{j,x,\lambda} &= \lambda^{4-n}\Delta^2\left[w_j\left(x + \frac{y-x}{\lambda^2}\right)\right]\\ &= \lambda^{4-n}\lambda^{-8}\Delta^2 w_j\left(x + \frac{y-x}{\lambda^2}\right)\\ &= \lambda^{-(n+4)} w_j^{\frac{n+4}{n-4}}\left(x + \frac{y-x}{\lambda^2}\right)\\ &= v_{j,x,\lambda}^{\frac{n+4}{n-4}}\end{aligned}$$

To show (4.9) let us for simplicity take $x = 0$ and let $v_j = v_{j,0}$.
Since $v_j(y) = |y|^2\left(|y|^{2-n} w_j\left(\frac{y}{|y|^2}\right)\right)$, we have

$$\Delta v_j(y) = \left(\Delta|y|^2\right)\left[|y|^{2-n} w_j\left(\frac{y}{|y|^2}\right)\right] + 2\nabla|y|^2 \cdot \nabla\left[|y|^{2-n} w_j\left(\frac{y}{|y|^2}\right)\right] + |y|^2\Delta\left[|y|^{2-n} w_j\left(\frac{y}{|y|^2}\right)\right]$$

using Kelvin's transformation rule

$$\begin{aligned}&= 2n\left[|y|^{2-n} w_j\left(\frac{y}{|y|^2}\right)\right] + 2\nabla|y|^2 \cdot \nabla\left[|y|^{2-n} w_j\left(\frac{y}{|y|^2}\right)\right] + |y|^{-n}\left(\Delta w_j\right)\left(\frac{y}{|y^2|}\right)\\ &= \frac{1}{|y|^n}\Delta w_j\left(\frac{y}{|y|^2}\right) - \frac{4}{|y|^{n-2}}\left\langle \nabla w_j\left(\frac{y}{|y|^2}\right), \frac{y}{|y|^2}\right\rangle - \frac{2(n-4)}{|y|^{2-n}} w_j\left(\frac{y}{|y|^2}\right).\end{aligned} \tag{4.11}$$

Furthermore

$$\begin{aligned}
\Delta^2 v_j(y) &= 2n|y|^{-2-n}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right) + 2\Delta\left(\nabla|y|^2\right)\cdot\nabla\left[|y|^{2-n}w_j\left(\frac{y}{|y|^2}\right)\right] \\
&\quad + 4\nabla\left(\nabla|y|^2\right)\cdot\nabla\left(\nabla\left[|y|^{2-n}w_j\left(\frac{y}{|y|^2}\right)\right]\right) \\
&\quad + 2\nabla|y|^2\cdot\nabla\left(\Delta\left[|y|^{2-n}w_j\left(\frac{y}{|y|^2}\right)\right]\right) + \Delta\left(|y|^{-2}\right)\left[|y|^{2-n}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right)\right] \\
&\quad + 2\nabla\left(|y|^{-2}\right)\cdot\nabla\left[|y|^{2-n}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right)\right] + |y|^{-2}\Delta\left[|y|^{2-n}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right)\right] \\
&= 16|y|^{-n-2}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right) + 2\nabla|y|^2\cdot\nabla\left[|y|^{2-n}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right)\right] \\
&\quad + 2\nabla|y|^{-2}\cdot\nabla\left[|y|^{2-n}\left(\Delta w_j\right)\left(\frac{y}{|y|^2}\right)\right] + |y|^{-4-n}\left(\Delta^2 w_j\right)\left(\frac{y}{|y|^2}\right) \\
&= |y|^{-4-n}\left(\Delta^2 w_j\right)\left(\frac{y}{|y|^2}\right)
\end{aligned}$$

in view of (4.7)

$$\begin{aligned}
&= |y|^{-4-n} w_j^{\frac{n+4}{n-4}}\left(\frac{y}{|y|^2}\right) \\
&= v_j^{\frac{n+4}{n-4}}
\end{aligned}$$

□

Essentially, the Method of Moving Spheres is made up of Kelvin's transformation and maximum principle. Thus our next step is to make sure that the maximum principle can be applied, namely, by showing that the $v_{j,x}$ and and $v_{j,x,\lambda}$ are positive superharmonic solutions of (4.1). By the definition of $v_{j,x}$ and $v_{j,x,\lambda}$ and the above Lemma, it is clear that they are positive solutions of (4.1). Moreover, $v_{j,x,\lambda}$ is clearly superharmonic, but the superharmonictiy of $v_{j,x}$ it is not as clear, and this is what we will prove next.

Looking again at (4.11), we see that

$$\Delta v_j(y) = -2(n-4)|y|^{2-n}w_j(0) - \sum_{i=1}^{n}\frac{a_i}{|y|^n}y_i + O\left(\frac{1}{|y|^n}\right)$$

at infinity, where $a_i = \frac{\partial w_j}{\partial x_i}(0)$. In particular, we have for large $|y|$, $\Delta v_j < 0$. Without loss of generality, we assume that

$$\Delta v_j(y) \leq 0 \qquad \text{for all} \;\; y \in \mathbb{R}^n \backslash B_1. \tag{4.12}$$

Next we will prove that $\Delta v_j$ is non-positive in $B_1 \backslash \overline{B_{\frac{1}{3\Gamma_j}}}$

**Lemma 4.4.** *Let $v_{j,x}$ be defined as above. Then*

$$\Delta v_{j,x}(y) \leq 0 \quad \textit{for all} \;\; y \in \mathbb{R}^n \backslash \overline{B_{\frac{1}{3\Gamma_j}}(x)}.$$

*Proof.* Like usual we take $x = 0$. Then by (4.12) and Lemma B.2. we will only need to show that for every non negative function $\varphi \in C^\infty(\Omega_j)$, $\varphi = 0$ on $\partial\Omega_j$ and every fixed $j$

$$\int_{\Omega_j} \Delta\varphi \Delta v_j \geq 0 \tag{4.13}$$

where $\Omega_j := B_1 \backslash \overline{B_{\frac{1}{3\Gamma_j}}}$.

Let $\eta_\epsilon \in C_0^\infty(\Omega_j)$ be such that for $i = 1, 2, 3, 4$

$$\eta_\epsilon(y) = 1 \;\; \text{for } |y| \geq \frac{1}{3\Gamma_j} + 2\epsilon, \;\; \eta_\epsilon(y) = 0 \;\; \text{for } |y| \leq \frac{1}{3\Gamma_j} + \epsilon, \;\; \text{and} \;\; \left| D^i \eta_\epsilon \right| \leq c\epsilon^{-i}$$

where $c$ is a positive constant independent of $\epsilon$.

