VARIATIONAL DISCRETIZATION OF LINEAR WAVE EQUATIONS ON EVOLVING SURFACES

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Abstract. A linear wave equation on a moving surface is derived from Hamilton's principle of stationary action. The variational principle is discretized with functions that are piecewise linear in space and time. This yields a discretization of the wave equation in space by evolving surface finite elements and in time by a variational integrator, a version of the leapfrog or Störmer–Verlet method. We study stability and convergence of the full discretization in the natural time-dependent norms under the same CFL condition that is required for a fixed surface. Using a novel modified Ritz projection for evolving surfaces, we prove optimal-order error bounds. Numerical experiments illustrate the behavior of the fully discrete method.

1. Introduction

In recent years, there have been significant advances in the numerical analysis of partial differential equations on fixed and moving surfaces. Concerning the latter, we refer to the review article by Deckelnick, Dziuk & Elliott [4] and to recent papers on linear parabolic equations on time-dependent surfaces discretized by evolving surface finite elements and various time discretizations [8, 9, 11, 19, 24], by finite volume methods [17], by a grid-based particle method [18] and by level set methods [1, 26], and to [10] for conservation laws on time-dependent surfaces.

In the present paper we consider a linear wave equation on a given time-dependent surface, which is the natural analog of the classical acoustic wave equation on a fixed spatial domain. We have no specific application in mind, but consider the problem as prototypical for dynamical problems on a moving surface that are described by Hamilton's principle of stationary action, a fundamental principle of mechanics. Just as the numerical analysis of the linear wave equation on a fixed domain has provided much insight into the numerical treatment of more complicated, linear and nonlinear, wave problems in a variety of application areas, we expect similar benefits from a thorough numerical analysis of the linear wave equation on evolving surfaces based on the variational formulation. Among novel analytic techniques developed here is a stability analysis of full discretizations with time-dependent mass and stiffness matrices in the natural time-dependent norms, and the use of appropriately modified Ritz projections to derive optimal-order error bounds. For example, our stability analysis could be similarly applied to moving-mesh methods for wave equations on a domain, and the newly introduced Ritz map on evolving

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surfaces is potentially useful in a much wider context than for the particular bilinear forms considered here.

In Section 2 we derive the wave equation on an evolving surface from the Hamilton variational principle with a Lagrangian that is the precise analog of the Lagrangian for the acoustic wave equation on a fixed domain. In Section 3 we describe a discretization of the variational principle by the piecewise linear evolving-surface finite elements of Dziuk and Elliott [8]. For discretization in time we use a variational integrator based on piecewise linear approximation, the Störmer–Verlet or leapfrog method, which is discussed in Section 4. We analyze the stability of the fully discrete scheme under the natural CFL condition in Section 5.

Our stability estimate is sufficiently strong to permit us to derive optimal-order error estimates. This is done in several steps. In Section 6 we bound the fully discrete error in terms of the residual of mappings of the exact solution onto the finite element space on the discretized surface. To estimate this residual, we need the preparatory Section 7 that provides known and new estimates for lifts of functions from the discretized to the original time-dependent surface. In Section 8 we introduce the Ritz map for evolving surfaces as the appropriate mapping of functions on the original surface to finite element functions on the discretized surface, and we study its approximation properties. This allows us, in Section 9, to give an optimal second-order bound of the residual that results when the Ritz map applied to the exact solution is inserted into the semidiscrete surface finite element equations.

Combining all the results obtained thus far, in Section 10 we finally obtain our main result, which states optimal-order convergence of the full discretization in the natural time-dependent norms under the CFL condition. We show second order of the error measured in the L^2 norm over the time-dependent surface for displacements and their material derivatives, and first order for the L^2 norm of the error in the surface gradient of the displacements, uniformly on bounded time intervals. We conclude the paper with numerical experiments in Section 11.

Throughout the paper, C and c denote generic constants (independent of the spatial meshwidth h and the time step size τ) that take on different values on different occurences.

2. The wave equation on evolving surfaces

2.1. **Basic notation.** Let $\Gamma(t)$, $t \in [0,T]$, be a smoothly evolving family of smooth m-dimensional compact closed hypersurfaces in $\mathbb{R}^{m+1}$ without boundary. We denote the corresponding space-time surface by $G_T = \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}$.

Let $v(x,t)$, for $x \in \Gamma(t)$ and $t \in [0,T]$, denote the given *velocity* of the surface, with the interpretation that a material point $x(t)$ on the surface moves with velocity $\dot{x}(t) = v(x(t),t)$.

We will work with the following time and space derivatives, for which we refer to [12] and [8] for a more detailed discussion. For a smooth function $u : G_T \to \mathbb{R}$ we let $\partial^\bullet u$ denote the *material derivative*, defined such that $\frac{d}{dt} u(x(t),t) = (\partial^\bullet u)(x(t),t)$, that is,

$$\partial^\bullet u = \frac{\partial u}{\partial t} + v \cdot \nabla u, \tag{2.1}$$

where $a \cdot b = \sum_{j=1}^{m+1} a_j b_j$ for vectors a and b in $\mathbb{R}^{m+1}$, and ∇u denotes the usual $(m+1)$-dimensional gradient of a smooth extension of $u(\cdot,t)$ to a neighborhood of

$\Gamma(t)$. The material derivative $\partial^\bullet u$ only depends on the values of the function u on the space-time surface G_T and is independent of the choice of the extension.

By ∇_Γ we denote the *tangential gradient* on the surface Γ, which is the projection of the $(m+1)$-dimensional gradient to the tangent space. For a smooth function u on a neighborhood of Γ we define

$$\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu\, \nu,$$

where ν is a normal vector field to Γ. The tangential gradient only depends on the values of u on the surface Γ and is independent of the extension.

The *Laplace-Beltrami operator* on Γ is the tangential divergence of the tangential gradient:

$$\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u = \sum_{j=1}^{m+1} (\nabla_\Gamma)_j (\nabla_\Gamma)_j u.$$

2.2. **Hamilton's principle of stationary action.** With the *Lagrangian* (kinetic energy minus potential energy)

$$\mathcal{L}(u, \partial^\bullet u, t) = \frac{1}{2}\int_{\Gamma(t)} |\partial^\bullet u|^2 - \frac{1}{2}\int_{\Gamma(t)} \left|\nabla_{\Gamma(t)} u\right|^2 \tag{2.2}$$

we consider the *action integral*

$$\mathcal{S}[u] = \int_0^T \mathcal{L}(u(t), \partial^\bullet u(t), t)\,\mathrm{d}t \tag{2.3}$$

for $u(t) = u(\cdot, t) \in H^1(\Gamma(t))$. The analogous action integral on a fixed domain Ω instead of moving surfaces $\Gamma(t)$ is minimized by solutions of the classical acoustic wave equation $\partial_t^2 u - \Delta u = 0$. In our situation we arrive at the following partial differential equation, which was first communicated to us by G. Dziuk.

Lemma 2.1. *If $u : G_T \to \mathbb{R}$ is a smooth function that extremizes the action integral $\mathcal{S}[u]$ among all smooth functions on G_T with given end-points $u(\cdot, 0)$ and $u(\cdot, T)$, then u is a solution of the Euler–Lagrange partial differential equation*

$$\partial^\bullet \partial^\bullet u(x,t) + \partial^\bullet u(x,t)\ \nabla_{\Gamma(t)} \cdot v(x,t) - \Delta_{\Gamma(t)} u(x,t) = 0 \tag{2.4}$$

for $x \in \Gamma(t)$ and $0 \le t \le T$.

We refer to (2.4) as the *wave equation on the evolving surface*. An inhomogeneity $f(x,t)$ on the right-hand side of (2.4) is obtained by adding the term $\int_{\Gamma(t)} fu$ to the Lagrangian.

Proof. The result is a consequence of the Leibniz formula on surfaces [8, Lemma 2.2]:

$$\frac{d}{dt}\int_\Gamma g = \int_\Gamma \partial^\bullet g + g \nabla_\Gamma \cdot v. \tag{2.5}$$

Computing variations of the action while keeping the endpoints of $u(.,t)$ fixed ($\delta u(0) = \delta u(T) = 0$), using (2.5) and partial integration, we get

$$\begin{aligned}\delta\mathcal{S}[u] &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} S\left(u+\epsilon\delta u\right) = \int_0^T \int_\Gamma \Big(\partial^\bullet u\, \partial^\bullet \delta u - \nabla_\Gamma u \nabla_\Gamma \delta u\Big)\, \mathrm{d}t \\ &= \int_0^T \frac{d}{dt}\int_\Gamma \partial^\bullet u\, \delta u\, \mathrm{d}t - \int_0^T \int_\Gamma \Big(\partial^\bullet \partial^\bullet u\, \delta u + \partial^\bullet u\, \delta u\, \nabla_\Gamma \cdot v + \nabla_\Gamma u \nabla_\Gamma \delta u\Big)\, \mathrm{d}t \\ &= -\int_0^T \int_\Gamma \left(\partial^\bullet \partial^\bullet u + \partial^\bullet u \nabla_\Gamma \cdot v - \Delta_\Gamma u\right) \delta u\, \mathrm{d}t = 0.\end{aligned}$$

With the fundamental lemma of the calculus of variations we obtain the result. □

Using the Leibniz formula on surfaces (2.5), a weak form of the wave equation (2.4) is readily obtained: for all smooth $\varphi : G_T \to \mathbb{R}$ and for almost every $t \in [0,T]$,

$$\frac{d}{dt}\int_\Gamma \partial^\bullet u\, \varphi + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_\Gamma \partial^\bullet u\, \partial^\bullet \varphi. \tag{2.6}$$

We will consider the *initial value problem* of the wave equation on the evolving surface, with given initial data $u(0) \in H^2(\Gamma(0))$ and $\partial^\bullet u(0) \in L^2(\Gamma(0))$. Wellposedness and regularity results are shown in [20].

3. Variational space discretization

3.1. **Recap: The evolving surface finite element method.**

Following [8], the smooth surface $\Gamma(t)$ is interpolated at nodes $a_i(t) \in \Gamma(t)$ ($i = 1, \ldots, J$) by a discrete polygonal surface $\Gamma_h(t)$, where h denotes the grid size. These nodes move with velocity $da_i(t)/dt = v(a_i(t), t)$. The discrete surface

$$\Gamma_h(t) = \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t)$$

is the union of m-dimensional simplices $E(t)$ that is assumed to form an admissible triangulation $\mathcal{T}_h(t)$; see [8] for details. The finite element space on the discrete surface $\Gamma_h(t)$ is chosen as

$$S_h(t) = \{\phi_h \in C^0(\Gamma_h(t)) \;:\; \phi_h|_E \in \mathbb{P}_1 \text{ for all } E \in \mathcal{T}_h(t)\},$$

where $\mathbb{P}_1$ denotes the space of polynomials of degree at most 1. Let $\chi_j(\cdot,t)$ ($j = 1, \ldots, J$) be the nodal basis of $S_h(t)$, given by $\chi_j(a_i(t), t) = \delta_{ji}$ for all i, so that

$$S_h(t) = \mathrm{span}\{\chi_1(\cdot,t), \ldots, \chi_J(\cdot,t)\}.$$

We define a velocity for material points $X(t)$ on the surface $\Gamma_h(t)$ by

$$\dot{X}(t) = V_h(X(t),t), \quad V_h(x,t) := \sum_{j=1}^{J} v(a_j(t),t)\chi_j(x,t), \quad x \in \Gamma_h(t). \tag{3.1}$$

Then the discrete material derivative on $\Gamma_h(t)$ is given by

$$\partial_h^\bullet \phi_h = \frac{\partial \phi_h}{\partial t} + V_h \cdot \nabla \phi_h. \tag{3.2}$$

The construction is such that the discrete material derivatives of the basis functions satisfy the *transport property* [8, Proposition 5.4]:

$$\partial_h^\bullet \chi_j = 0. \tag{3.3}$$

The discrete surface gradient is defined piecewise as

$$\nabla_{\Gamma_h} g = \nabla g - \nabla g \cdot \nu_h \nu_h,$$

where ν_h denotes the normal to the discrete surface.

3.2. **The semi-discrete Hamilton principle.** We replace the Lagrangian (2.2) with the Lagrangian on the discretized surface

$$\mathcal{L}_h\left(U_h, \partial_h^\bullet U_h, t\right) = \frac{1}{2}\int_{\Gamma_h(t)} |\partial_h^\bullet U_h|^2 - \frac{1}{2}\int_{\Gamma_h(t)} |\nabla_{\Gamma_h} U_h|^2 \tag{3.4}$$

and minimize the action integral

$$\mathcal{S}_h[U_h] = \int_0^T \mathcal{L}_h\left(U_h(t), \partial_h^\bullet U_h(t), t\right) \,\mathrm{d}t \tag{3.5}$$

for $U_h(t) = U_h(\cdot,t) \in S_h(t)$. This turns out to be equivalent to the Galerkin discretization of (2.6): for all temporally smooth ϕ_h with $\phi_h(\cdot,t) \in S_h(t)$ and for all t,

$$\frac{d}{dt}\int_{\Gamma_h} \partial_h^\bullet U_h \, \phi_h + \int_{\Gamma_h} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h = \int_{\Gamma_h} \partial_h^\bullet U_h \partial_h^\bullet \phi_h. \tag{3.6}$$

3.3. **Matrix-vector formulation and Hamiltonian ODE system.** We denote the discrete solution

$$U_h(\cdot,t) = \sum_{j=1}^J q_j(t)\chi_j(\cdot,t) \ \in S_h(t)$$

and define $\mathbf{q}(t) \in \mathbb{R}^J$ as the nodal vector with entries $q_j(t) = U_h(a_j(t),t)$. Then by the transport property (3.3), we have

$$\partial_h^\bullet U_h(\cdot,t) = \sum_{j=1}^J \dot q_j(t)\chi_j(\cdot,t) \ \in S_h(t),$$

where $\dot q_j = dq_j/dt$. We often abbreviate $U_h(t) = U_h(\cdot,t)$, $\partial_h^\bullet U_h(t) = \partial_h^\bullet U_h(\cdot,t)$, $\chi_j(t) = \chi_j(\cdot,t)$, etc.