Multiplying (4.9) by $\varphi\eta_\epsilon$, we get

$$0 < \int_{\Omega_j} \varphi\eta_\epsilon \Delta^2 v_j = \int_{\Omega_j} \varphi(y)\eta_\epsilon(y) v_j^{\frac{n+4}{n-4}}(y) \; \mathrm{d}y$$

two times integrations by parts and observing that $\varphi\eta_\epsilon$ vanish on the boundary of $\Omega_j$, we obtain

$$\begin{aligned} &= \int \Delta\left(\varphi\eta_\epsilon\right) \Delta v_j \\ &= \int \Delta v_j \left[ \Delta\varphi\eta_\epsilon + \underbrace{2\nabla\varphi\nabla\eta_\epsilon + \varphi\Delta\eta_\epsilon}_{:=\psi} \right] \end{aligned} \tag{4.14}$$

By the definition of $\psi$ we have

$$\begin{cases} \psi(y) = 0 & \text{for } \; |y| \leq \frac{1}{3\Gamma_j} + \epsilon \;\; \text{and} \;\; |y| \geq \frac{1}{3\Gamma_j} + 2\epsilon \\ \Delta\psi \leq c\epsilon^{-4}. \end{cases}$$

After two integrations by parts, observing that $\psi$ vanishes on the boundary of $\Omega_j$ and triangle inequality, we get

$$\left| \int \Delta v_j \psi \right| \leq \int v_j \left| \Delta\psi \right|$$

Since $1 - \frac{n-4}{n+4} = \frac{8}{n+4}$; by Hölder's inequality

$$\leq c\epsilon^{-4}\left[\int_{\frac{1}{3\Gamma_j}+\epsilon\leq|y|\leq\frac{1}{3\Gamma_j}+2\epsilon} v_j^{\frac{n+4}{n-4}}\right]^{\frac{n-4}{n+4}} \epsilon^{\frac{8n}{n+4}}$$

thanks to Lemma B.2 we have $v_j^{\frac{n+4}{n-4}} \in L^1(\Omega_j)$. Thus, since $\frac{8n}{n+4} = 4\frac{n+4}{n-4} + 4 > 4$

$$\leq c\epsilon^{\frac{8n}{n+4}-4} \longrightarrow 0 \quad \text{as } \epsilon \to 0.$$

Therefore, in view of (4.14), we have

$$\begin{aligned}\int_{\Omega_j} \Delta v_j \Delta\varphi &= \lim_{\epsilon\to 0}\int_{\Omega_j} \eta_\epsilon \Delta v_j \Delta\varphi \\ &= \int_{\Omega_j} \varphi v_j^{\frac{n+4}{n-4}} > 0.\end{aligned}$$

Estimate (4.13) is established. □

Now we define

$$f_{j,x,\lambda} = v_{j,x} - v_{j,x,\lambda}$$

Then in view of (4.9) and (4.10), we obtain

$$\Delta^2 f_{j,x,\lambda} = v_{j,x}^{\frac{n+4}{n-4}} - v_{j,x,\lambda}^{\frac{n+4}{n-4}}. \tag{4.15}$$

Our next lemma says that the method of moving spheres can get started.

**Lemma 4.5.** *For every $x \in \mathbb{R}^n$, there exists $\lambda_{j,x} \geq 0$ large enough such that*

$$f_{j,x,\lambda}(y) \geq 0 \quad \textit{and} \quad -\Delta f_{j,x,\lambda}(y) \geq 0$$

*for all $\lambda > \lambda_{j,x}$ and $y \in B_\lambda(x)\backslash\overline{B_{\frac{1}{2\Gamma_j}}(x)}$.*

*Proof.* Without loss of generality we may take $x = 0$ and let $f_{j,\lambda} = f_{j,0,\lambda}$. Then we will prove this lemma by four steps.

- *Step 1.* There exists $R_{j,0} > \frac{1}{2\Gamma_j}$ such that for $R_{j,0} \leq |y| \leq \frac{1}{2}\lambda$, we have $f_{j,\lambda}(y) \geq 0$. We easily see that

$$A = \lim_{|y|\to\infty} |y|^{n-4} v_j(y) = w_j(0) = 1.$$

By direct computation we get for all $R_{j,0} \leq |y| \leq \frac{1}{2}\lambda$, that

$$\begin{aligned} f_{j,\lambda}(y) &= v_j(y) - \left(\frac{\lambda}{|y|}\right)^{n-4} v_j\left(\frac{\lambda^2}{|y|^2}y\right) \\ &= |y|^{4-n} w_j\left(\frac{y}{|y|^2}\right) - \lambda^{4-n}\left[\left|\frac{\lambda^2 y}{|y|^2}\right|^{n-4} v_j\left(\frac{\lambda^2 y}{|y|^2}\right)\right] \\ &= |y|^{4-n}\left(A + O\left(\frac{1}{|y|}\right)\right) - \lambda^{4-n}\left(A + O\left(\frac{|y|}{\lambda^2}\right)\right) \end{aligned}$$

if $R_{j,0}$ large enough

$$\geq |y|^{4-n}\left(A\left(1-\frac{1}{2}\right) - O\left(\frac{1}{R_{j,0}}\right)\right) > 0.$$

- *Step 2.* There exits $R_{j,1} > R_{j,0}$, such that, for $R_{j,1} \leq \frac{1}{2}\lambda \leq |y| \leq \lambda$, we have $f_{j,\lambda}(y) \geq 0$ and $-\Delta f_{j,\lambda}(y) \geq 0$.
  We get with a simple calculation and (4.11)

$$\begin{aligned} -\Delta f_{j,\lambda} = |y|^{-2}\Bigg[&\frac{|y|^2}{\lambda^n}(\Delta w_j)\left(\frac{y}{\lambda^2}\right) - |y|^{2-n}(\Delta w_j)\left(\frac{y}{|y|^2}\right) \\ &+ 4|y|^{4-n}\left(\sum_{i=1}^{n}\frac{y_i}{|y|^2}\cdot\frac{\partial w_j}{\partial y_i}\left(\frac{y}{|y|^2}\right)\right) + 2(n-4)v_j(y)\Bigg]. \end{aligned} \tag{4.16}$$

Let $C = \max_{B_1}|\nabla w_j|$. Then by Lemma 4.2 $C < \infty$, therefore we get for $R_{j,1}$ sufficiently large

$$\left|4\sum_{i=1}^{n}\frac{y_i}{|y|^2}\cdot w_j\left(\frac{y}{|y|^2}\right)\right| \leq \frac{4C}{|y|} \leq \frac{1}{2}(n-4)w_j(0).$$

Now let $B = \max_{B_1}|\Delta w_j|$. Then we have

$$\begin{aligned} \left|\frac{|y|^2}{\lambda^n}(\Delta w_j)\left(\frac{y}{\lambda^2}\right) - |y|^{2-n}(\Delta w_j)\left(\frac{y}{|y|^2}\right)\right| &\leq \frac{|y|^2}{\lambda^n}B + |y|^{2-n}B \\ &\leq B\left[\frac{1}{\lambda^{n-2}} + \frac{1}{|y|^{n-2}}\right] \\ &\leq \frac{2B}{\lambda^{n-2}}. \end{aligned}$$

It is clear that for $|y|$ large enough we have $w_j\left(\frac{y}{|y|^2}\right) \geq \frac{1}{2}w_j(0)$.
Thus $v_j(y) \geq \frac{1}{2}|y|^{4-n}$.
Therefore by (4.16), if $R_{j,1} > R_{j,0}$ is sufficiently large, then

$$-\Delta f_{j,\lambda} \geq 0 \quad \text{for} \quad R_{j,1} \leq \frac{1}{2}\lambda \leq |y| \leq \lambda.$$

And in view of *Step 1* and the definition of $f_{j,\lambda}$ we have

$$\begin{cases} f_{j,\lambda} & \geq 0 \quad \text{on } |y| = \frac{1}{2}\lambda \\ f_{j,\lambda} & = 0 \quad \text{on } |y| = \lambda \\ -\Delta f_{j,\lambda} & \geq 0 \quad \text{in } \frac{1}{2}\lambda \leq |y| \leq \lambda \end{cases}$$

thus thanks to the Strong Maximum Principle, follows

$$f_{j,\lambda}(y) \geq 0 \quad \text{for } R_{j,1} \leq \frac{1}{2}\lambda \leq |y| \leq \lambda.$$

- *Step 3.* There exists $R_{j,2} > R_{j,1}$ such that $f_{j,\lambda} \geq 0$ for $y \in B_{R_{j,0}} \backslash \overline{B_{\frac{1}{2\Gamma_j}}}$ and $-\Delta f_{j,\lambda}(y) \geq 0$ for $y \in B_{\frac{1}{2}} \backslash \overline{B_{\frac{1}{2\Gamma_j}}}$ for all $\lambda \geq R_{j,2}$.

By Lemma 4.1, we know that there is a constant $c_j > 0$ such that $w_j\left(\frac{y}{|y|^2}\right) \geq c_j |y|^{n-4}$ thus $v_j(y) \geq c_j$ for $y \in \overline{B_{R_{j,0}}} \backslash B_{\frac{1}{2\Gamma_j}}$. By the definition of $v_{j,\lambda} = \lambda^{4-n} w_j\left(\frac{y}{\lambda^2}\right)$, we also have for all $\frac{1}{2\Gamma_j} \leq |y| \leq R_{j,0}$

$$v_{j,\lambda} \longrightarrow 0 \qquad \text{as } \lambda \to \infty.$$

Therefore, for $\lambda$ large enough, we have

$$f_{j,\lambda} = v_j - v_{j,\lambda} \geq 0 \quad \text{for } \frac{1}{2\Gamma_j} \leq |y| \leq R_{j,0}.$$

By lemma 4.4 and lemma 4.3 we have $-\Delta v_j$ ist non-negative and superharmonic for all $|y| \geq \frac{1}{3\Gamma_j}$. Thus by the maximum principle, it is clear

$$-\Delta v_j \geq b_j > 0 \quad \text{for all } \frac{1}{2\Gamma_j} \leq |y| \leq \frac{1}{2},$$

where $b_j = \min\left\{ \min_{\partial B_{\frac{1}{2}}} (-\Delta v_j) \ , \ \min_{\partial B_{\frac{1}{2\Gamma_j}}} (-\Delta v_j) \right\}$.

On the other hand we have for $\frac{1}{2\Gamma_j} \leq |y| \leq \frac{1}{2}$ that

$$-\Delta v_{j,\lambda} = -\lambda^{-n} (\Delta w_j) \left(\frac{y}{\lambda^2}\right) \to 0 \quad \text{as } \lambda \to \infty.$$

Thus, for $\lambda$ large enough, we get

$$-\Delta f_{j,\lambda} = -\Delta v_j + \Delta v_{j,\lambda} \geq 0 \quad \text{for } \frac{1}{2\Gamma_j} \leq |y| \leq \frac{1}{2}.$$

- *Step 4.* Combining Steps 1 to 3, we conclude for $\lambda$ sufficiently large

$$f_{j,\lambda} = v_j - v_{j,\lambda} \geq 0 \quad \text{for } \frac{1}{2\Gamma_j} \leq |y| \leq \lambda.$$

Therefore by lemma 4.3

$$-\Delta\left(-\Delta f_{j,\lambda}\right) = v_j^{\frac{n+4}{n-4}} - v_{j,\lambda}^{\frac{n+4}{n-4}} \geq 0 \text{ for } \frac{1}{2\Gamma_j} \leq |y| \leq \lambda.$$

Moreover we have by Step 2 and Step 3

$$\begin{cases} -\Delta f_{j,\lambda} \geq 0 & \text{on } |y| = \frac{1}{4} \\ -\Delta f_{j,\lambda} \geq 0 & \text{on } |y| = \frac{3}{4}\lambda \end{cases}$$

Thus by the maximum principle, Step 1 and Step 3, for $\lambda$ large enough

$$-\Delta f_{j,\lambda} \geq 0 \quad \text{for } \frac{1}{2\Gamma_j} \leq |y| \leq \lambda$$

Lemma 4.5 is established. □

Now we can define for every fixed $j$ and $x \in \mathbb{R}^n$,

$$\overline{\lambda_j}(x) := \inf\left\{\mu > 0 : f_{j,\lambda} \geq 0 \text{ and } -\Delta f_{j,\lambda} \geq 0 \text{ for all } \frac{1}{\Gamma_j} \leq |y-x| \leq \lambda, \text{ for } \mu < \lambda < +\infty\right\}$$

**Lemma 4.6.** *For every $x \in \mathbb{R}^n$,*

$$\lim_{j\to\infty} \overline{\lambda_j}(x) = 0.$$

*Proof.* Without loss of generality, we take $x = 0$ and let $\lambda_j = \lambda_j(0)$. Suppose not, then for some positive constant $\epsilon > 0$ and along a subsequence (still denoted $\overline{\lambda_j}$), we have

$$\overline{\lambda_j} > 2\epsilon > \frac{1}{\gamma_j}. \tag{4.17}$$

Here we have used the fact $\gamma_j \to \infty$ (see (4.5)). Before we start trying to reach a contradiction, let us do some preparation. By the definition of $\overline{\lambda_j}$,

$$f_{j,\overline{\lambda_j}} = v_j - v_{j,\overline{\lambda_j}} \geq 0 \text{ in } \Sigma_{j,\overline{\lambda_j}} := B_{\overline{\lambda_j}} \backslash B_{\frac{1}{\Gamma_j}}$$

and

$$-\Delta f_{j,\overline{\lambda_j}} \geq 0 \text{ in } \Sigma_{j,\overline{\lambda_j}}.$$

We easily see that

$$f_{j,\lambda_j} = 0 \quad \text{on } |y| = \lambda_j.$$

By direct computation we get

$$\max_{\partial B_{\frac{1}{\Gamma_j}}} v_{j,\overline{\lambda_j}} = \max_{\partial B_{\frac{1}{\Gamma_j}}} \overline{\lambda_j}^{4-n} w_j \left( \frac{\cdot}{\overline{\lambda_j}^2} \right),$$

by (4.17), $\frac{1}{\Gamma_j \overline{\lambda_j}^2} \leq \frac{\gamma_j^2}{\Gamma_j} < \gamma_j$, follows

$$\leq (2\epsilon)^{4-n} \max_{B_{\gamma_j}} w_j$$

using (4.7), we deduce

$$\leq D \tag{4.18}$$

for some D independent of $j$.
We also have in view of (4.8)

$$\min_{\partial B_{\frac{1}{\Gamma_j}}} v_j = \Gamma_j^{n-4} \min_{\partial B_{\frac{1}{\Gamma_j}}} w_j \left( \frac{y}{|y|^2} \right) = \Gamma_j^{n-4} \min_{\partial B_{\Gamma_j}} w_j > j. \tag{4.19}$$