The evolving mass matrix $\mathbf{M}(t)$ and the stiffness matrix $\mathbf{A}(t)$ are defined by

$$\mathbf{M}(t)_{ij} = \int_{\Gamma_h(t)} \chi_i(t)\chi_j(t), \qquad \mathbf{A}(t)_{ij} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\chi_i(t) \cdot \nabla_{\Gamma_h(t)}\chi_j(t)$$

for $i,j = 1,\dots,J$. The mass matrix is symmetric and positive definite. The stiffness matrix is symmetric and only positive semidefinite. Its null-space is spanned by the vector $(1,\dots,1)^{\mathrm{T}}$ because we consider closed surfaces.

With these matrices, the discrete Lagrangian becomes

$$\mathcal{L}_h\big(U_h, \partial_h^\bullet U_h, t\big) = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}(t)\dot{\mathbf{q}} - \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{A}(t)\mathbf{q} =: \mathcal{L}_h(\mathbf{q}, \dot{\mathbf{q}}, t) \tag{3.7}$$

with an obvious doubling of notation. The minimizer of the action integral

$$\mathcal{S}_h[\mathbf{q}] = \int_0^T \mathcal{L}_h(\mathbf{q}(t), \dot{\mathbf{q}}(t), t)\,\mathrm{d}t \tag{3.8}$$

is a solution of the Euler-Lagrange equation

$$\frac{d}{dt}\left(\mathbf{M}(t)\dot{\mathbf{q}}(t)\right) + \mathbf{A}(t)\mathbf{q}(t) = 0. \tag{3.9}$$

By introducing the conjugate *momenta*

$$\mathbf{p}(t) := \frac{\partial \mathcal{L}_h}{\partial \dot{\mathbf{q}}}(\mathbf{q}(t), \dot{\mathbf{q}}(t), t) = \mathbf{M}(t)\dot{\mathbf{q}}(t),$$

we reformulate (3.9) as the Hamiltonian system

$$\dot{\mathbf{p}}(t) = -\mathbf{A}(t)\mathbf{q}(t) \tag{3.10a}$$

$$\dot{\mathbf{q}}(t) = \mathbf{M}(t)^{-1}\mathbf{p}(t) \tag{3.10b}$$

corresponding to the time-dependent Hamiltonian

$$H(\mathbf{q}, \mathbf{p}, t) = \frac{1}{2}\mathbf{p}^{\mathrm{T}}\mathbf{M}(t)^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{A}(t)\mathbf{q}.$$

We work with the norms

$$|\mathbf{q}|^2_{\mathbf{M}(t)} = \langle \mathbf{q} \,|\mathbf{M}(t)|\, \mathbf{q}\rangle = \mathbf{q}^{\mathrm{T}}\mathbf{M}(t)\mathbf{q}, \qquad \mathbf{q} \in \mathbb{R}^J,$$
$$|\mathbf{p}|^2_{\mathbf{M}(t)^{-1}} = \big\langle \mathbf{p} \,\big|\mathbf{M}(t)^{-1}\big|\, \mathbf{p}\big\rangle = \mathbf{p}^{\mathrm{T}}\mathbf{M}(t)^{-1}\mathbf{p}, \qquad \mathbf{p} \in \mathbb{R}^J,$$

and the semi-norm

$$|\mathbf{q}|^2_{\mathbf{A}(t)} = \langle \mathbf{q} \,|\mathbf{A}(t)|\, \mathbf{q}\rangle = \mathbf{q}^{\mathrm{T}}\mathbf{A}(t)\mathbf{q}, \qquad \mathbf{q} \in \mathbb{R}^J.$$

Note that for finite-element functions $U_h(t) = \sum_{j=1}^J q_j(t)\chi_j(t) \in S_h(t)$ with the vector of nodal values $\mathbf{q}(t) = (q_j(t)) \in \mathbb{R}^J$ and $\mathbf{p}(t) = \mathbf{M}(t)\dot{\mathbf{q}}(t)$ we have

$$\begin{aligned} &|\mathbf{q}(t)|_{\mathbf{M}(t)} = \|U_h(t)\|_{L_2(\Gamma_h(t))}, \quad |\mathbf{q}(t)|_{\mathbf{A}(t)} = \|\nabla_{\Gamma_h(t)}U_h(t)\|_{L_2(\Gamma_h(t))} \\ &|\mathbf{p}(t)|_{\mathbf{M}(t)^{-1}} = |\dot{\mathbf{q}}(t)|_{\mathbf{M}(t)} = \|\partial_h^\bullet U_h(t)\|_{L_2(\Gamma_h(t))}. \end{aligned} \tag{3.11}$$

The following result from [11, Lemma 4.1] and [19, Lemma 2.2] provides basic estimates.

Lemma 3.1. *There are constants μ, κ (independent of the meshwidth h) such that*

$$\mathbf{w}^{\mathrm{T}}\big(\mathbf{M}(s) - \mathbf{M}(t)\big)\mathbf{z} \le \mu|s-t|\; |\mathbf{w}|_{\mathbf{M}(t)}\, |\mathbf{z}|_{\mathbf{M}(t)} \tag{3.12}$$

$$\mathbf{w}^{\mathrm{T}}\big(\mathbf{M}^{-1}(s) - \mathbf{M}(t)^{-1}\big)\mathbf{z} \le \mu|s-t|\; |\mathbf{w}|_{\mathbf{M}(t)^{-1}}\, |\mathbf{z}|_{\mathbf{M}(t)^{-1}} \tag{3.13}$$

$$\mathbf{w}^{\mathrm{T}}\big(\mathbf{A}(s) - \mathbf{A}(t)\big)\mathbf{z} \le \kappa|s-t|\; |\mathbf{w}|_{\mathbf{A}(t)}\, |\mathbf{z}|_{\mathbf{A}(t)} \tag{3.14}$$

for all $\mathbf{w}, \mathbf{z} \in \mathbb{R}^J$ and $s, t \in [0, T]$.

Apart from the fact that $\mathbf{M}(t)$, $\mathbf{M}(t)^{-1}$ and $\mathbf{A}(t)$ are symmetric positive semi-definite, the inequalities (3.12)–(3.14) are the only properties of the evolving surface finite elements that will be used in the stability analysis of the full discretization.

4. Variational time discretization

For a given set of discrete time points $0 = t_0 < t_1 < \cdots < t_N = T$, for simplicity assumed equidistant with step size τ, we compute approximations $\mathbf{q}_n, \dot{\mathbf{q}}_n$ to the solution $\mathbf{q}(t_n), \dot{\mathbf{q}}(t_n)$ of the Euler-Lagrange equation (3.9) at time t_n. Here this is done by minimizing an approximate action.

4.1. **Recap: Variational integrators.** We give a brief review of variational integrators which have been studied by Suris [22], Veselov [23] and in a series of papers by Marsden and coauthors. For a comprehensive discussion of variational integrators we refer the reader to Marsden and West [21] and [14, Section VI.6].

We use an approximation

$$\mathcal{L}_{h,\tau}\left(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n\right) \approx \int_{t_n}^{t_{n+1}} \mathcal{L}_h\left(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\right)\, \mathrm{d}t. \tag{4.1}$$

Then the action integral over the whole time interval is approximated by the *discrete action sum*

$$\mathcal{S}_{h,\tau}\big(\{\mathbf{q}_n\}_0^N\big) = \sum_{n=0}^{N-1} \mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n).$$

Computing variations of this discrete action sum with the boundary points $\mathbf{q}_0$ and $\mathbf{q}_N$ held fixed, gives the *discrete Euler-Lagrange equations*

$$D_2\mathcal{L}_{h,\tau}(\mathbf{q}_{n-1}, \mathbf{q}_n, t_{n-1}) + D_1\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n) = 0, \qquad 1 \le n \le N-1, \tag{4.2}$$

where D_1 and D_2 denote the partial derivative with respect to the first and second argument of $\mathcal{L}_{h,\tau}$, respectively. If we take initial conditions $(\mathbf{q}_0, \mathbf{q}_1)$ then the discrete Euler-Lagrange equations (4.2) implicitly define a two-step integrator

$$(\mathbf{q}_{n-1}, \mathbf{q}_n) \to (\mathbf{q}_n, \mathbf{q}_{n+1})$$

that calculates recursively the sequence $\{\mathbf{q}_n\}_0^N$ by solving in every step the discrete Euler-Lagrange equations.

Since we rewrote our problem (3.10) in a Hamiltonian position-momenta form, we want to have an integrator also in this form. We define the *discrete momenta* at every time step n as

$$\mathbf{p}_n := D_2\mathcal{L}_{h,\tau}(\mathbf{q}_{n-1}, \mathbf{q}_n, t_{n-1}) = -D_1\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n),$$

where the second equality holds in view of (4.2). With this definition the variational integrator in the position-momenta form is written as the one-step method

$$\mathbf{p}_n = -D_1\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n) \tag{4.3a}$$

$$\mathbf{p}_{n+1} = D_2\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n). \tag{4.3b}$$

If we take initial conditions $(\mathbf{q}_0, \mathbf{p}_0)$, then we solve the first equation for $\mathbf{q}_1$, then evaluate the second equation to get $\mathbf{p}_1$, and repeat this procedure to get the full sequence $\{\mathbf{q}_n\}_0^N$.

4.2. **The leapfrog or Störmer–Verlet method.** For a given stepsize τ, we choose $\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n)$ by approximating $\mathbf{q}(t)$ as the linear interpolant of $\mathbf{q}_n$ and $\mathbf{q}_{n+1}$ and approximating the first part of the integral (4.1) with the two terms of (3.7) by the midpoint rule and the second part by the trapezoidal rule. This gives

$$\begin{aligned}\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n) = {} & \frac{\tau}{2}\left\langle \dot{\mathbf{q}}_{n+\frac{1}{2}} \,\middle|\, \mathbf{M}_{n+\frac{1}{2}} \,\middle|\, \dot{\mathbf{q}}_{n+\frac{1}{2}} \right\rangle \\ & - \frac{\tau}{4}\Big(\langle \mathbf{q}_n \,|\mathbf{A}_n|\, \mathbf{q}_n \rangle + \langle \mathbf{q}_{n+1} \,|\mathbf{A}_{n+1}|\, \mathbf{q}_{n+1} \rangle \Big)\end{aligned}$$

with $\dot{\mathbf{q}}_{n+\frac{1}{2}} = (\mathbf{q}_{n+1} - \mathbf{q}_n)/\tau,\ \mathbf{A}_n = \mathbf{A}(t_n)$ and $\mathbf{M}_{n+1/2} = \mathbf{M}(t_n + \frac{1}{2}\tau)$.

Then we compute the scheme (4.3),

$$\mathbf{p}_n = -D_1\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n) = \mathbf{M}_{n+\frac12}\dot{\mathbf{q}}_{n+\frac12} + \frac{\tau}{2}\mathbf{A}_n\mathbf{q}_n$$
$$\mathbf{p}_{n+1} = D_2\mathcal{L}_{h,\tau}(\mathbf{q}_n, \mathbf{q}_{n+1}, t_n) = \mathbf{M}_{n+\frac12}\dot{\mathbf{q}}_{n+\frac12} - \frac{\tau}{2}\mathbf{A}_{n+1}\mathbf{q}_{n+1}.$$

Inserting the term of $\dot{\mathbf{q}}_{n+\frac12}$ and solving the first equation for $\mathbf{q}_{n+1}$, we obtain a version of the *leapfrog* or *Störmer–Verlet* method (see, e.g., [13]):

$$\mathbf{q}_{n+1} = \mathbf{q}_n + \tau\mathbf{M}^{-1}_{n+\frac12}\mathbf{p}_n - \frac12\tau^2\mathbf{M}^{-1}_{n+\frac12}\mathbf{A}_n\mathbf{q}_n \tag{4.4a}$$
$$\mathbf{p}_{n+1} = \mathbf{p}_n - \frac{\tau}{2}\mathbf{A}_n\mathbf{q}_n - \frac{\tau}{2}\mathbf{A}_{n+1}\mathbf{q}_{n+1}, \tag{4.4b}$$

or equivalently

$$\mathbf{p}_{n+1/2} = \mathbf{p}_n - \frac{\tau}{2}\mathbf{A}_n\mathbf{q}_n \tag{4.5a}$$
$$\mathbf{q}_{n+1} = \mathbf{q}_n + \tau\mathbf{M}^{-1}_{n+\frac12}\mathbf{p}_{n+1/2} \tag{4.5b}$$
$$\mathbf{p}_{n+1} = \mathbf{p}_{n+1/2} - \frac{\tau}{2}\mathbf{A}_{n+1}\mathbf{q}_{n+1}. \tag{4.5c}$$

The scheme is explicit except for solving a linear system with the mass matrix in each time step.

From the vectors $\mathbf{q}_n = (q_j^n)$ and $\dot{\mathbf{q}}_n = (\dot{q}_j^n) := \mathbf{M}(t_n)^{-1}\mathbf{p}_n$ we obtain the finite element functions on the discrete surface $\Gamma_h(t_n)$

$$U_h^n = \sum_{j=1}^{J} q_j^n \chi_j(t_n), \quad \partial_h^\bullet U_h^n = \sum_{j=1}^{J} \dot{q}_j^n \chi_j(t_n) \tag{4.6}$$

as approximations to $u(t_n)$ and $\partial^\bullet u(t_n)$, respectively.

5. Stability analysis of the full discretization

5.1. **Defects and errors.** Let $\widetilde{\mathbf{q}}_n$ and $\widetilde{\mathbf{p}}_n$ be reference values that we want to compare with $\mathbf{q}_n$ and $\mathbf{p}_n$, respectively (e.g., $\widetilde{\mathbf{q}}_n = \mathbf{q}(t_n)$ and $\widetilde{\mathbf{p}}_n = \mathbf{p}(t_n)$). Inserted into (4.4) they yield defects $\mathbf{d}^q_{n+1}$ and $\mathbf{d}^p_{n+1}$ in

$$\widetilde{\mathbf{q}}_{n+1} = \widetilde{\mathbf{q}}_n + \tau\mathbf{M}^{-1}_{n+\frac12}\widetilde{\mathbf{p}}_n - \frac12\tau^2\mathbf{M}^{-1}_{n+\frac12}\mathbf{A}_n\widetilde{\mathbf{q}}_n + \mathbf{d}^q_{n+1} \tag{5.1a}$$
$$\widetilde{\mathbf{p}}_{n+1} = \widetilde{\mathbf{p}}_n - \frac{\tau}{2}\mathbf{A}_n\widetilde{\mathbf{q}}_n - \frac{\tau}{2}\mathbf{A}_{n+1}\widetilde{\mathbf{q}}_{n+1} + \mathbf{d}^p_{n+1}. \tag{5.1b}$$

For the errors we use the notation

$$\mathbf{e}^q_n = \mathbf{q}_n - \widetilde{\mathbf{q}}_n \tag{5.2a}$$
$$\mathbf{e}^p_n = \mathbf{p}_n - \widetilde{\mathbf{p}}_n \tag{5.2b}$$

and subtract to get the *error equation*

$$\mathbf{e}^q_{n+1} = \mathbf{e}^q_n + \tau\mathbf{M}^{-1}_{n+\frac12}\mathbf{e}^p_n - \frac12\tau^2\mathbf{M}^{-1}_{n+\frac12}\mathbf{A}_n\mathbf{e}^q_n - \mathbf{d}^q_{n+1} \tag{5.3a}$$
$$\mathbf{e}^p_{n+1} = \mathbf{e}^p_n - \frac{\tau}{2}\mathbf{A}_n\mathbf{e}^q_n - \frac{\tau}{2}\mathbf{A}_{n+1}\mathbf{e}^q_{n+1} - \mathbf{d}^p_{n+1}. \tag{5.3b}$$

5.2. **The CFL condition.** From now on we assume that the step size τ fulfills the following restriction:

$$\frac{1}{4}\tau^2\rho\left(\mathbf{M}(t)^{-1/2}\mathbf{A}(t)\mathbf{M}(t)^{-1/2}\right) \le 1-\theta \tag{5.4}$$

for all $0 \le t \le T$ and for a fixed $0 < \theta < 1$, where $\rho(\cdot)$ denotes the spectral radius. For a quasi-uniform triangulation we have $\rho\left(\mathbf{M}(t)^{-1/2}\mathbf{A}(t)\mathbf{M}(t)^{-1/2}\right) \sim h^{-2}$, so that we have a time step restriction $\tau \le ch$.

Under the CFL condition (5.4), the symmetric matrix

$$\widehat{\mathbf{A}}(t) = \mathbf{A}(t) - \frac{1}{4}\tau^2\mathbf{A}(t)\mathbf{M}(t)^{-1}\mathbf{A}(t) \quad \text{is positive semidefinite,} \tag{5.5}$$

and there exists C_θ such that for every $\mathbf{e}^q \in \mathbb{R}^J$ we have

$$\langle \mathbf{e}^q | \widehat{\mathbf{A}}(t) | \mathbf{e}^q \rangle \le \langle \mathbf{e}^q | \mathbf{A}(t) | \mathbf{e}^q \rangle \le C_\theta \langle \mathbf{e}^q | \widehat{\mathbf{A}}(t) | \mathbf{e}^q \rangle. \tag{5.6}$$

5.3. **Stability estimate.** We use a time-dependent modified energy norm on $\mathbb{R}^{2J}$: for $\mathbf{e} = (\mathbf{e}^q, \mathbf{e}^p) \in \mathbb{R}^{2J}$,

$$\|\mathbf{e}\|_t^2 = \langle \mathbf{e}^q | \mathbf{M}(t) + \widehat{\mathbf{A}}(t) | \mathbf{e}^q \rangle + \langle \mathbf{e}^p | \mathbf{M}(t)^{-1} | \mathbf{e}^p \rangle. \tag{5.7}$$

We denote by $\mathbf{e}_n = (\mathbf{e}_n^q, \mathbf{e}_n^p)$ the error vector at time t_n and by $\mathbf{d}_n = (\mathbf{d}_n^q, \mathbf{d}_n^p)$ the defect vector in (5.3). With this notation we prove the following stability result.

Lemma 5.1. *There exists $\tau_0 > 0$ (depending only on μ and κ of Lemma 3.1 and on θ of* (5.4)*) such that for step sizes $\tau \le \tau_0$ satisfying the CFL condition* (5.4)*, the error is bounded, for $t_n = n\tau \le T$, by*

$$\|\mathbf{e}_n\|_{t_n} \le C\Big(\|\mathbf{e}_0\|_{t_0} + \sum_{k=1}^{n} \|\mathbf{d}_k\|_{t_k}\Big).$$

The constant C is independent of h, τ, and n subject to the stated conditions (but depends on μ, κ, θ, and T).

Proof. We prove the lemma in three steps.
(*a*) *Local error:* Here we analyze the error after *one* step, starting with $\mathbf{e}_n = 0$. Thus the error equation (5.3) simply reads

$$\mathbf{e}_{n+1}^q = -\mathbf{d}_{n+1}^q \tag{5.8a}$$

$$\mathbf{e}_{n+1}^p = -\frac{\tau}{2}\mathbf{A}_{n+1}\mathbf{e}_{n+1}^q - \mathbf{d}_{n+1}^p. \tag{5.8b}$$

Using the semi-norm equivalence (5.6) for the first equation of (5.8) yields

$$\begin{aligned}\left\langle \mathbf{e}_{n+1}^q \left| \mathbf{A}_{n+1} + \mathbf{M}_{n+1} \right| \mathbf{e}_{n+1}^q \right\rangle &= \left\langle \mathbf{d}_{n+1}^q \left| \mathbf{M}_{n+1} + \mathbf{A}_{n+1} \right| \mathbf{d}_{n+1}^q \right\rangle \\ &\le C_\theta \left\langle \mathbf{d}_{n+1}^q \left| \mathbf{M}_{n+1} + \widehat{\mathbf{A}}_{n+1} \right| \mathbf{d}_{n+1}^q \right\rangle. \end{aligned} \tag{5.9}$$

Furthermore we get by the second equation of (5.8)

$$\left\langle \mathbf{e}_{n+1}^p + \mathbf{d}_{n+1}^p \left| \mathbf{M}_{n+1}^{-1} \right| \mathbf{e}_{n+1}^p + \mathbf{d}_{n+1}^p \right\rangle = \frac{1}{4}\tau^2 \left\langle \mathbf{e}_{n+1}^q \left| \mathbf{A}_{n+1}\mathbf{M}_{n+1}^{-1}\mathbf{A}_{n+1} \right| \mathbf{e}_{n+1}^q \right\rangle.$$

Thus we obtain

$$\begin{aligned}\left\langle \mathbf{e}_{n+1}^p \left| \mathbf{M}_{n+1}^{-1} \right| \mathbf{e}_{n+1}^p \right\rangle = \frac{1}{4}\tau^2 &\left\langle \mathbf{e}_{n+1}^q \left| \mathbf{A}_{n+1}\mathbf{M}_{n+1}^{-1}\mathbf{A}_{n+1} \right| \mathbf{e}_{n+1}^q \right\rangle \\ &- 2\left\langle \mathbf{e}_{n+1}^p \left| \mathbf{M}_{n+1}^{-1} \right| \mathbf{d}_{n+1}^p \right\rangle - \left\langle \mathbf{d}_{n+1}^p \left| \mathbf{M}_{n+1}^{-1} \right| \mathbf{d}_{n+1}^p \right\rangle.\end{aligned}$$

We estimate the second term on the right-hand side by the Cauchy–Schwarz inequality and Young's inequality to obtain

$$\frac{1}{2}\left\langle \mathbf{e}^p_{n+1} \left|\mathbf{M}^{-1}_{n+1}\right| \mathbf{e}^p_{n+1}\right\rangle - \frac{1}{4}\tau^2 \left\langle \mathbf{e}^q_{n+1} \left|\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+1}\mathbf{A}_{n+1}\right| \mathbf{e}^q_{n+1}\right\rangle \le 2\left\langle \mathbf{d}^p_{n+1} \left|\mathbf{M}^{-1}_{n+1}\right| \mathbf{d}^p_{n+1}\right\rangle . \quad (5.10)$$

Therefore, adding (5.9) to (5.10) yields

$$\|\mathbf{e}_{n+1}\|_{t_{n+1}} \le C_\theta \ \ \|\mathbf{d}_{n+1}\|_{t_{n+1}} . \quad (5.11)$$

(*b*) *Error propagation:* We now consider one step of the error equations without defects and estimate $\|\mathbf{e}_{n+1}\|_{t_{n+1}}$ in terms of $\|\mathbf{e}_n\|_{t_n}$:

$$\mathbf{e}^q_{n+1} = \mathbf{e}^q_n + \tau\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{e}^p_n - \frac{1}{2}\tau^2\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_n\mathbf{e}^q_n \quad (5.12a)$$

$$\mathbf{e}^p_{n+1} = \mathbf{e}^p_n - \frac{\tau}{2}\mathbf{A}_n\mathbf{e}^q_n - \frac{\tau}{2}\mathbf{A}_{n+1}\mathbf{e}^q_{n+1}. \quad (5.12b)$$

We start by direct computation taking the squared $\mathbf{A}$-seminorm of $\mathbf{e}^q_{n+1}$ at time t_{n+1} and the squared $\mathbf{M}^{-1}$-norm of $\mathbf{e}^p_{n+1}$ at time $t_{n+\frac{1}{2}}$ to find

$$\begin{aligned}
&\left\langle \mathbf{e}^q_{n+1} \left|\mathbf{A}_{n+1}\right| \mathbf{e}^q_{n+1}\right\rangle \\
&\quad = \left\langle \mathbf{e}^q_n \left|\mathbf{A}_{n+1}\right| \mathbf{e}^q_n\right\rangle + 2\tau\left\langle \mathbf{e}^q_n \left|\mathbf{A}_{n+1}\right| \mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{e}^p_n\right\rangle - \tau^2\left\langle \mathbf{e}^q_n \left|\mathbf{A}_{n+1}\right| \mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_n\mathbf{e}^q_n\right\rangle \\
&\quad + \tau^2\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_n\right\rangle - \tau^3\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_n\mathbf{e}^q_n\right\rangle \\
&\quad + \frac{1}{4}\tau^4\left\langle \mathbf{e}^q_n \left|\mathbf{A}_n\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_n\right| \mathbf{e}^q_n\right\rangle . \\
&\left\langle \mathbf{e}^p_{n+1} \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_{n+1}\right\rangle \\
&\quad = \left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_n\right\rangle - \tau\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_n\mathbf{e}^q_n\right\rangle - \tau\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_{n+1}\mathbf{e}^q_{n+1}\right\rangle \\
&\quad + \frac{1}{4}\tau^2\left\langle \mathbf{e}^q_n \left|\mathbf{A}_n\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_n\right| \mathbf{e}^q_n\right\rangle + \frac{1}{2}\tau^2\left\langle \mathbf{A}_n\mathbf{e}^q_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_{n+1}\mathbf{e}^q_{n+1}\right\rangle \\
&\quad + \frac{1}{4}\tau^2\left\langle \mathbf{e}^q_{n+1} \left|\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\right| \mathbf{e}^q_{n+1}\right\rangle .
\end{aligned}$$

Expressing $\mathbf{e}^q_{n+1}$ by (5.12a), it follows that

$$\begin{aligned}
&\left\langle \mathbf{e}^p_{n+1} \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_{n+1}\right\rangle \\
&= \left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_n\right\rangle - \tau\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_n\mathbf{e}^q_n\right\rangle - \tau\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_{n+1}\mathbf{e}^q_n\right\rangle \\
&- \tau^2\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_n\right\rangle + \frac{1}{2}\tau^3\left\langle \mathbf{e}^p_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_n\mathbf{e}^q_n\right\rangle \\
&+ \frac{1}{4}\tau^2\left\langle \mathbf{e}^q_n \left|\mathbf{A}_n\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_n\right| \mathbf{e}^q_n\right\rangle + \frac{1}{2}\tau^2\left\langle \mathbf{A}_n\mathbf{e}^q_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{A}_{n+1}\mathbf{e}^q_n\right\rangle \\
&+ \frac{1}{2}\tau^3\left\langle \mathbf{A}_n\mathbf{e}^q_n \left|\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\right| \mathbf{e}^p_n\right\rangle - \frac{1}{4}\tau^4\left\langle \mathbf{e}^q_n \left|\mathbf{A}_n\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_n\right| \mathbf{e}^q_n\right\rangle \\
&+ \frac{1}{4}\tau^2\left\langle \mathbf{e}^q_{n+1} \left|\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac{1}{2}}\mathbf{A}_{n+1}\right| \mathbf{e}^q_{n+1}\right\rangle .
\end{aligned}$$

Adding both expressions leads to

$$\begin{aligned}
&\left\langle \mathbf{e}^q_{n+1} \left| \mathbf{A}_{n+1} - \frac{1}{4}\tau^2 \mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac12}\mathbf{A}_{n+1} \right| \mathbf{e}^q_{n+1} \right\rangle + \left\langle \mathbf{e}^p_{n+1} \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{e}^p_{n+1} \right\rangle \\
&\quad = \left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} \right| \mathbf{e}^q_n \right\rangle + \left\langle \mathbf{e}^p_n \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{e}^p_n \right\rangle - \frac12 \tau^2 \left\langle \mathbf{A}_{n+1}\mathbf{e}^q_n \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{A}_n \mathbf{e}^q_n \right\rangle \\
&\qquad + \frac14 \tau^2 \left\langle \mathbf{A}_n\mathbf{e}^q_n \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{A}_n \mathbf{e}^q_n \right\rangle + \left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} - \mathbf{A}_n \right| \tau \mathbf{M}^{-1}_{n+\frac12} \mathbf{e}^p_n \right\rangle . \qquad (5.13)
\end{aligned}$$

We estimate the terms on the right-hand side of (5.13) separately, starting by the first and second term, then the third and the fourth together, and in the end the last term.