Here we like to note that (4.19) still hold for $x \neq 0$. Since we would still have $B_{\Gamma_j}(x) \subset B_{\frac{3}{2}\Gamma_j}$ for $j$ large enough ($\frac{1}{2}\Gamma_j \to \infty$).
Therefore, by (4.18) and (4.19), for large $j$, we get

$$\min_{\partial B_{\frac{1}{\Gamma_j}}} (v_j - v_{j,\overline{\lambda_j}}) \geq \min_{\partial B_{\frac{1}{\Gamma_j}}} v_j - \max_{\partial B_{\frac{1}{\Gamma_j}}} v_{j,\overline{\lambda_j}} > j - D > 0.$$

Thus, by the superharmonicity of $f_{j,\overline{\lambda_j}}$ we have

$$f_{j,\overline{\lambda_j}} > 0 \quad \text{in } \overline{\Sigma_{j,\overline{\lambda_j}}} \backslash \partial B_{\overline{\lambda_j}}. \tag{4.20}$$

and letting

$$a_j := \min \left\{ \min_{\partial B_{\frac{\overline{\lambda_j}}{2}}} f_{j,\overline{\lambda_j}} \ , \ \min_{\partial B_{\frac{1}{\Gamma_j}}} f_{j,\overline{\lambda_j}} \right\}$$

we get

$$f_{j,\overline{\lambda_j}}(y) \geq a_j > 0 \quad \text{for all } \ y \in B_{\frac{\overline{\lambda_j}}{2}} \backslash B_{\frac{1}{\Gamma_j}} \tag{4.21}$$

and also, since $f_{j,\overline{\lambda_j}}(y) = 0$ for $|y| = \overline{\lambda_j}$, using the Hopf Lemma we conclude

$$\frac{d}{dr} f_{j,\overline{\lambda_j}}(y) < 0 \quad \text{for } \ y \in \partial B_{\overline{\lambda_j}}. \tag{4.22}$$

On the other hand, by (4.16) we have for $|y| = \overline{\lambda_j}$

$$-\Delta f_{j,\overline{\lambda_j}} = 4\overline{\lambda_j}^{2-n} \left\langle \nabla w_j \left( \frac{y}{|y|^2} \right) , \frac{y}{|y|^2} \right\rangle + 2(n-4)\overline{\lambda_j}^{-2} v_j(y)$$

by direct computation and (4.22) follows

$$= -2\overline{\lambda_j} \frac{d}{dr} f_{j,\overline{\lambda_j}}(y) > 0 \tag{4.23}$$

On the other hand by Lemma 4.5 and the definition of $\overline{\lambda_j}$ we have for $j$ large enough that $-\Delta f_{j,\overline{\lambda_j}}(y)$ is positive for $|y| = \frac{1}{\Gamma_j}$. Thus by (4.23) and the superharmonicity of $-\Delta f_{j,\overline{\lambda_j}}$, we have

$$-\Delta f_{j,\overline{\lambda_j}} > 0 \quad \text{in } \overline{\Sigma_{j,\overline{\lambda_j}}}.$$

Then, as above we can easily find $b_j > 0$ such that

$$-\Delta f_{j,\overline{\lambda_j}}(y) \geq b_j > 0 \quad \text{for all } y \in B_{\frac{\overline{\lambda_j}}{2}} \backslash B_{\frac{1}{\Gamma_j}}. \tag{4.24}$$

Now we are ready to discuss, the two cases, that we observe by the definition of $\overline{\lambda_j}$ and (4.17).

- **Case 1:** There exists for every fixed $j$ a sequence $\{\lambda_k\}$ such that

$$\lambda_k \longrightarrow \overline{\lambda_j} \text{ as } k \to \infty,$$

$$\frac{1}{\Gamma_j} < \lambda_k < \overline{\lambda_j},$$

$$\inf_{\Sigma_{j,\lambda_k}} f_{j,\lambda_k} < 0.$$

We can easily see from (4.21), that for $k$ large enough, we get

$$f_{j,\lambda_k}(y) \geq \frac{1}{2} a_j \quad \text{for } y \in B_{\frac{\overline{\lambda_j}}{2}} \backslash B_{\frac{1}{\Gamma_j}}.$$

It follows that there exists $y_k \in \overline{B_{\lambda_k}} \backslash B_{\frac{\overline{\lambda_j}}{2}}$ such that

$$f_{j,\lambda_k}(y_k) = \min_{\overline{\Sigma_{j,\lambda_k}}} f_{j,\lambda_k} < 0.$$

Since $f_{j,\lambda_k}(y) = 0$ for $|y| = \lambda_k$, we get

$$\frac{1}{2}\overline{\lambda_j} < |y_k| < \lambda_k$$

$$\nabla f_{j,\lambda_k}(y_k) = 0$$
$$\Delta f_{j,\lambda_k}(y_k) \geq 0.$$

After passing to a subsequence (still denoted by $y_k$), $y_k \to y_0$. It follows

$$f_{j,\overline{\lambda_j}}(y_0) = 0 \quad \text{thus } |y_0| = \overline{\lambda_j}$$

and

$$\nabla_{f_{j,\overline{\lambda_j}}}(y_0) = 0 \ , \ \Delta f_{j,\overline{\lambda_j}}(y_0) \geq 0$$

a contradictory of (4.23). Thus **case 1** can not occur.

- **Case 2:** For all $\lambda \geq \overline{\lambda_j} - \epsilon_j$

$$f_{j,\lambda}(y) \geq 0 \quad \text{for all } \ y \in B_\lambda \backslash B_{\frac{1}{\Gamma_j}},$$

and for every fixed $j$ exists a sequence $\{\lambda_k\}$ such that

$$\lambda_k \longrightarrow \overline{\lambda_j} \ \text{ as } k \to \infty,$$

$$\frac{1}{\Gamma_j} < \lambda_k < \overline{\lambda_j},$$

$$\inf_{\Sigma_{j,\lambda_k}} \left(-\Delta f_{j,\lambda_k}\right) < 0.$$

Since

$$f_{j,\lambda_k} \geq 0 \quad \text{on } B_{\lambda_k} \backslash B_{\frac{1}{\Gamma_j}}$$

$$f_{j,\lambda_k} = 0 \quad \text{for } |y| = \lambda_k$$

we have

$$\frac{d}{dr} f_{j,\lambda_k}(y) \leq 0 \quad \text{for } |y| = \lambda_k.$$

Therefore like by (4.23), we get

$$-\Delta f_{j,\lambda_k}(y) = -2\lambda_k \frac{d}{dr} f_{j,\lambda_k} \geq 0 \ \text{ for } |y| = \lambda_k.$$

It follows from (4.24), that for $k$ sufficiently large, we have

$$-\Delta f_{j,\lambda_k}(y) \geq \frac{1}{2} b_j \quad \text{for all} y \in B_{\frac{\overline{\lambda_j}}{2}} \backslash B_{\frac{1}{\Gamma_j}}.$$

Furthermore, since $f_{j,\lambda_k} \geq 0$ we get

$$-\Delta \left(-\Delta f_{j,\lambda_k}\right) \geq 0 \quad \text{on } \ B_{\lambda_k} \backslash B_{\frac{1}{\Gamma_j}}.$$

Thus by applying the maximum principle to the function $-\Delta f_{j,\lambda_k}$ follows

$$-\Delta f_{j,\lambda_k} \geq 0 \quad \text{on } \Sigma_{j,\lambda_k} = B_{\lambda_k} \backslash B_{\frac{1}{\Gamma_j}}$$

which contradicts to the above assumption. Thus **case 2** does not occur too. Lemma 4.6 is established.

□

By lemma 4.2 we know that along a subsequence

$$w_j \longrightarrow w \quad \text{in } C^4_{loc}(\mathbb{R}^n)$$

for some solution $w$ of

$$\begin{aligned} \Delta^2 w &= w^{\frac{n+4}{n-4}} \quad \text{on } \mathbb{R}^n \\ w(0) &= 1. \end{aligned} \tag{4.25}$$

By the convergence of $w_j$ to $w$ and the fact that $\overline{\lambda_j}(x) \to 0$ for every $x \in \mathbb{R}^n$, we have

$$|y-x|^{4-n} w\left(x + \frac{y-x}{|y-x|^2}\right) \leq \lambda^{4-n} w\left(x + \frac{y-x}{\lambda^2}\right) \quad \forall \lambda > 0, \ |y-x| \leq \lambda.$$

It follows thanks to Lemma B.3, $w \equiv w(0) \equiv 1$. Which is impossible because of (4.25).
Theorem 4 is established. □

Now with the help of *Theorem 4* and the Green representation theorem, we give a

*Proof of Corollary 4.1.* Clearly, we only need to establish it for $R = 1$. Let $G$ be the Green's function of the Biharmonic operator on $B_3$, i.e.

$$\begin{cases} \Delta^2 G(x,\cdot) = \delta_x & \text{in } B_3 \\ \\ G(x,\cdot) = -\Delta G(x,\cdot) = 0 & \text{on } \partial B_3. \end{cases}$$

First of all we see by the strong maximum principle and the Hopf Lemma that

$$\begin{cases} -\Delta G(x,\cdot) \geq 0 & \text{in } B_3 \\ \\ \frac{\partial(-\Delta G)}{\partial \nu}(y,s) < 0, \end{cases} \tag{4.26}$$

for every fixed $y \in B_3$ and $s \in \partial B_3$.
Therefore in view of (4.26), it is clear by the maximum principle that there exists a constant $C \geq 1$ such that for all $y$ and $\eta \in B_2$ we have

$$G(y,\eta) \geq C^{-1}$$

and by the Hopf Lemma for every fixed $y \in B_3$ and $s \in \partial B_3$

$$\frac{\partial G(y,s)}{\partial \nu} < 0.$$

Now we let for $y \in \overline{B_2}$

$$u(y) := \min_{\overline{B_2}} u.$$

Then we get by the Green's representation formula,

$$\begin{aligned} u(y) &= \int_{B_3} G(y,\eta)\left(\Delta^2 u\right) \mathrm{d}\eta - \int_{\partial B_3} \frac{\partial\left(-\Delta G(y,s)\right)}{\partial \nu} u(s)\,\mathrm{d}s - \int_{\partial B_3} \frac{\partial G(y,s)}{\partial \nu}\left(-\Delta u\right) \mathrm{d}s \\ &\geq \int_{B_3} G(y,\eta) u^{\frac{n+4}{n-4}}(\eta)\,\mathrm{d}\eta \\ &\geq C^{-1} \int_{B_1} u^{\frac{n+4}{n-4}}(\eta)\,\mathrm{d}\eta \end{aligned}$$

Therefore by Theorem 4

$$\begin{aligned} \int_{B_1} u^{\frac{2n}{n-4}} &\leq \max_{\overline{B_1}} u \int_{B_1} u^{\frac{n+4}{n-4}} \\ &\leq C\left(\max_{\overline{B_1}} u\right)\left(\min_{\overline{B_2}} u\right) \\ &\leq C. \end{aligned}$$

□

# A

# APPENDIX

In this chapter we present the calculs lemmas, that we used in the chapters 1-3. The first Lemma shows one of the important properties of the Kelvin transform, and the main reason behind its creation.

**Lemma A.1.** *Let*

$$\Delta u = f \qquad in \;\; \Omega \subset \mathbb{R}^n.$$

*Then the Kelvin transform of u defined by*

$$u_\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n-2} u\left(\frac{\lambda^2}{|y|^2}y\right) \quad for \;\; \frac{\lambda^2}{|y|^2}y \in \Omega$$

*satisfies*

$$\Delta u_\lambda(y) = \left(\frac{\lambda}{|y|}\right)^{n+2} f\left(\frac{\lambda^2}{|y|^2}y\right)$$

*Proof.* Writing $r = |y|, \omega = \frac{y}{|y|}$ and $\Delta_\omega$ the Laplace-Beltrami operator on $\partial B_1$. Then

$$\begin{aligned}
\Delta u_\lambda(r,\omega) &= \left(r^{1-n}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega\right) u_\lambda(r,\omega) \\
&= \left(r^{1-n}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\omega\right)\left(\left(\frac{\lambda}{r}\right)^{n-2} u\left(\frac{\lambda^2}{r},\omega\right)\right) \\
&= \lambda^{n-2}r^{1-n}\frac{\partial}{\partial r}r^{n-1}\frac{\partial}{\partial r}r^{2-n}u\left(\frac{\lambda^2}{r},\omega\right) + \left(\frac{\lambda}{r}\right)^{n+2}\frac{1}{(\lambda^2/r)^2}\Delta_\omega u\left(\frac{\lambda^2}{r},\omega\right) \\
&= \left(\frac{\lambda}{r}\right)^{n+2}\left((u_{rr})\left(\frac{\lambda^2}{r},\omega\right) + \frac{n-1}{\lambda^2/r}(u_r)\left(\frac{\lambda^2}{r},\omega\right) + \frac{1}{(\lambda^2/r)^2}\Delta_\omega\left(\frac{\lambda^2}{r},\omega\right)\right) \\
&= \left(\frac{\lambda}{r}\right)^{n+2}(\Delta u)\left(\frac{\lambda^2}{r},\omega\right)
\end{aligned}$$

□

**Lemma A.2.** *Let* $f \in C^1(\mathbb{R}^n), n \geq 1, \nu > 0$. *Assume that*

$$\left(\frac{\lambda}{|y-x|}\right)^{\nu} f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \quad \forall \lambda > 0, \quad x \in \mathbb{R}^n, |y-x| \geq \lambda.$$