• In the first and second term on the right hand side of (5.13) we write $\mathbf{A}_{n+1} = (\mathbf{A}_{n+1} - \mathbf{A}_n) + \mathbf{A}_n$ and $\mathbf{M}^{-1}_{n+\frac12} = \left(\mathbf{M}^{-1}_{n+\frac12} - \mathbf{M}^{-1}_n\right) + \mathbf{M}^{-1}_n$ respectively. Then conditions (3.14) and (3.13) yield

$$\begin{aligned}
\left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} \right| \mathbf{e}^q_n \right\rangle &= \left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} - \mathbf{A}_n \right| \mathbf{e}^q_n \right\rangle + \left\langle \mathbf{e}^q_n \left| \mathbf{A}_n \right| \mathbf{e}^q_n \right\rangle \\
&\le (1+\kappa\tau) \left\langle \mathbf{e}^q_n \left| \mathbf{A}_n \right| \mathbf{e}^q_n \right\rangle \qquad (5.14) \\
\left\langle \mathbf{e}^p_n \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{e}^p_n \right\rangle &\le (1+\mu\tau) \left\langle \mathbf{e}^p_n \left| \mathbf{M}^{-1}_n \right| \mathbf{e}^p_n \right\rangle . \qquad (5.15)
\end{aligned}$$

• In the third term of (5.13) we also write $\mathbf{A}_{n+1} = (\mathbf{A}_{n+1} - \mathbf{A}_n) + \mathbf{A}_n$ and add it to the fourth term on the right side of (5.13) to get

$$\begin{aligned}
&-\frac12 \tau^2 \left\langle \mathbf{A}_{n+1}\mathbf{e}^q_n \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{A}_n \mathbf{e}^q_n \right\rangle + \frac14 \tau^2 \left\langle \mathbf{A}_n\mathbf{e}^q_n \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{A}_n \mathbf{e}^q_n \right\rangle \\
&= \left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} - \mathbf{A}_n \right| - \frac12 \tau^2 \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \mathbf{e}^q_n \right\rangle - \frac14 \tau^2 \left\langle \mathbf{e}^q_n \left| \mathbf{A}_n \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \right| \mathbf{e}^q_n \right\rangle . \qquad (5.16)
\end{aligned}$$

We start by the first term on the right hand side and use condition (3.14) and Young's inequality to get

$$\begin{aligned}
\left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} - \mathbf{A}_n \right| - \frac12 \tau^2 \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \mathbf{e}^q_n \right\rangle &\le C\tau \, |\mathbf{e}^q_n|_{\mathbf{A}_n} \left| \frac12 \tau^2 \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \mathbf{e}^q_n \right|_{\mathbf{A}_n} \\
&\le C\tau \left(|\mathbf{e}^q_n|^2_{\mathbf{A}_n} + \frac14 \left| \frac12 \tau^2 \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \mathbf{e}^q_n \right|^2_{\mathbf{A}_n} \right).
\end{aligned}$$

Using the CFL condition (5.4), similar arguments to those used for (5.14) and (5.15), and (5.6) yield that this is further bounded by

$$\begin{aligned}
\left\langle \mathbf{e}^q_n \left| \mathbf{A}_{n+1} - \mathbf{A}_n \right| - \frac12 \tau^2 \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \mathbf{e}^q_n \right\rangle &\le C\tau \left(|\mathbf{e}^q_n|^2_{\mathbf{A}_n} + \frac14 \left| \frac12 \tau^2 \mathbf{M}^{-1}_{n+\frac12} \mathbf{A}_n \mathbf{e}^q_n \right|^2_{\mathbf{A}_{n+\frac12}} \right) \\
&\le C\tau \left(|\mathbf{e}^q_n|^2_{\mathbf{A}_n} + \frac14 \tau^2 \left| \mathbf{A}_n \mathbf{e}^q_n \right|^2_{\mathbf{M}^{-1}_{n+\frac12}} \right) \\
&\le C\tau \left(|\mathbf{e}^q_n|^2_{\mathbf{A}_n} + \frac14 \tau^2 \left| \mathbf{A}_n \mathbf{e}^q_n \right|^2_{\mathbf{M}^{-1}_n} \right) \\
&\le C_\theta \, \tau \left\langle \mathbf{e}^q_n \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}^q_n \right\rangle .
\end{aligned}$$

For the last term of (5.16) we write $\mathbf{M}_{n+\frac12}^{-1} = \left(\mathbf{M}_{n+\frac12}^{-1} - \mathbf{M}_n^{-1}\right) + \mathbf{M}_n^{-1}$ and use condition (3.13), the CFL condition (5.4) and (5.6) to get

$$
\begin{aligned}
&-\frac14\tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n\mathbf{M}_{n+\frac12}^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle \\
&\qquad = -\frac14\tau^2\left\langle \mathbf{A}_n\mathbf{e}_n^q \left| \mathbf{M}_{n+\frac12}^{-1} - \mathbf{M}_n^{-1} \right| \mathbf{A}_n\mathbf{e}_n^q\right\rangle - \frac14\tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n\mathbf{M}_n^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle \\
&\qquad \le C\tau \left\langle \frac14\tau^2\mathbf{M}_n^{-1}\mathbf{A}_n\mathbf{e}_n^q \left| \mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle - \frac14\tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n\mathbf{M}_n^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle \\
&\qquad \le C_\theta\ \tau\left\langle \mathbf{e}_n^q \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}_n^q\right\rangle - \frac14\tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n\mathbf{M}_n^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle .
\end{aligned}
$$

Combining the above bounds yields

$$
\begin{aligned}
&-\frac12\tau^2\left\langle \mathbf{A}_{n+1}\mathbf{e}_n^q \left| \mathbf{M}_{n+\frac12}^{-1} \right| \mathbf{A}_n\mathbf{e}_n^q\right\rangle + \frac14\tau^2\left\langle \mathbf{A}_n\mathbf{e}_n^q \left| \mathbf{M}_{n+\frac12}^{-1} \right| \mathbf{A}_n\mathbf{e}_n^q\right\rangle \\
&\qquad\qquad \le -\frac14\tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n\mathbf{M}_n^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle + C_\theta\ \tau\left\langle \mathbf{e}_n^q \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}_n^q\right\rangle . \qquad (5.17)
\end{aligned}
$$

• For the last term on the right-hand side of (5.13), we use condition (3.14), Young's inequality, the CFL condition (5.4) to estimate

$$
\begin{aligned}
\left\langle \mathbf{e}_n^q \left| \mathbf{A}_{n+1} - \mathbf{A}_n \right| \tau\mathbf{M}_{n+\frac12}^{-1}\mathbf{e}_n^p\right\rangle &\le C\tau\left(\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle + \frac14\left\langle \tau\mathbf{M}_{n+\frac12}^{-1}\mathbf{e}_n^p \left| \mathbf{A}_n \right| \tau\mathbf{M}_{n+\frac12}^{-1}\mathbf{e}_n^p\right\rangle\right) \\
&\le C\tau\left(\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle + \left\langle \mathbf{e}_n^p \left| \mathbf{M}_{n+\frac12}^{-1} \right| \mathbf{e}_n^p\right\rangle\right) \\
&\le C_\theta\ \tau\left(\left\langle \mathbf{e}_n^q \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}_n^q\right\rangle + \left\langle \mathbf{e}_n^p \left| \mathbf{M}_n^{-1} \right| \mathbf{e}_n^p\right\rangle\right). \qquad (5.18)
\end{aligned}
$$

Now we take the squared $\mathbf{M}$-norm of $\mathbf{e}_{n+1}^q$ at time $t_{n+\frac12}$ to find

$$
\begin{aligned}
&\left\langle \mathbf{e}_{n+1}^q \left| \mathbf{M}_{n+\frac12} \right| \mathbf{e}_{n+1}^q\right\rangle \\
&\quad = \left\langle \mathbf{e}_n^q \left| \mathbf{M}_{n+\frac12} \right| \mathbf{e}_n^q\right\rangle + 2\tau\left\langle \mathbf{e}_n^q | \mathbf{e}_n^p\right\rangle - \tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle + \tau^2\left\langle \mathbf{e}_n^p \left| \mathbf{M}_{n+\frac12}^{-1} \right| \mathbf{e}_n^p\right\rangle \\
&\qquad\qquad - \tau^3\left\langle \mathbf{e}_n^p \left| \mathbf{M}_{n+\frac12}^{-1} \right| \mathbf{A}_n\mathbf{e}_n^q\right\rangle + \frac14\tau^4\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n\mathbf{M}_{n+\frac12}^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle .
\end{aligned}
$$

The Cauchy–Schwarz inequality, the CFL condition (5.4) and the bound (3.12) yield

$$
\begin{aligned}
\left\langle \mathbf{e}_n^q \left| \mathbf{M}_{n+\frac12} \right| \mathbf{e}_n^q\right\rangle &\le (1+\mu\tau)\left\langle \mathbf{e}_n^q \left| \mathbf{M}_n \right| \mathbf{e}_n^q\right\rangle \\
2\tau\left\langle \mathbf{e}_n^q | \mathbf{e}_n^p\right\rangle &\le \tau\left(|\mathbf{e}_n^q|_{\mathbf{M}_n}^2 + |\mathbf{e}_n^p|_{\mathbf{M}_n^{-1}}^2\right) \\
-\tau^2\left\langle \mathbf{e}_n^q \left| \mathbf{A}_n - \frac14\tau^2\mathbf{A}_n\mathbf{M}_{n+\frac12}^{-1}\mathbf{A}_n \right| \mathbf{e}_n^q\right\rangle &\le \left(-\tau^2 + C_\theta\tau^3\right)\left\langle \mathbf{e}_n^q \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}_n^q\right\rangle \\
\tau^3\left\langle \mathbf{e}_n^p \left| \mathbf{M}_{n+\frac12}^{-1} \right| \mathbf{A}_n\mathbf{e}_n^q\right\rangle &\le \tau^3 |\mathbf{e}_n^p|_{\mathbf{M}_{n+\frac12}^{-1}}\, |\mathbf{A}_n\mathbf{e}_n^q|_{\mathbf{M}_{n+\frac12}^{-1}} \\
&\le C\tau\left(|\mathbf{e}_n^p|_{\mathbf{M}_n^{-1}}^2 + C_\theta\left\langle \mathbf{e}_n^q \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}_n^q\right\rangle\right).
\end{aligned}
$$

Thus we have

$$
\left\langle \mathbf{e}_{n+1}^q \left| \mathbf{M}_{n+\frac12} \right| \mathbf{e}_{n+1}^q\right\rangle \le (1+C\tau)\,|\mathbf{e}_n^q|_{\mathbf{M}_n}^2 + C\tau|\mathbf{e}_n^p|_{\mathbf{M}_n^{-1}}^2 + C_\theta\tau\left\langle \mathbf{e}_n^q \left| \widehat{\mathbf{A}}_n \right| \mathbf{e}_n^q\right\rangle . \qquad (5.19)
$$

Combining (5.13)-(5.18) and the above bound (5.19) yields

$$\left\langle \mathbf{e}^q_{n+1} \left| \mathbf{M}_{n+\frac12} + \mathbf{A}_{n+1} - \frac{\tau^2}{4}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac12}\mathbf{A}_{n+1} \right| \mathbf{e}^q_{n+1} \right\rangle + \left\langle \mathbf{e}^p_{n+1} \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{e}^p_{n+1} \right\rangle \le (1 + C_\theta \tau) \left\| \mathbf{e}_n \right\|^2_{t_n}. \quad (5.20)$$

This is almost the desired estimate, except that we have here $\mathbf{M}_{n+1/2}$ instead of $\mathbf{M}_{n+1}$. It remains to show that we have a bound of the same type also with $\mathbf{M}_{n+1}$. Since by (3.12) and (3.13),

$$\left\langle \mathbf{e}^q_{n+1} \left| \mathbf{M}_{n+1} - \mathbf{M}_{n+\frac12} \right| \mathbf{e}^q_{n+1} \right\rangle \le \mu\tau \left\langle \mathbf{e}^q_{n+1} \left| \mathbf{M}_{n+1/2} \right| \mathbf{e}^q_{n+1} \right\rangle$$
$$\left\langle \mathbf{e}^p_{n+1} \left| \mathbf{M}^{-1}_{n+1} - \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{e}^p_{n+1} \right\rangle \le \mu\tau \left\langle \mathbf{e}^p_{n+1} \left| \mathbf{M}^{-1}_{n+1/2} \right| \mathbf{e}^p_{n+1} \right\rangle$$

and by (3.13) and (5.5),

$$\left\langle \mathbf{e}^q_{n+1} \left| \frac{\tau^2}{4}\mathbf{A}_{n+1}(\mathbf{M}^{-1}_{n+1} - \mathbf{M}^{-1}_{n+\frac12})\mathbf{A}_{n+1} \right| \mathbf{e}^q_{n+1} \right\rangle \le \mu\tau \left\langle \mathbf{e}^q_{n+1} \left| \frac{\tau^2}{4}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+1}\mathbf{A}_{n+1} \right| \mathbf{e}^q_{n+1} \right\rangle \le \mu\tau \left\langle \mathbf{e}^q_{n+1} \left| \mathbf{A}_{n+1} \right| \mathbf{e}^q_{n+1} \right\rangle,$$

we obtain

$$\left\| \mathbf{e}_{n+1} \right\|^2_{t_{n+1}} \le (1+\mu\tau)\Big(\left\langle \mathbf{e}^q_{n+1} \left| \mathbf{M}_{n+\frac12} + \mathbf{A}_{n+1} - \frac{\tau^2}{4}\mathbf{A}_{n+1}\mathbf{M}^{-1}_{n+\frac12}\mathbf{A}_{n+1} \right| \mathbf{e}^q_{n+1} \right\rangle + \left\langle \mathbf{e}^p_{n+1} \left| \mathbf{M}^{-1}_{n+\frac12} \right| \mathbf{e}^p_{n+1} \right\rangle\Big),$$

which together with (5.20) finally yields

$$\left\| \mathbf{e}_{n+1} \right\|_{t_{n+1}} \le (1 + C\tau) \left\| \mathbf{e}_n \right\|_{t_n}.$$

(*c*) *Error accumulation:* A standard application of Lady Windermere's fan (see [15, 16]) completes the proof. □

6. Bounding the error in terms of the semidiscrete residual

We compare the numerical solution U_h^n and $\partial_h^\bullet U_h^n$ given in (4.6), which are finite element functions on the discretized surface $\Gamma_h(t_n)$, with a near-identity mapping of the PDE solution $u(t)$ to the finite element space $S_h(t)$ at $t = t_n$:

$$P_h(t)u(t) = \sum_{j=1}^J \widetilde{q}_j(t)\chi_j(t), \qquad \partial_h^\bullet(P_h u)(t) = \sum_{j=1}^J \dot{\widetilde{q}}_j(t)\chi_j(t).$$

The map $P_h(t) : H^1(\Gamma(t)) \to S_h(t) \subset H^1(\Gamma_h(t))$ is arbitrary in this section and will later be chosen as the Ritz map of Section 8. The finite element function $P_h(t)u(t)$ on $\Gamma_h(t)$ has a residual $R_h(t) = \sum_{j=1}^J r_j(t)\chi_j(t) \in S_h(t)$ when inserted into Equation (3.6) of the spatial semidiscretization: for all temporally smooth ϕ_h with $\phi_h(\cdot,t) \in S_h(t)$ and for all t,

$$\frac{d}{dt}\int_{\Gamma_h} \partial_h^\bullet(P_h u)\,\phi_h + \int_{\Gamma_h} \nabla_{\Gamma_h}(P_h u)\cdot\nabla_{\Gamma_h}\phi_h = \int_{\Gamma_h} \partial_h^\bullet(P_h u)\,\partial_h^\bullet\phi_h + \int_{\Gamma_h} R_h\phi_h, \quad (6.1)$$

or equivalently, on inserting the nodal vector $\widetilde{\mathbf{q}}(t) = (\widetilde{q}_j(t))$ of $P_h(t)u(t)$ into (3.9),

$$\frac{d}{dt}\Big(\mathbf{M}(t)\dot{\widetilde{\mathbf{q}}}(t)\Big) + \mathbf{A}(t)\widetilde{\mathbf{q}}(t) = \mathbf{M}(t)\mathbf{r}(t), \tag{6.2}$$

where $\mathbf{r}(t) = (r_j(t)) \in \mathbb{R}^J$.