*Then*

$$f \equiv constant$$

*Proof.* For $x \in \mathbb{R}^n, \lambda > 0$, set

$$g_{x,\lambda}(z) = f(x+z) - \left(\frac{\lambda}{|z|}\right)^{\nu} f\left(x + \frac{\lambda^2 z}{|z|^2}\right), \quad |z| \geq \lambda.$$

Then we have

$$g_{x,|z|}(z) = f(x+z) - \left(\frac{|z|}{|z|}\right)^{\nu} f\left(x + \frac{|z|^2 z}{|z|}\right) = 0, \tag{A.1}$$

Using the assumtion for $y = x + rz, \lambda = |z|$ we get

$$\frac{|z|}{|rz|} f\left(x + \frac{|z|^2 rz}{|rz|^2}\right) \leq f(x+rz), \quad \forall r \geq 1,$$

Therefore

$$g_{x,|z|}(rz) \geq 0, \quad \forall r \geq 1. \tag{A.2}$$

By (A.1) and (A.2), it follows that

$$\frac{d}{dr}\left\{g_{x,|z|}(rz)\right\}_{|r=1} \geq 0.$$

By direct computation we get

$$\begin{aligned}\frac{d}{dr}\left\{g_{x,|z|}(rz)\right\}_{|r=1} &= \frac{d}{dr}\left\{f(x+rz) - \left(\frac{1}{r}\right)^{\nu} f(x+\frac{z}{r})\right\}_{|r=1} \\ &= \left\{\nabla f(x+rz)\cdot z + \nu\left(\frac{1}{r}\right)^{\nu-1}\frac{1}{r^2} f(x+\frac{z}{r}) + \left(\frac{1}{r}\right)^{\nu}\nabla f(x+\frac{z}{r})\cdot\frac{z}{r^2}\right\}_{|r=1} \\ &= 2\nabla f(x+z)\cdot z + \nu f(x+z) \geq 0.\end{aligned}$$

Since $z$ and $x$ are arbitrary, by a change of variables, we have

$$2\nabla f(y)\cdot(y-x) + \nu f(y) = 2\nabla f(y)\cdot y - 2\nabla f(y)\cdot x + \nu f(y) \geq 0.$$

Dividing the above by $|x|$ and sending $|x| \to \infty$, we have,

$$\nabla f(y)\cdot\theta \leq 0 \qquad \forall x \in \mathbb{R}^n \text{ and } \theta \in S^{n-1}.$$

It follows that

$$\nabla f \equiv 0$$

□

**Lemma A.3.** *Let $f \in C^1(\mathbb{R}^n_+), n \geq 1, \nu > 0$. Assume that*

$$\left(\frac{\lambda}{|y-x|}\right)^{\nu} f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \quad \forall \lambda > 0, \quad x \in \partial\mathbb{R}^n_+, |y-x| \geq \lambda, \, y \in \mathbb{R}^n_+.$$

*Then*

$$f(y) = f(y', t) = f(0, t) \quad \forall y = (y', t) \in \mathbb{R}^n_+.$$

*Proof.* For $x \in \partial\mathbb{R}^n_+, \lambda > 0$, set

$$g_{x,\lambda}(z) = f(x+z) - \left(\frac{\lambda}{|z|}\right)^{\nu} f\left(x + \frac{\lambda^2 z}{|z|^2}\right), \quad z \in \mathbb{R}^n_+, \, |z| \geq \lambda.$$

As in the proof of Lemma A.2, we have

$$2\nabla f(x+z) \cdot z + \nu f(x+z) \geq 0, \quad \forall x \in \partial\mathbb{R}^n_+, \, z \in \mathbb{R}^n_+.$$

Making a change of variables, we have

$$2\partial_{y'} f(y', t) \cdot (y' - x') + 2\partial_t f(y', t)t + \nu f(y', t) \geq 0, \; \forall x', y' \in \mathbb{R}^{n-1}, t > 0.$$

Dividing the above by $|x'|$ and sending $|x'| \to \infty$, we have,

$$\partial_{y'} f(y', t) \cdot \theta \leq 0 \qquad \forall (y', t) \in \mathbb{R}^n_+ \;\text{ and }\; \theta \in S^{n-2}.$$

It follows that

$$\partial_{y'} f(y', t) \equiv 0.$$

□

**Lemma A.4.** *Let $g$ be a positive continuous function on $(0, \infty)$ satisfying*

$$\liminf_{s \to \infty} g(s) > 0.$$

*Then*

$$u''(t) + g(u(t)) = 0, \quad 0 \leq t < \infty$$

*does not have any positive solution $u$.*

*Proof.* Let $v = u'$, then we have

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -g(u) \end{pmatrix} \tag{A.3}$$

Next we discuss three cases:

- If $v(0) < 0$, then we have from $\frac{d}{dt}v = -g(u) < 0$ and the fundamental theorem of calculus, that
$$v(t) < v(0) \qquad \forall t > 0.$$
Again by the fundamental theorem of calculus and the first equation of (A.3)
$$u(t) \leq u(0) + \underbrace{v(0)}_{<0} t.$$
This is impossible for large $t$ since $u$ is positive.

- If $v(0) = 0$, then by the second equation, $v(t) < 0$ for all $t > 0$. Therefore by the same argument as above, we get for all $t > 1$
$$u(t) \leq u(1) + \underbrace{v(1)}_{<0} t.$$
which is also impossible for large $t$ since $u$ is positive

- So we only need to eliminate the possibility that $v(t) > 0$ for all $t \geq 0$. In this case, by the first equation of (A.3) and the fundamental theorem of calculus, follows
$$u(t) > u(0) > 0 \quad \text{for all } \ t > 0.$$
Therefore, by the hypothesis on $g$ and the second equation, there exists some $\delta > 0$ such that
$$-v'(t) = g\left(u(t)\right) \geq \delta \quad \text{for all } \ t > 0.$$
Thus by the fundamental theorem of calculus
$$v(t) \leq v(0) - \delta t \quad \text{for all } \ t > 0.$$
This is impossible since $v$ is assumed to be positive all the time.

□

In the following, we let $\Omega$ be a domain of $\mathbb{R}^n, n \geq 2$ with the origin 0 on its boundary. Assume that near 0 the boundary consists of two transversally intersection $C^2$ hypersurfaces $\rho = 0$ and $\sigma = 0$. Also we suppose $\rho, \sigma > 0$ in $\Omega$. Let $\nu(y)$ be the unit outer normal of the surface $\{\sigma = 0\} \cap \partial\Omega$ at $y$. Then we get

**Lemma A.5.** *Let $u \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$ be positive in $\overline{\Omega} \setminus \{0\}, u(0) = 0$, and satisfy, for some positive constant $A$, that*
$$\begin{cases} -\Delta u \geq -Au & \text{in } \ \Omega \\ \frac{\partial u}{\partial \nu} \geq -Au & \text{on } \ \{\sigma = 0, \rho > 0\}, \end{cases}$$
*where $\nu$ denotes the unit outer normal. Then we have*
$$\frac{\partial u}{\partial \nu'}(0) = 0,$$
*where $\nu'$ is any vector in the tangent space $\{\sigma = 0\}$ that enters into $\{\rho > 0\}$.*