Using the stability bound of the previous section and translating it back into a function-space framework, we will show the following result which reduces the problem of estimating the fully discrete error to estimating the semidiscrete residual.

Theorem 6.1. *Under the CFL condition* (5.4) *and suitable regularity conditions on the exact solution u of the wave equation* (2.4)*, the errors $E_h^n = U_h^n - P_h(t_n)u(t_n)$ and $\partial_h^\bullet E_h^n = \partial_h^\bullet U_h^n - \partial_h^\bullet(P_hu)(t_n)$ are bounded for sufficiently small $h \le h_0$ and for $t_n = n\tau \le T$ by*

$$\begin{aligned}
&\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \|\nabla_{\Gamma_h}E_h^n\|_{L^2(\Gamma_h(t_n))} + \|\partial_h^\bullet E_h^n\|_{L^2(\Gamma_h(t_n))} \\
&\quad \le C\Big(\big\|E_h^0\big\|_{L^2(\Gamma_h(t_0))} + \big\|\nabla_{\Gamma_h}E_h^0\big\|_{L^2(\Gamma_h(t_0))} + \big\|\partial_h^\bullet E_h^0\big\|_{L^2(\Gamma_h(t_0))}\Big) \\
&\qquad\qquad + C\beta_h\tau^2 + C\tau\sum_{k=0}^{n}\|R_h(t_k)\|_{L^2(\Gamma_h(t_k))}.
\end{aligned}$$

Here C is independent of h (but depends on T and θ), and

$$\beta_h = \int_0^T \sum_{\ell=1}^{4}\Big(\|(P_hu)^{(\ell)}(t)\|_{L^2(\Gamma_h(t))} + \|\nabla_{\Gamma_h}(P_hu)^{(\ell)}(t)\|_{L^2(\Gamma_h(t))}\Big)\,\mathrm{d}t,$$

where the superscript (ℓ) denotes the ℓth discrete material derivative.

Proof. We reformulate (6.2) as

$$\dot{\widetilde{\mathbf{p}}}(t) = -\mathbf{A}(t)\widetilde{\mathbf{q}}(t) + \mathbf{M}(t)\mathbf{r}(t) \tag{6.3a}$$

$$\dot{\widetilde{\mathbf{q}}}(t) = \mathbf{M}(t)^{-1}\widetilde{\mathbf{p}}(t). \tag{6.3b}$$

Considering the errors

$$\begin{aligned}
\mathbf{e}_n^q &= \mathbf{q}_n - \widetilde{\mathbf{q}}(t_n) \\
\mathbf{e}_n^p &= \mathbf{p}_n - \widetilde{\mathbf{p}}(t_n),
\end{aligned}$$

the defects appearing in the error equation (5.3) satisfy

$$\begin{aligned}
\mathbf{d}_{n+1}^q &= \widetilde{\mathbf{q}}\,(t_{n+1}) - \widetilde{\mathbf{q}}\,(t_n) - \tau\mathbf{M}_{n+\frac12}^{-1}\widetilde{\mathbf{p}}\,(t_n) + \frac12\tau^2\mathbf{M}_{n+\frac12}^{-1}\mathbf{A}_n\widetilde{\mathbf{q}}\,(t_n) \\
\mathbf{d}_{n+1}^p &= \widetilde{\mathbf{p}}\,(t_{n+1}) - \widetilde{\mathbf{p}}\,(t_n) + \frac{\tau}{2}\mathbf{A}_n\widetilde{\mathbf{q}}\,(t_n) + \frac{\tau}{2}\mathbf{A}_{n+1}\widetilde{\mathbf{q}}\,(t_{n+1}).
\end{aligned}$$

By (6.3) and Taylor expansion, we obtain

$$\begin{aligned}
\mathbf{d}_{n+1}^q = \frac16\tau^2\int_{t_n}^{t_{n+1}}\dddot{\widetilde{\mathbf{q}}}(t)\,\mathrm{d}t + \frac14\tau^2\int_{t_n}^{t_{n+\frac12}}\dddot{\widetilde{\mathbf{q}}}(t)\,\mathrm{d}t + \frac18\tau^3\mathbf{M}_{n+\frac12}^{-1}\ddot{\widetilde{\mathbf{p}}}(t_{n+\frac12}) \\
+ \frac{5}{24}\tau^3\mathbf{M}_{n+\frac12}^{-1}\int_{t_n}^{t_{n+\frac12}}\dddot{\widetilde{\mathbf{p}}}(t)\,\mathrm{d}t + \frac12\tau^2\mathbf{M}_{n+\frac12}^{-1}\mathbf{M}_n\mathbf{r}_n
\end{aligned} \tag{6.4}$$

$$\mathbf{d}_{n+1}^p = -\frac{1}{12}\tau^2\int_{t_n}^{t_{n+1}}\dddot{\widetilde{\mathbf{p}}}(t)\,\mathrm{d}t + \frac{\tau}{2}\mathbf{M}_n\mathbf{r}_n + \frac{\tau}{2}\mathbf{M}_{n+1}\mathbf{r}_{n+1}. \tag{6.5}$$

Using Lemma 3.1 and the norm identity (3.11) we first have

$$
\begin{aligned}
&|\dddot{\widetilde{\mathbf{q}}}(t)|_{\mathbf{M}(s)} + |\dddot{\widetilde{\mathbf{q}}}(t)|_{\mathbf{A}(s)} \\
&\qquad \le \sqrt{2}\left(\|(P_h u)^{(3)}(t)\|_{L^2(\Gamma_h(t))} + \|\nabla_{\Gamma_h}(P_h u)^{(3)}(t)\|_{L^2(\Gamma_h(t))}\right)
\end{aligned}
\tag{6.6}
$$

provided that $\mu|t-s| \le 1$ and $\kappa|t-s| \le 1$. Now by Lemma 3.1 and the CFL condition (5.4) we estimate for $t \in [t_n, t_{n+1}]$

$$
\begin{aligned}
|\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}_{n+1}} &\le \sqrt{2}|\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}_{n+\frac12}} \le 2|\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}(t)^{-1}} \\
\tau^3|\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{A}_{n+1}} &\le C_\theta\tau^2|\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}_{n+\frac12}^{-1}} \le 2C_\theta\tau^2|\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}(t)^{-1}}.
\end{aligned}
$$

Therefore in view of (6.3) we find for sufficiently small τ:

$$
\tau^3\left(|\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}_{n+1}} + |\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{A}_{n+1}}\right) \le C\tau^2|(\mathbf{M}\widetilde{\mathbf{q}})^{(4)}(t)|_{\mathbf{M}(t)^{-1}}. \tag{6.7}
$$

Lemma 9.2 of [11] shows that for $w_h(t) = \sum_{j=1}^J w_j(t)\chi_j(t)$ with $\mathbf{w}(t) = (w_j(t))$:

$$
|(\mathbf{M}\mathbf{w})^{(k)}|_{\mathbf{M}^{-1}}^2 \le c\sum_{j=0}^k \|w_h^{(\ell)}\|_{L^2(\Gamma_h)}^2 \tag{6.8}
$$

$$
|\mathbf{M}^{-1}(\mathbf{M}\mathbf{w})^{(k)}|_{\mathbf{A}}^2 \le c\sum_{j=0}^k \left(\|w_h^{(\ell)}\|_{L^2(\Gamma_h)}^2 + \|\nabla_{\Gamma_h} w_h^{(\ell)}\|_{L^2(\Gamma_h)}^2\right). \tag{6.9}
$$

Thus, (6.8) and (6.7) yield

$$
\tau^3\left(|\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{M}_{n+1}} + |\mathbf{M}_{n+\frac12}^{-1}\dddot{\widetilde{\mathbf{p}}}(t)|_{\mathbf{A}_{n+1}}\right) \le C\tau^2\sum_{\ell=1}^4 \|(P_h u)^{(\ell)}(t)\|_{L^2(\Gamma_h(t))}.
$$

Similarly we estimate

$$
\begin{aligned}
&\tau^3\left(|\mathbf{M}_{n+\frac12}^{-1}\ddot{\widetilde{\mathbf{p}}}(t_{n+\frac12})|_{\mathbf{M}_{n+1}} + |\mathbf{M}_{n+\frac12}^{-1}\ddot{\widetilde{\mathbf{p}}}(t_{n+\frac12})|_{\mathbf{A}_{n+1}}\right) \\
&\quad \le C\tau^3\left(\sum_{\ell=1}^3 \|(P_h u)^{(\ell)}(t_{n+\frac12})\|_{L^2(\Gamma_h(t_{n+\frac12}))} + \|\nabla_{\Gamma_h}(P_h u)^{(\ell)}(t_{n+\frac12})\|_{L^2(\Gamma_h(t_{n+\frac12}))}\right),
\end{aligned}
$$

where we use (6.9) instead of the CFL condition to estimate the second term.

Again by Lemma 3.1 and the CFL condition (5.4) used similarly to (6.7), and by the norm identity (3.11), we get the bound

$$
\tau^2\left(|\mathbf{M}_{n+\frac12}^{-1}\mathbf{M}_n\mathbf{r}_n|_{\mathbf{M}_{n+1}} + |\mathbf{M}_{n+\frac12}^{-1}\mathbf{M}_n\mathbf{r}_k|_{\mathbf{A}_{n+1}}\right) \le C\tau\|R_h(t_n)\|_{L^2(\Gamma_h(t_n))}. \tag{6.10}
$$

Combining (6.6)–(6.10), we thus have by (6.4)

$$
\sum_{k=1}^n |\mathbf{d}_k^q|_{M_k} + |\mathbf{d}_k^q|_{A_k} \le C\tau^2\beta_h + C\tau\sum_{k=0}^n \|R_h(t_k)\|_{L^2(\Gamma_h(t_k))}. \tag{6.11}
$$

For $\mathbf{d}_{k+1}^p$ of (6.5) we use the same arguments (Lemma 3.1 and (6.8)) as above, to find

$$
\sum_{k=1}^n |\mathbf{d}_k^p|_{M_k^{-1}} \le C\tau^2\beta_h + C\tau\sum_{k=0}^n \|R_h(t_k)\|_{L^2(\Gamma_h(t_k))}. \tag{6.12}
$$

Inserting the bounds (6.11) and (6.12) into Lemma 5.1 and using the norm identity (3.11) completes the proof. □

7. Lifts

In this section we summarize some results from [7, 8, 9] and show some others about lifts of functions from the discretized to the original surface.

7.1. **Estimates between surface finite elements and their lifts.** We denote by $d(x,t), x \in \mathbb{R}^{m+1}, t \in [0,T]$ the *signed distance function* to the smooth closed surface $\Gamma(t)$ and let $\mathcal{N}(t)$ be a neighbourhood of $\Gamma(t)$ such that for every $x \in \mathcal{N}(t)$ and $t \in [0,T]$ there exists a unique $p(x,t) \in \Gamma(t)$ which is the normal projection of x onto $\Gamma(t)$, i.e.

$$x - p(x,t) = d(x,t)\nu(p(x,t),t). \tag{7.1}$$

We assume $\Gamma_h(t) \subset \mathcal{N}(t)$. Thus for each triangle $E(t)$ in $\Gamma_h(t)$ there is a unique curved triangle $e(t) = p(E(t),t) \subset \Gamma(t)$, and this induces an exact triangulation of $\Gamma(t)$ with curved edges. Furthermore we assume that $\Gamma_h(t)$ consists of triangles $E(t)$ in $\mathcal{T}_h(t)$ with inner radius bounded below by $\sigma_h \geq ch$ for some $c > 0$.

For any continuous function $\eta_h : \Gamma_h \to \mathbb{R}$ we define its lift $\eta_h^l : \Gamma \to \mathbb{R}$ by

$$\eta_h^l(p,t) = \eta_h(x,t), \qquad p \in \Gamma(t),$$

where $x \in \Gamma_h(t)$ is such that $p = p(x,t)$. Then we have the lifted finite element space

$$S_h^l(t) = \{\varphi_h = \phi_h^l \ : \ \phi_h \in S_h(t)\}.$$

Note that $\chi_j^l(\cdot,t)(j = 1,\ldots,J)$ form a basis of $S_h^l(t)$.

We denote by δ_h the quotient between the smooth and discrete surface measures dA and dA_h, defined by $\delta_h dA_h = dA$.

We further introduce Pr and Pr_h as the projections onto the tangent planes of Γ and Γ_h respectively and the Weingarten map $\mathcal{H}\,(\mathcal{H}_{ij} = \partial_{x_j}\nu_i)$. Defining $\mathcal{Q}_h = \frac{1}{\delta_h}(I - d\mathcal{H})PrPr_hPr(I - d\mathcal{H})$ we get the relation [9, Lemma 5.5]

$$\nabla_{\Gamma_h}\eta(x) \cdot \nabla_{\Gamma_h}\phi(x) = \delta_h \mathcal{Q}_h \nabla_\Gamma \eta^l(p) \cdot \nabla_\Gamma \phi^l(p). \tag{7.2}$$

Lemma 7.1. *Assume $\Gamma(t)$ and $\Gamma_h(t)$ satisfy the requirements stated above. Then we have*

$$\|d\|_{L^\infty(\Gamma_h)} \leq ch^2, \quad \|1-\delta_h\|_{L^\infty(\Gamma_h)} \leq ch^2, \quad \|\nu - \nu_h\|_{L^\infty(\Gamma_h)} \leq ch,$$
$$\|Pr - \mathcal{Q}_h\|_{L^\infty(\Gamma_h)} \leq ch^2, \quad \left\|\partial_h^{(\ell)} d\right\|_{L^\infty(\Gamma_h)} \leq ch^2, \quad \left\|\partial_h^{(\ell)}\delta_h\right\|_{L^\infty(\Gamma_h)} \leq ch^2,$$
$$\left\|Pr(\partial_h^{(\ell)}\mathcal{Q}_h)Pr\right\|_{L^\infty(\Gamma_h)} \leq ch^2,$$

where the superscript (ℓ) denotes the ℓth discrete material derivative.

Proof. A proof for the first four estimates can be found in [8, Lemma 5.1]. To prove the other estimates, we consider a single element $E(t) \subset \Gamma_h(t)$, and w.l.o.g. we assume $E \in \mathbb{R}^2 \times \{0\}$. Since $\partial_h^{(\ell)} d = 0$ in the vertices of the triangle E, the linear interpolant $I_h \partial_h^{(\ell)} d$ vanishes on E. By the standard interpolation estimates it follows that

$$\left\|\partial_h^{(\ell)} d\right\|_{L^\infty(E)} = \left\|\partial_h^{(\ell)} d - I_h\partial_h^{(\ell)} d\right\|_{L^\infty(E)} \leq ch^2 \left\|\partial_h^{(\ell)} d\right\|_{W^{2,\infty}(E)} \leq ch^2.$$

Similarly,

$$\left\|\partial_{x_j}(\partial_h^{(\ell)} d)\right\|_{L^\infty(E)} \le ch \qquad \text{for} \quad j=1,2.$$

Since $\nu_j = \partial_{x_j} d$ and $\partial_h^\bullet(\partial_{x_j} f) = \partial_{x_j}(\partial_h^\bullet f) - \partial_{x_j} V_h \cdot \nabla f$, we obtain recursively

$$\left\|\partial_h^{(\ell)} \nu_j\right\|_{L^\infty(E)} \le ch \qquad \text{for} \quad j=1,2.$$

For $x=(x_1,x_2,0)\in E$ we have by (7.1)

$$p_{x_j} = e_j - \nu_j \nu - d\nu_{x_j} \qquad (j=1,2),$$

where $e_j \in \mathbb{R}^3$ denotes the jth standard basis vector. Then direct computation yields

$$\delta_h = \|p_{x_1} \times p_{x_2}\| = |\nu_3| + dR(\nu,\nu_{x_1},\nu_{x_2}) = \sqrt{1-\nu_1^2-\nu_2^2} + dR(\nu,\nu_{x_1},\nu_{x_2})$$

with some smooth remainder function R. Since $|d|, |\partial_h^{(\ell)} d| = \mathcal{O}(h^2)$ and $|\nu_j|, |\partial_h^{(\ell)}\nu_j| = \mathcal{O}(h)$ for $j=1,2$, it follows that $|\partial_h^{(\ell)}\nu_3| \le ch^2$ and

$$\left\|\partial_h^{(\ell)} \delta_h\right\|_{L^\infty(E)} \le ch^2.$$

Let us now prove the last estimate for $\ell=1$. The general case follows recursively with similar arguments. We note that for $\mathcal{Q}_h$ in (7.2), we have

$$\mathcal{Q}_h = \frac{1}{\delta_h} Pr Pr_h Pr + dR(\delta_h, Pr, Pr_h, \mathcal{H})$$

with some smooth remainder function R. Since $|d|, |\partial_h^\bullet d| = \mathcal{O}(h^2)$, $\delta_h = 1 + \mathcal{O}(h^2)$ and $|\partial_h^\bullet \delta_h| = \mathcal{O}(h^2)$, we find

$$Pr\,(\partial_h^\bullet \mathcal{Q}_h)\,Pr = Pr\partial_h^\bullet\,(Pr Pr_h Pr)\,Pr + \mathcal{O}(h^2). \tag{7.3}$$

Using the fact that $\partial_h^\bullet \nu \cdot \nu = 0$, we get

$$\begin{aligned} Pr\partial_h^\bullet\,(Pr Pr_h Pr)\,Pr &= Pr\partial_h^\bullet\,(Pr Pr_h Pr - Pr)\,Pr \\ &= -Pr\partial_h^\bullet \left(Pr \nu_h \nu_h^{\mathsf{T}} Pr\right) Pr. \end{aligned} \tag{7.4}$$

We keep in mind that in our situation $\nu_h = e_3$. Thus

$$|Pr\nu_h| = |\nu_h - (\nu_h\cdot\nu)\nu| = |e_3 - \nu_3\nu| = \sqrt{1-\nu_3^2} = \sqrt{\nu_1^2+\nu_2^2} = \mathcal{O}(h), \tag{7.5a}$$

$$|\partial_h^\bullet(Pr\nu_h)| = |-(\partial_h^\bullet\nu_3)\nu - \nu_3\partial_h^\bullet\nu| = \mathcal{O}(h). \tag{7.5b}$$

Inserting the bounds (7.5) into (7.4) and finally into (7.3) completes the proof. □

7.2. **Error bound of the lifted interpolation.** We shall make use of the following interpolation estimate given in [7, Lemma 5]:

Lemma 7.2. *For a given* $\eta \in H^2(\Gamma)$*,*

$$\|\eta - I_h\eta\|_{L^2(\Gamma)} + h\,\|\nabla_\Gamma(\eta - I_h\eta)\|_{L^2(\Gamma)} \le ch^2 \left(\left\|\nabla_\Gamma^2\eta\right\|_{L^2(\Gamma)} + h\,\|\nabla_\Gamma\eta\|_{L^2(\Gamma)}\right),$$

where $I_h\eta \in S_h^l$ *is the lift of the pointwise linear interpolation* $\widetilde{I}_h\eta \in S_h$*.*

7.3. **Velocity of lifted material points and material derivatives.** By the definition (3.1) of the discrete material velocity V_h for a material point $X(t)$ on $\Gamma_h(t)$, we get the associated material velocity on $\Gamma(t)$: for $y(t) = p(X(t),t)$, we have

$$\dot{y}(t) = v_h(y(t),t)$$

with

$$\begin{aligned} v_h\,(y,t) &= \frac{\partial p}{\partial t}\,(x,t) + V_h\,(x,t)\cdot \nabla p\,(x,t) \\ &= (Pr - d\mathcal{H})(x,t)V_h(x,t) - \partial_t d(x,t)\nu(x,t) - d(x,t)\partial_t\nu(x,t), \end{aligned} \tag{7.6}$$

for $y = p(x,t)$. We note that $-\partial_t d(x,t)\nu(x,t)$ is just the normal component of $v(p,t)$, and the other two terms on the right-hand side of (7.6) are tangent to $\Gamma(t)$ in p. It follows that

$$v_h - v \text{ is a tangent vector.} \tag{7.7}$$

The discrete material derivatives on $\Gamma_h(t)$ and $\Gamma(t)$ then read

$$\begin{aligned} \partial_h^\bullet \phi_h &= \frac{\partial \phi_h}{\partial t} + V_h\cdot\nabla\phi_h, \\ \partial_h^\bullet \varphi_h &= \frac{\partial \varphi_h}{\partial t} + v_h\cdot\nabla\varphi_h. \end{aligned}$$

It was shown in [9, Lemma 4.1] that the basis functions of $S_h^l(t)$ also satisfy the *transport property*

$$\partial_h^\bullet \varphi_j = \partial_h^\bullet \phi_j^l = 0. \tag{7.8}$$

Therefore the discrete material derivative and the lifting process commute in the following sense: For $\varphi_h = \phi_h^l \in S_h^l$,

$$\partial_h^\bullet \varphi_h = (\partial_h^\bullet \phi_h)^l = \sum_{j=1}^{J} \dot{\phi}_{h,j}\chi_j^l,$$

where $\phi_{h,j}(t) = \phi_h(a_j(t),t) = \varphi_h(a_j(t),t)$.

We have the following bounds for the difference between the different velocities:

Lemma 7.3. *The error between the continuous velocity v and the lifted discrete velocity v_h on the smooth surface Γ satisfies the bounds, for $\ell \geq 0$,*

$$\|\partial_h^{(\ell)}(v - v_h)\|_{L^\infty(\Gamma)} + h\|\nabla_\Gamma \partial_h^{(\ell)}(v - v_h)\|_{L^\infty(\Gamma)} \leq C_\ell\, h^2. \tag{7.9}$$

Proof. The definition (7.6) of v_h together with the fact that $V_h = I_h v$ give (see [9, Lemma 5.6])

$$|v(p,t) - v_h(p,t)| = |Pr(v - I_h v)(p,t) + d(\mathcal{H} I_h v(p,t) + \partial_t \nu)| \leq Ch^2.$$

For $\ell = 1$, we have by the transport property (7.8) and Lemma 7.1

$$\begin{aligned} |\partial_h^\bullet (v - v_h)| &\leq |(\partial_h^\bullet Pr)(v - I_h v)| + |Pr(\partial_h^\bullet v - I_h \partial_h^\bullet v)| \\ &\quad + |(\partial_h^\bullet d)(\mathcal{H} I_h v + \partial_t\nu)| + |d\partial_h^\bullet(\mathcal{H} I_h v + \partial_t\nu)| \\ &\leq Ch^2. \end{aligned}$$

Using the fact that $\nabla_\Gamma d = \nabla_\Gamma \partial_h^\bullet d = 0$ and Lemma 7.1, we obtain

$$
\begin{aligned}
|\nabla_\Gamma (v - v_h)| &\leq c\,|v - I_h v| + c\,|\nabla_\Gamma (v - I_h v)| + ch^2 \leq ch, \\
|\nabla_\Gamma \partial_h^\bullet (v - v_h)| &\leq c\,|v - I_h v| + c\,|\nabla_\Gamma (v - I_h v)| + c\,|\partial_h^\bullet v - I_h \partial_h^\bullet v| \\
&\quad + c\,|\nabla_\Gamma (\partial_h^\bullet v - I_h \partial_h^\bullet v)| + ch^2 \\
&\leq ch.
\end{aligned}
$$

For $\ell > 1$ the proof uses the same arguments. □

7.4. **Lifts and bilinear forms.** We define the bilinear forms for $w, \varphi \in H^1(\Gamma)$ as

$$a(w,\varphi) = \int_\Gamma \nabla_\Gamma w \cdot \nabla_\Gamma \varphi, \tag{7.10a}$$

$$m(w,\varphi) = \int_\Gamma w\varphi, \tag{7.10b}$$

where the forms also depend on time t. We write $a(w,\varphi;t)$ etc. when we want to make the dependence on t explicit.

The discrete analogs of the above bilinear forms for $W_h, \phi_h \in S_h$ are defined by

$$a_h(W_h,\phi_h) = \sum_{E\in\mathcal{T}_h} \int_E \nabla_{\Gamma_h} W_h \cdot \nabla_{\Gamma_h} \phi_h, \tag{7.11a}$$

$$m_h(W_h,\phi_h) = \int_{\Gamma_h} W_h \phi_h. \tag{7.11b}$$

We are interested in the time derivatives of these bilinear forms. For this we need some more bilinear forms:

$$g(v;w,\varphi) = \int_\Gamma (\nabla_\Gamma \cdot v)\, w\varphi, \tag{7.12a}$$

$$b(v;w,\varphi) = \int_\Gamma \mathcal{B}(v) \nabla_\Gamma w \cdot \nabla_\Gamma \varphi \tag{7.12b}$$

with the matrix

$$\mathcal{B}(v)_{ij} = \delta_{ij} \nabla_\Gamma \cdot v - \left((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i \right), \quad i,j = 1,\ldots,m+1.$$

Their discrete analogs read

$$g_h(V_h;W_h,\phi_h) = \int_{\Gamma_h} (\nabla_{\Gamma_h} \cdot V_h)\, W_h \phi_h, \tag{7.13a}$$

$$b_h(V_h;W_h,\phi_h) = \sum_{E\in\mathcal{T}_h} \int_E \mathcal{B}_h(V_h) \nabla_{\Gamma_h} W_h \cdot \nabla_{\Gamma_h} \phi_h \tag{7.13b}$$

with

$$\mathcal{B}_h(V_h)_{ij} = \delta_{ij} \nabla_{\Gamma_h} \cdot V_h - \left((\nabla_{\Gamma_h})_i V_{hj} + (\nabla_{\Gamma_h})_j V_{hi} \right), \quad i,j = 1,\ldots,m+1.$$

We shall make use of the following transport lemma [9, Lemma 4.2].

Lemma 7.4. *For $\varphi, w, \partial^\bullet \varphi, \partial^\bullet w, \partial_h^\bullet \varphi, \partial_h^\bullet w \in H^1(\Gamma)$ we have:*

$$
\begin{aligned}
\frac{d}{dt} m(w,\varphi) &= m(\partial^\bullet w, \varphi) + m(w, \partial^\bullet \varphi) + g(v;w,\varphi), \\
\frac{d}{dt} a(w,\varphi) &= a(\partial^\bullet w, \varphi) + a(w, \partial^\bullet \varphi) + b(v;w,\varphi).
\end{aligned}
$$

The same formulas hold when $\partial^\bullet$ and v are replaced with $\partial_h^\bullet$ and v_h, respectively. Furthermore for $W_h, \phi_h \in S_h$ we have the following analogs:

$$\frac{d}{dt} m_h(W_h,\phi_h) = m_h(\partial_h^\bullet W_h,\phi_h) + m_h(W_h,\partial_h^\bullet \phi_h) + g_h(V_h;W_h,\phi_h),$$
$$\frac{d}{dt} a_h(W_h,\phi_h) = a_h(\partial_h^\bullet W_h,\phi_h) + a_h(W_h,\partial_h^\bullet \phi_h) + b_h(V_h;W_h,\phi_h).$$

We show the following bounds for the lifting process.