*Proof.* Since the hypotheses and conclusions are invariant under change of coordinates, and of the choices of the particular $\rho$ and $\sigma$ representing the bounding hypersurfaces. We may assume without loss of generality that $\rho(y) = y_1$ and $\sigma(y) = y_2$.
Choosing $\epsilon$ small enough so that

$$\{y_1 > 0\} \cap \{y_2 > 0\} \cap B_{2\epsilon} \subset \Omega.$$

We wish to construct a function $\phi > 0$ in $\Omega$ such that

$$\begin{cases} -\Delta\phi \leq -A\phi & \text{in } \ \Omega \cap B_\epsilon, \\ \phi = 0 & \text{in } \ \{y_1 = 0\} \cap B_\epsilon, \\ \frac{\partial\phi}{\partial\nu} \leq -A\phi & \text{on } \ \{y_2 = 0, \, y_1 > 0\} \cap B_\epsilon, \\ \phi \leq u & \text{on } \ \partial B_\epsilon \cap \overline{\Omega}, \\ \frac{\partial\phi}{\partial\nu'}(0) > 0. \end{cases}$$

Once such $\phi$ is constructed, Lemma A.5 can be proved as follows.
Let $\omega = u - \phi$, then by a direct calculation, we see that $w$ satisfies

$$\begin{cases} -\Delta\omega + A\omega \geq 0 & \text{in } \ \Omega \cap B_\epsilon, \\ \omega \geq 0 & \text{on } \ \{y_1 = 0\} \cap B_\epsilon \text{ and } \partial B_\epsilon \cap \overline{\Omega}, \\ \frac{\partial\omega}{\partial\nu} + A\omega \geq 0 & \text{on } \ \{y_2 = 0, \, y_1 > 0\} \cap B_\epsilon. \end{cases}$$

An application of the Strong Maximum Principle, yields to

$$\omega \geq 0 \quad \text{on } \overline{\Omega \cap B_\epsilon}.$$

Therefore by $\omega(0) = 0$, we have

$$\frac{\partial\omega}{\partial\nu'}(0) \geq 0.$$

Consequently

$$\frac{\partial u}{\partial\nu'}(0) \geq \frac{\partial\phi}{\partial\nu'}(0) > 0.$$

Such a $\phi$ can be given explicitly by setting

$$\phi(y) = \delta\left(e^{\alpha^2 y_1 - 1}\right) e^{\alpha y_2} \quad \text{for } y \in \Omega,$$

where $\alpha > 1$ will be large and $\delta > 0$ will be chosen small.
By direct computation, we get, for large $\alpha$,

$$\Delta\phi(y) = \delta\alpha^4 e^{\alpha y_2} e^{\alpha^2 y_1} + \alpha^2\phi(y) \geq A\phi(y).$$

On $\{y_2 = 0\}$, for large $\alpha$,

$$\frac{\partial \phi}{\partial y_2} = \alpha\phi \geq A\phi,$$

i.e. on $\{\sigma = 0\}$,

$$\frac{\partial \phi}{\partial y} \leq -A\phi$$

Now we fix the value of $\alpha$. Since $u > 0$ on $\overline{\Omega} \setminus \{0\}$, we chose $\delta > 0$ small enough such that

$$u > \phi \quad \text{on } \partial B_\epsilon \cap \overline{\Omega}.$$

Finally it is immediate to check that $\frac{\partial \phi}{\partial y_1}(0) = \delta\alpha^2 > 0$, so all the desired properties are satisfied. Lemma A.5 is established. $\square$

# B

## APPENDIX

In this chapter we present some calculs lemmas, that we used in chapter 4.

**Lemma B.1.** *Let $\Omega \subset\subset \mathbb{R}^n$ open and connected, $f \in C^2(\Omega)$ and*

$$\int_\Omega f\Delta\varphi \, dx \geq 0, \tag{B.1}$$

*for all non-negative $\varphi \in C^2(\Omega)$ such that $\varphi = 0$ on $\partial\Omega$.*
*Then*

$$f \leq 0 \quad in \ \Omega.$$

*Proof.* We argue by contradiction. Suppose that $f > 0$ over some ball $B \subset \Omega$. We fix any nonzero, non-negative function $\psi \in C(B)$ and denote by $\widetilde{\psi}$ its extension by zero. Now considering $\varphi \in C^2(\Omega)$, such that

$$\begin{cases} -\Delta\varphi & = \widetilde{\psi} \ \text{ in } \ \Omega \\ \varphi & = 0 \ \text{ on } \ \partial\Omega \end{cases}$$

Then by the strong maximum principle, $\varphi \geq 0$ in $\Omega$, and we get

$$0 < \int_B \psi f \, \mathrm{d}x = \int_\Omega \widetilde{\psi} f \, \mathrm{d}x = -\int_\Omega f\Delta\varphi \, \mathrm{d}x.$$

This contradicts (B.1)

□

**Lemma B.2.** *Let $u$ be a smooth positive solution of*

$$\Delta^2 u = u^{\frac{n+4}{n-4}} \qquad in \ \ B_1 \setminus \{0\},$$

*where $n \geq 5$.*
*Then*

$$u^{\frac{n+4}{n-4}} \in L^1\left(\overline{B_{\frac{1}{2}}}\right).$$

*Proof.* We argue by contradiction. Let $p := \frac{n+4}{n-4}$ and suppose that $u^p \notin L^1\left(\overline{B_{\frac{1}{2}}}\right)$. Then by applying the Green's formula, we have for all $0 < s \leq r \leq \frac{1}{2}$

$$0 < \int_{B_r \setminus B_s} u^p = \int_{\partial B_r} \frac{\partial}{\partial r}(\Delta u) \ \mathrm{d}\ \sigma - \int_{\partial B_s} \frac{\partial}{\partial r}(\Delta u)\ \mathrm{d}\ \sigma. \tag{B.2}$$

Since the left hand side of (B.2) tends to $+\infty$ as $s \to 0$, there exists $r_1 > 0$ and $0 < r_2 < r_1$ such that for $0 < r \leq r_2$

$$⨍_{\partial B_r} \frac{\partial}{\partial r}(\Delta u)\ \mathrm{d}\sigma \leq -c_1 r^{1-n} \int_{B_{\frac{1}{2}} \setminus B_r} u^p. \tag{B.3}$$