Lemma 7.5. *For any $(W_h,\phi_h) \in S_h \times S_h$ with the corresponding lifts $(w_h,\varphi_h) \in S_h^l \times S_h^l$ we have*

$$|m(w_h,\varphi_h) - m_h(W_h,\phi_h)| \le ch^2 \|w_h\|_{L^2(\Gamma)} \|\varphi_h\|_{L^2(\Gamma)},$$
$$|a(w_h,\varphi_h) - a_h(W_h,\phi_h)| \le ch^2 \|\nabla_\Gamma w_h\|_{L^2(\Gamma)} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma)},$$
$$|g(v_h;w_h,\varphi_h) - g_h(V_h;W_h,\phi_h)| \le ch^2 \|w_h\|_{L^2(\Gamma)} \|\varphi_h\|_{L^2(\Gamma)},$$
$$|b(v_h;w_h,\varphi_h) - b_h(V_h;W_h,\phi_h)| \le ch^2 \|\nabla_\Gamma w_h\|_{L^2(\Gamma)} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma)}.$$

Proof. The first two estimates have been shown in [9, Lemma 5.5]. To prove the third estimate, we apply the Transport Lemma 7.4 once on Γ_h and a second time on Γ, to get the following identities:

$$\begin{aligned}\frac{d}{dt} m(w_h,\varphi_h) &= \frac{d}{dt} m_h(W_h,\phi_h\cdot\delta_h)\\ &= m_h(\partial_h^\bullet W_h,\phi_h\cdot\delta_h) + m_h(W_h,\partial_h^\bullet\phi_h\cdot\delta_h) + m_h(W_h,\phi_h\cdot\partial_h^\bullet\delta_h)\\ &\quad + g_h(V_h;W_h,\phi_h\cdot\delta_h)\\ &= m(\partial_h^\bullet w_h,\varphi_h) + m(w_h,\partial_h^\bullet\varphi_h) + g(v_h;w_h,\varphi_h).\end{aligned}$$

Due to the fact that $\partial_h^\bullet w_h = (\partial_h^\bullet W_h)^l$, using Lemma 7.1 and the equivalence of norms between the continuous and discrete surface, it follows

$$\begin{aligned}|g(v_h;w_h,\varphi_h) - g_h(V_h;W_h,\phi_h)| &= |m_h(W_h,\phi_h\cdot\partial_h^\bullet\delta_h) + g_h(V_h;W_h,\phi_h\cdot(\delta_h-1))|\\ &\le c\Big(\|\partial_h^\bullet\delta_h\|_{L^\infty(\Gamma_h)} + \|\delta_h-1\|_{L^\infty(\Gamma_h)}\Big)\|w_h\|_{L^2(\Gamma)}\|\varphi_h\|_{L^2(\Gamma)}\\ &\le ch^2\|w_h\|_{L^2(\Gamma)}\|\varphi_h\|_{L^2(\Gamma)}.\end{aligned}$$

Similarly we prove the last estimate. We use Lemma 7.4 and the relation (7.2) to find

$$\begin{aligned}&\frac{d}{dt}\int_{\Gamma_h}\nabla_{\Gamma_h}W_h\nabla_{\Gamma_h}\phi_h = \int_\Gamma \mathcal{Q}_h^l\nabla_\Gamma w_h\nabla_\Gamma\varphi_h\\ &= \int_\Gamma \mathcal{Q}_h^l\nabla_\Gamma\partial_h^\bullet w_h\nabla_\Gamma\varphi_h + \int_\Gamma \mathcal{Q}_h^l\nabla_\Gamma w_h\nabla_\Gamma\partial_h^\bullet\varphi_h + \int_\Gamma \partial_h^\bullet\mathcal{Q}_h^l\nabla_\Gamma w_h\nabla_\Gamma\varphi_h\\ &\quad + \int_\Gamma \mathcal{B}(v_h)\mathcal{Q}_h^l\nabla_\Gamma w_h\nabla_\Gamma\varphi_h\\ &= \int_{\Gamma_h}\nabla_{\Gamma_h}\partial_h^\bullet W_h\nabla_{\Gamma_h}\phi_h + \int_{\Gamma_h}\nabla_{\Gamma_h}W_h\nabla_{\Gamma_h}\partial_h^\bullet\phi_h + \int_{\Gamma_h}\mathcal{B}_h(v_h)\nabla_{\Gamma_h}W_h\nabla_{\Gamma_h}\phi_h.\end{aligned}$$

Therefore, the relation $\partial_h^\bullet w_h = (\partial_h^\bullet W_h)^l$, (7.2) and Lemma 7.1 yield

$$\begin{aligned}
&|b_h(V_h; W_h, \phi_h) - b(v_h; w_h, \varphi_h)| \\
&= \left| \int_\Gamma \partial_h^\bullet \mathcal{Q}_h^l \nabla_\Gamma w_h \nabla_\Gamma \varphi_h + \int_\Gamma \mathcal{B}(v_h) \left(\mathcal{Q}_h^l - I\right) \nabla_\Gamma w_h \nabla_\Gamma \varphi_h \right| \\
&\le ch^2 \|\nabla_\Gamma w_h\|_{L^2(\Gamma)} \|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma)},
\end{aligned}$$

which completes the proof. □

8. The Ritz map for evolving surfaces

8.1. **A modified Ritz projection.** It turns out to be convenient in the error analysis to use a modified Ritz projection $\widetilde{\mathcal{P}}_h(t) : H^1(\Gamma(t)) \to S_h(t)$ defined in the following way, where we use the bilinear forms of Section 7.4 and the lifted discrete velocity of Section 7.3. To motivate the definition, we rewrite the weak form (2.6) of the wave equation in terms of the bilinear forms,

$$\frac{d}{dt} m(\partial^\bullet u, \varphi) + a(u, \varphi) = m(\partial^\bullet u, \partial^\bullet \varphi),$$

and use the Leibniz formula with the discrete material derivative $\partial_h^\bullet$ on Γ and note $\partial_h^\bullet \varphi = \partial^\bullet \varphi + (v_h - v) \cdot \nabla_\Gamma \varphi$, because $v_h - v$ is a tangent vector (see (7.7)). Then this equation becomes

$$\begin{aligned}
m(\partial_h^\bullet \partial_h^\bullet u, \varphi) + g(v_h; \partial^\bullet u, \varphi) + m(\partial_h^\bullet \partial^\bullet u - \partial_h^\bullet \partial_h^\bullet u, \varphi)& \\
+ m(\partial^\bullet u, (v - v_h) \cdot \nabla_\Gamma \varphi) + a(u, \varphi) = 0.& \qquad (8.1)
\end{aligned}$$

We now define a Ritz map that collects the last two terms on the left-hand side of this equation, which are the only terms that contain the surface gradient of the test function φ. Since $a(\cdot,\cdot)$ is only positive semi-definite, we consider the positive definite bilinear forms

$$\begin{aligned}
a^*(w, \varphi) &= a(w, \varphi) + m(w, \varphi), && w, \varphi \in H^1(\Gamma) \\
a_h^*(W_h, \phi_h) &= a_h(W_h, \phi_h) + m_h(W_h, \phi_h), && W_h, \phi_h \in S_h.
\end{aligned}$$

We note that $a^*(w, w) = \|w\|_{H^1(\Gamma)}^2$. We write $a^*(w, \varphi; t)$ etc. to make the dependence on t explicit.

Definition 8.1. For given $z \in H^1(\Gamma(t))$ and $\partial^\bullet z \in L^2(\Gamma(t))$, there is a unique $\widetilde{\mathcal{P}}_h(t) z \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ we have, with the corresponding lift $\varphi_h = \phi_h^l$,

$$a_h^*(\widetilde{\mathcal{P}}_h(t) z, \phi_h; t) = a^*(z, \varphi_h; t) + m(\partial^\bullet z, (v(\cdot, t) - v_h(\cdot, t)) \cdot \nabla_{\Gamma(t)} \varphi_h; t). \qquad (8.2)$$

We define $\mathcal{P}_h(t) z \in S_h^l(t)$ as the lift of $\widetilde{\mathcal{P}}_h(t) z$, i.e., $\mathcal{P}_h(t) z = (\widetilde{\mathcal{P}}_h(t) z)^l$.

8.2. **Error in the Ritz map.**

Theorem 8.2. *The error in the Ritz map satisfies the bounds, for $0 \le t \le T$ and $h \le h_0$ with sufficiently small h_0,*

$$\begin{aligned}
\|z - \mathcal{P}_h(t) z\|_{L^2(\Gamma(t))} + h \left\| \nabla_{\Gamma(t)} (z - \mathcal{P}_h(t) z) \right\|_{L^2(\Gamma(t))}& \\
\le Ch^2 \left(\|z\|_{H^2(\Gamma(t))} + \|\partial^\bullet z\|_{L^2(\Gamma(t))} \right).& \qquad (8.3)
\end{aligned}$$

Proof. We omit the omnipresent argument t in the following. We first note that in view of (7.9) and Lemma 7.5, we have for all $\varphi_h \in S_h^l$:

$$a^*(z - \mathcal{P}_h z, \varphi_h) = a_h^*(\widetilde{\mathcal{P}}_h z, \phi_h) - a^*(\mathcal{P}_h z, \varphi_h) - m(\partial^\bullet z, (v - v_h) \cdot \nabla_\Gamma \varphi_h)$$
$$\leq Ch^2 \|\mathcal{P}_h z\|_{H^1(\Gamma)} \|\varphi_h\|_{H^1(\Gamma)} + Ch^2 \|\partial^\bullet z\|_{L^2(\Gamma)} \|\varphi_h\|_{H^1(\Gamma)} . \quad (8.4)$$

This relation will serve as a substitute for the Galerkin orthogonality in standard finite element theory on fixed domains. Together with the interpolation error bound of Lemma 7.2 this yields

$$\begin{aligned}\|z - \mathcal{P}_h z\|_{H^1(\Gamma)}^2 &= a^*(z - \mathcal{P}_h z, z - I_h z) + a^*(z - \mathcal{P}_h z, I_h z - \mathcal{P}_h z)\\ &\leq \|z - \mathcal{P}_h z\|_{H^1(\Gamma)} \|z - I_h z\|_{H^1(\Gamma)}\\ &\quad + Ch^2(\|\mathcal{P}_h z\|_{H^1(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}) \|I_h z - \mathcal{P}_h z\|_{H^1(\Gamma)}\\ &\leq Ch \|z - \mathcal{P}_h z\|_{H^1(\Gamma)} \|z\|_{H^2(\Gamma)}\\ &\quad + Ch^2 \Big(\|\mathcal{P}_h z\|_{H^1(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\Big) \|I_h z - \mathcal{P}_h z\|_{H^1(\Gamma)} .\end{aligned}$$

Using once more Lemma 7.2 we estimate

$$\begin{aligned}&\Big(\|\mathcal{P}_h z\|_{H^1(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\Big) \|I_h z - \mathcal{P}_h z\|_{H^1(\Gamma)}\\ &\quad\leq \Big(\|\mathcal{P}_h z - z\|_{H^1(\Gamma)} + \|z\|_{H^1(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\Big) \cdot \Big(Ch \|z\|_{H^2(\Gamma)} + \|z - \mathcal{P}_h z\|_{H^1(\Gamma)}\Big)\\ &\quad\leq 2 \|z - \mathcal{P}_h z\|_{H^1(\Gamma)}^2 + \|z\|_{H^1(\Gamma)}^2 + \|\partial^\bullet z\|_{L^2(\Gamma)}^2 + Ch^2 \|z\|_{H^2(\Gamma)}^2 .\end{aligned}$$

Combining both of the above inequalities yields, for sufficiently small h,

$$\|z - \mathcal{P}_h z\|_{H^1(\Gamma)}^2 \leq Ch^2 \Big(\|z\|_{H^2(\Gamma)}^2 + \|\partial^\bullet z\|_{L^2(\Gamma)}^2\Big) , \quad (8.5)$$

which implies the gradient estimate in (8.3).

Now we use the Aubin-Nitsche trick to prove the $O(h^2)$ bound of the $L^2(\Gamma)$ error, and solve the problem

$$-\Delta_\Gamma w + w = z - \mathcal{P}_h z \quad \text{on } \Gamma.$$

Then by the elliptic theory on smooth surfaces (see [2] and [25] for more details), $w \in H^2(\Gamma)$ satisfies the bound

$$\|w\|_{H^2(\Gamma)} \leq c \|z - \mathcal{P}_h z\|_{L^2(\Gamma)} . \quad (8.6)$$

The Cauchy–Schwarz inequality, the interpolation estimate of Lemma 7.2 and the bounds (8.5) and (8.4) yield

$$\begin{aligned}\|z - \mathcal{P}_h z\|_{L^2(\Gamma)}^2 &= a^*(z - \mathcal{P}_h z, w)\\ &= a^*(z - \mathcal{P}_h z, w - I_h w) + a^*(z - \mathcal{P}_h z, I_h w)\\ &\leq Ch^2 \Big(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\Big) \|w\|_{H^2(\Gamma)}\\ &\quad + Ch^2\Big(\|\mathcal{P}_h z\|_{H^1(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\Big) \|I_h w\|_{H^1(\Gamma)} .\end{aligned}$$

Noting, from (8.5) and Lemma 7.2,

$$\begin{aligned}\|\mathcal{P}_h z\|_{H^1(\Gamma)} &\leq \|z\|_{H^1(\Gamma)} + Ch \Big(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\Big)\\ \|I_h w\|_{H^1(\Gamma)} &\leq \|w\|_{H^1(\Gamma)} + Ch \|w\|_{H^2(\Gamma)} ,\end{aligned}$$

we obtain

$$\|z-\mathcal{P}_h z\|^2_{L^2(\Gamma)} \le Ch^2\left(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)}\right)\|w\|_{H^2(\Gamma)}\,.$$

Applying the bound (8.6) completes the proof. □

8.3. **Error in the material derivatives of the Ritz map.** In general, $\partial_h^\bullet \mathcal{P}_h z \neq \mathcal{P}_h \partial_h^\bullet z$, but we have the following result.