Integrating the identity above along $[s, r]$, we get

$$\int_{\partial B_r} \Delta u\ \mathrm{d}\sigma - \int_{\partial B_s} \Delta u\ \mathrm{d}\sigma \leq -c_1 \int_s^r \tau^{1-n} \int_{B_{\frac{1}{2}} \setminus B_r} u^p \mathrm{d}y\, \mathrm{d}\tau, \tag{B.4}$$

and

$$\int_{\partial B_r} \Delta u\ \mathrm{d}\sigma \geq c_2 r^{-n+2}. \tag{B.5}$$

Next by letting

$$\overline{u} = ⨍_{\partial B_r} u\ \mathrm{d}\sigma,$$

we get in view of (B.5)

$$\left(r^{n-1}\overline{u}'(r)\right)' \geq c_2 r. \tag{B.6}$$

If $\lim_{r\to 0} r^{n-1}\overline{u}'(r) \geq 0$, then we have for any $r > 0$

$$r^{n-1}\overline{u}'(r) \geq \frac{1}{2} c_2 r^2 \tag{B.7}$$

integrating along $r$ and observing that $u$ is positive, we get

$$\overline{u}(r) \geq \int_0^r \overline{u}(t)\ \mathrm{d}t \geq \frac{1}{2} c_2 \int_0^r t^{3-n}\ \mathrm{d}t = +\infty,$$

which a contradiction with its definition.
Therefore we may assume there exists $0 < r_3 < r_2$ such that for all $r \leq r_3$, we have

$$r^{n-1}\overline{u}'(r) \leq -c_3, \tag{B.8}$$

where $c_3$ is a positive constant. Thus

$$\overline{u}(r) \geq c_4 r^{2-n}. \tag{B.9}$$

Suppose $\overline{u}(r) \geq c_4 r^{-s}$ for some $s \geq n-2$. Then by (B.3) and (B.4), we have for small $r > 0$,

$$(\Delta \overline{u}(r))' \leq -c_1 r^{-ps-n+2} \tag{B.10}$$

$$\Delta \overline{u}(r) \geq c_2 r^{-ps-n+3} \tag{B.11}$$

and since $ps \geq \frac{n+4}{n-4}(n-2) > n+4 > 9$ for $n \geq 5$

$$r^{n-1}\overline{u}'(r) \leq -c_3 r^{-ps+3} \tag{B.12}$$

and

$$\overline{u}(r) \geq c_4 r^{-ps-n+5} \tag{B.13}$$

Since $\overline{u}' \leq 0$, we have $0 < \Delta\overline{u}(r) \leq \overline{u}''$ and $\overline{u}^{(3)}(r) \leq (\Delta\overline{u})' < 0$. Thus after a finite time of iterations, there exists $r_0$ such that for $0 \leq r \leq 2r_0$ and $\beta = \frac{4}{p-1}$, we have

$$\overline{u}(r) \geq r^{-(1+\beta)} \tag{B.14}$$

$$\overline{u}'(r) \leq -r^{-(2+\beta)} \tag{B.15}$$

$$\overline{u}''(r) \geq r^{-(3+\beta)} \tag{B.16}$$

$$\overline{u}'''(r) \leq -r^{-(4+\beta)} \tag{B.17}$$

Therefore by (B.15) $\backsim$ (B.17), we get by Jensen inequality

$$\overline{u}^{(4)}(r) \geq \Delta^2 \overline{u}(r) \geq \overline{u}^p(r). \tag{B.18}$$

For $r_0 \leq r \leq 2r_0$ and $A > 0$, let

$$v(r) = A(r - r_0)^{-\beta}.$$

Then by direct computations follows

$$v^{(4)} = A\beta(\beta+1)(\beta+2)(\beta+3)(r-r_0)^{-(\beta+4)}$$

and since $\beta \cdot p = \beta + 4$

$$= A^{1-p}\beta(\beta+1)(\beta+2)(\beta+3)v^p(r)$$

taking $A$ large

$$< v^p(r)$$

If $r_0$ is sufficiently small, then by (B.14) $\backsim$ (B.17), we have for all $r_0 \le r \le 2r_0$

$$v(r) \le \overline{u}(r).$$

However

$$\lim_{r\to r_0} \overline{u}(r) \ge \lim_{r\to r_0} v(r) = +\infty$$

which also a contradiction with its definition.
Lemma B.2 is established. □

**Lemma B.3.** *Let $f \in C^1(\mathbb{R}^n), n \ge 1, \nu > 0$. Assume that*

$$\left(\frac{1}{|y-x|}\right)^{\nu} f\left(x + \frac{(y-x)}{|y-x|^2}\right) \le \left(\frac{1}{\lambda}\right)^{\nu} f\left(x + \frac{(y-x)}{\lambda^2}\right), \quad \forall \lambda > 0, \quad x \in \mathbb{R}^n, |y-x| \le \lambda.$$

*Then*

$$f \equiv constant$$

*Proof.* For all $x \in \mathbb{R}^n, \lambda > 0$, set

$$g_{x,\lambda}(z) = |z|^{-\nu} f\left(x + \frac{z}{|z|^2}\right) - \lambda^{-\nu} f\left(x + \frac{z}{\lambda^2}\right), \quad |z| \le \lambda.$$

Then we clearly have

$$g_{x,|z|}(z) = 0 \tag{B.19}$$

and using the assumption for $y = x + rz, \lambda = |z|$ we get

$$\left(\frac{1}{r|z|}\right)^{\nu} f\left(x + \frac{rz}{|rz|^2}\right) \ge \left(\frac{1}{|z|}\right)^{\nu} f\left(x + \frac{rz}{|z|^2}\right) \quad \forall r \le 1.$$

Thus

$$g_{x,|z|}(rz) \ge 0 \quad \forall r \le 1. \tag{B.20}$$

By (B.19) and (B.20), it follows that

$$\frac{d}{dr}\left\{g_{x,|z|}(rz)\right\}_{|r=1} \le 0$$

which means that

$$\begin{aligned}
\frac{d}{dr}\left\{g_{x,|z|}(rz)\right\}_{|r=1} &= \frac{d}{dr}\left\{\left(\frac{1}{r|z|}\right)^{\nu} f\left(x + \frac{rz}{|rz|^2}\right) - \left(\frac{1}{|z|}\right)^{\nu} f\left(x + \frac{rz}{|z|^2}\right)\right\}_{|r=1} \\
&= \left\{-\nu\left(\frac{1}{r|z|}\right)^{\nu-1} \cdot \frac{1}{r^2|z|} f\left(x + \frac{z}{r|z|^2}\right) - \left(\frac{1}{r|z|}\right)^{\nu} \nabla f\left(x + \frac{z}{r|z|^2}\right) \cdot \frac{z}{r^2|z|^2}\right. \\
&\quad \left. - \left(\frac{1}{|z|}\right)^{\nu} \nabla f\left(x + \frac{rz}{|z|^2}\right) \cdot \frac{z}{|z|^2}\right\}_{r=1} \\
&= -\nu\left(\frac{1}{|z|}\right)^{\nu} f\left(x + \frac{z}{|z|^2}\right) - 2\left(\frac{1}{|z|}\right)^{\nu} \nabla f\left(x + \frac{z}{|z|^2}\right) \cdot \frac{z}{|z|^2} \le 0.
\end{aligned}$$

Since $z$ and $x$ are arbitrary, by a change of variables, we have

$$2\nabla f(y) \cdot (y - x) + \nu f(y) = 2\nabla f(y) \cdot y - 2\nabla f(y) \cdot x + \nu f(y) \geq 0.$$

Dividing the above by $|x|$ and sending $|x| \to \infty$, we have,

$$\nabla f(y) \cdot \theta \leq 0 \qquad \forall x \in \mathbb{R}^n \ \text{ and } \ \theta \in S^{n-1}.$$

It follows that

$$\nabla f \equiv 0$$

□

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