Theorem 8.3. *The error in the material derivatives of the Ritz map satisfies the bounds, for $\ell \ge 1$, $0 \le t \le T$ and $h \le h_0$ with sufficiently small h_0,*

$$\left\|\partial_h^{(\ell)}\left(z-\mathcal{P}_h z\right)(t)\right\|_{L^2(\Gamma(t))} + h\left\|\nabla_\Gamma\left(\partial_h^{(\ell)}\left(z-\mathcal{P}_h z\right)(t)\right)\right\|_{L^2(\Gamma(t))}$$
$$\le C_\ell\, h^2 \sum_{i=0}^{\ell}\left(\left\|\partial^{(i)} z(t)\right\|_{H^2(\Gamma(t))} + \left\|\partial^{(i+1)} z(t)\right\|_{L^2(\Gamma(t))}\right). \qquad (8.7)$$

Proof. We prove this bound only for $\ell = 1$. The general case follows with similar arguments.

We take the time derivative of Equation (8.2) and use the Transport Lemma 7.4, the relation

$$\partial_h^\bullet \nabla_\Gamma f = \nabla_\Gamma \partial_h^\bullet f - \mathcal{D}(v_h)\nabla_\Gamma f \quad \text{with} \quad \mathcal{D}(v_h)_{ij} = (\nabla_\Gamma)_i v_{hj} - \sum_{l=1}^{m+1} \nu_l \nu_i (\nabla_\Gamma)_j v_{h,l},$$

which is proved in [10, Lemma 2.6], and the definition of the Ritz projection (8.2) to arrive at

$$\begin{aligned} a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \varphi_h) = &- b(v_h; z-\mathcal{P}_h z, \varphi_h) - g(v_h; z-\mathcal{P}_h z, \varphi_h) \\ &+ F_1(\varphi_h) + F_2(\varphi_h) \end{aligned} \qquad (8.8)$$

for $\varphi_h \in S_h^l$,where

$$\begin{aligned} F_1(\varphi_h) &= a_h^*(\partial_h^\bullet \widetilde{\mathcal{P}}_h z, \phi_h) - a^*(\partial_h^\bullet \mathcal{P}_h z, \varphi_h) + b_h(V_h; \widetilde{\mathcal{P}}_h z, \phi_h) - b(v_h; \mathcal{P}_h z, \varphi_h) \\ &\quad + g_h(V_h; \widetilde{\mathcal{P}}_h z, \phi_h) - g(v_h; \mathcal{P}_h z, \varphi_h), \\ F_2(\varphi_h) &= -\, m(\partial_h^\bullet \partial^\bullet z, (v-v_h)\cdot\nabla_\Gamma \varphi_h) - g(v_h; \partial^\bullet z, (v-v_h)\cdot\nabla_\Gamma\varphi_h) \\ &\quad - m(\partial^\bullet z, (\partial_h^\bullet[v-v_h])\cdot\nabla_\Gamma\varphi_h) + m(\partial^\bullet z, (v-v_h)\cdot\mathcal{D}(v_h)\nabla_\Gamma\varphi_h). \end{aligned}$$

We start by bounding $F_1(\varphi_h)$ and $F_2(\varphi_h)$. Lemma 7.5 yields

$$|F_1(\varphi_h)| \le Ch^2\left(\|\partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)} + \|\mathcal{P}_h z\|_{H^1(\Gamma)}\right)\|\varphi_h\|_{H^1(\Gamma)}\,. \qquad (8.9)$$

In view of (7.9) and the fact that $(v-v_h)$ is a tangent vector, it follows that

$$\begin{aligned} \|\partial_h^\bullet \partial^\bullet z\|_{L^2(\Gamma)} &\le \|\partial_h^\bullet\partial^\bullet z - \partial^\bullet\partial^\bullet z\|_{L^2(\Gamma)} + \|\partial^\bullet\partial^\bullet z\|_{L^2(\Gamma)} \\ &= \|(v_h - v)\cdot\nabla_\Gamma\partial^\bullet z\|_{L^2(\Gamma)} + \|\partial^\bullet\partial^\bullet z\|_{L^2(\Gamma)} \\ &\le ch^2\,\|\nabla_\Gamma\partial^\bullet z\|_{L^2(\Gamma)} + \|\partial^\bullet\partial^\bullet z\|_{L^2(\Gamma)}\,. \end{aligned}$$

Thus we get the bound

$$|F_2(\varphi_h)| \le Ch^2\left(\|\partial^\bullet\partial^\bullet z\|_{L^2(\Gamma)} + \|\partial^\bullet z\|_{H^1(\Gamma)}\right)\|\varphi_h\|_{H^1(\Gamma)}\,. \qquad (8.10)$$

We use the relation (7.9) to find

$$\|\partial_h^\bullet z\|_{H^1(\Gamma)} \le \|\partial^\bullet z\|_{H^1(\Gamma)} + ch\,\|z\|_{H^2(\Gamma)}\,.$$

Then inserting $\varphi_h = \partial_h^\bullet \mathcal{P}_h z$ in (8.8), and using Theorem 8.2, (8.9) and (8.10), for $h \le h_0$, we obtain

$$\|\partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)} \le C \|\partial^\bullet z\|_{H^1(\Gamma)} + Ch \|z\|_{H^2(\Gamma)} + Ch^2 \|\partial^\bullet \partial^\bullet z\|_{L^2(\Gamma)} .$$

Combining the above bounds, we estimate the right hand side of (8.8), for sufficiently small $h \le h_0$, as follows:

$$\begin{aligned} &a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \varphi_h) \\ &\quad \le Ch \Big(\|z\|_{H^2(\Gamma)} + h \|\partial^\bullet \partial^\bullet z\|_{L^2(\Gamma)} + h \|\partial^\bullet z\|_{H^1(\Gamma)} \Big) \|\varphi_h\|_{H^1(\Gamma)} . \end{aligned} \tag{8.11}$$

So we obtain

$$\begin{aligned} &\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)}^2 \\ &\quad = a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \partial_h^\bullet z - I_h \partial^\bullet z) + a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z) \\ &\quad \le \|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)} \|\partial_h^\bullet z - I_h \partial^\bullet z\|_{H^1(\Gamma)} \\ &\qquad + Ch \Big(\|z\|_{H^2(\Gamma)} + h \|\partial^\bullet \partial^\bullet z\|_{L^2(\Gamma)} + h \|\partial^\bullet z\|_{H^1(\Gamma)} \Big) \|I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)} \end{aligned} \tag{8.12}$$

The interpolation error bound of Lemma 7.2 and (7.9) yield

$$\begin{aligned} \|\partial_h^\bullet z - I_h \partial^\bullet z\|_{H^1(\Gamma)} &= \|\partial_h^\bullet z - \partial^\bullet z\|_{H^1(\Gamma)} + \|\partial^\bullet z - I_h \partial^\bullet z\|_{H^1(\Gamma)} \\ &\le Ch \|z\|_{H^2(\Gamma)} + Ch \|\partial^\bullet z\|_{H^2(\Gamma)} , \end{aligned}$$

and similarly

$$\begin{aligned} \|I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)} &\le Ch \|\partial^\bullet z\|_{H^2(\Gamma)} + Ch \|z\|_{H^2(\Gamma)} \\ &\quad + \|I_h \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)} . \end{aligned}$$

Applying the last two estimates to (8.12) and using Young's inequality, for $h \le h_0$, yield

$$\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{H^1(\Gamma)}^2 \le Ch^2 \Big(\|z\|_{H^2(\Gamma)}^2 + \|\partial^\bullet z\|_{H^2(\Gamma)}^2 + \|\partial^\bullet \partial^\bullet z\|_{L^2(\Gamma)}^2 \Big) , \tag{8.13}$$

which implies the gradient estimate in (8.7).

To prove the $L^2(\Gamma)$ estimate, we use as before the Aubin-Nitsche trick and solve the problem

$$-\Delta_\Gamma w + w = \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z \quad \text{on } \Gamma.$$

Then by the elliptic theory it follows

$$\|w\|_{H^2(\Gamma)} \le c \|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{L^2(\Gamma)} . \tag{8.14}$$

A calculation that is nearly identical to [9, proof of Theorem 6.2] gives

$$-b(v_h; z - \mathcal{P}_h z, I_h w) \le Ch^2 \Big(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{L^2(\Gamma)} \Big) \|w\|_{H^2(\Gamma)} .$$

Therefore combining (8.8), (8.9), (8.10) and Theorem 8.2 yields

$$a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, I_h w) \le Ch^2 \Big(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{H^1(\Gamma)} + \|\partial^\bullet \partial^\bullet z\|_{L^2(\Gamma)} \Big) \|w\|_{H^2(\Gamma)} .$$

Together with (8.13) and Lemma 7.2 this yields

$$\begin{aligned}\|\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z\|_{L^2(\Gamma)}^2 &= a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, w)\\ &= a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, w - I_h w) + a^*(\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, I_h w)\\ &\leq Ch^2 \left(\|z\|_{H^2(\Gamma)} + \|\partial^\bullet z\|_{H^2(\Gamma)} + \|\partial^\bullet\partial^\bullet z\|_{L^2(\Gamma)}\right)\|w\|_{H^2(\Gamma)}.\end{aligned}$$

Finally applying the bound (8.14) completes the proof. □

9. Bound of the semidiscrete residual

We show an optimal-order bound of the residual $R_h(t) \in S_h(t)$ of (6.1) when we take the mapping $P_h(t)$ to be the Ritz map $\widetilde{\mathcal{P}}_h(t)$ defined in (8.2).

Lemma 9.1. *Assume that the solution u of the wave equation is sufficiently smooth. Then, there exist $C > 0$ and $h_0 > 0$ such that for $h \leq h_0$ and $0 \leq t \leq T$,*

$$\|R_h(t)\|_{L^2(\Gamma_h(t))} \leq Ch^2. \tag{9.1}$$

Proof. We start by rewriting the residual equation (6.1) for $R_h \in S_h$ with $P_h = \widetilde{\mathcal{P}}_h$ as

$$\begin{aligned}m_h(R_h,\phi_h) &= \frac{d}{dt} m_h(\partial_h^\bullet \widetilde{\mathcal{P}}_h u,\phi_h) + a_h(\widetilde{\mathcal{P}}_h u,\phi_h) - m_h(\partial_h^\bullet \widetilde{\mathcal{P}}_h u, \partial_h^\bullet\phi_h)\\ &= m_h(\partial_h^\bullet\partial_h^\bullet \widetilde{\mathcal{P}}_h u,\phi_h) + g_h(V_h;\partial_h^\bullet \widetilde{\mathcal{P}}_h u,\phi_h) + a_h(\widetilde{\mathcal{P}}_h u,\phi_h),\end{aligned}$$

where we have used the Transport Lemma 7.4. Combining this equation with (8.1) and using the definition of the Ritz map (8.2) yield

$$m_h(R_h,\phi_h) = F_1(\varphi_h) + F_2(\varphi_h) + F_3(\varphi_h), \qquad \varphi_h = \phi_h^l \in S_h^l, \tag{9.2}$$

where

$$\begin{aligned}F_1(\varphi_h) &= m_h(\partial_h^\bullet\partial_h^\bullet \widetilde{\mathcal{P}}_h u,\phi_h) - m(\partial_h^\bullet\partial_h^\bullet u,\varphi_h),\\ F_2(\varphi_h) &= g_h(V_h;\partial_h^\bullet \widetilde{\mathcal{P}}_h u,\phi_h) - g(v_h;\partial^\bullet u,\varphi_h),\\ F_3(\varphi_h) &= m(\partial_h^\bullet\partial^\bullet u - \partial_h^\bullet\partial_h^\bullet u,\varphi_h).\end{aligned}$$

Applying Lemma 7.5, using $(\partial_h^\bullet\partial_h^\bullet \widetilde{\mathcal{P}}_h u)^l = \partial_h^\bullet\partial_h^\bullet \mathcal{P}_h u$ and applying Theorem 8.3 with $\ell = 2$ yields

$$\begin{aligned}|F_1(\varphi_h)| &= m_h(\partial_h^\bullet\partial_h^\bullet \widetilde{\mathcal{P}}_h u,\phi_h) - m(\partial_h^\bullet\partial_h^\bullet \mathcal{P}_h u,\varphi_h) + m(\partial_h^\bullet\partial_h^\bullet \mathcal{P}_h u - \partial_h^\bullet\partial_h^\bullet u,\varphi_h)\\ &\leq ch^2\|\varphi_h\|_{L^2(\Gamma)},\end{aligned}$$

and with the same arguments

$$|F_2(\varphi_h)| \leq ch^2\|\varphi_h\|_{L^2(\Gamma)}.$$

Furthermore Lemma 7.3 yields

$$|F_3(\varphi_h)| = m(\partial_h^\bullet[(v - v_h)\cdot\nabla_\Gamma u],\varphi_h) \leq ch^2\|\varphi_h\|_{L^2(\Gamma)}.$$

Inserting the above bounds into (9.2) with $\phi_h = R_h$ and noting the equivalence of L^2 norms between the original and discretized surfaces completes the proof. □

10. Error bound for the full discretization

We consider the lifts of the fully discrete numerical solution and its numerical material derivative as determined by (4.6),

$$u_h^n := (U_h^n)^l = \sum_{j=1}^J q_j^n \chi_j^l(t_n), \qquad \partial_h^\bullet u_h^n := (\partial_h^\bullet U_h^n)^l = \sum_{j=1}^J \dot q_j^n \chi_j^l(t_n),$$

which are lifted finite element functions defined on the surface $\Gamma(t_n)$. This will be compared with the solution $u(t_n)$ of the wave equation (2.4) and its material derivative $\partial^\bullet u(t_n)$.

We rewrite the error by subtracting and adding the Ritz map applied to the exact solution,

$$u_h^n - u(t_n) = u_h^n - \mathcal{P}_h(t_n)u(t_n) + \mathcal{P}_h(t_n)u(t_n) - u(t_n).$$

We use Theorem 6.1 together with Lemma 9.1 (residual bound) and Theorem 8.3 (for estimating β_h) to bound the first difference and note the equivalence of norms on the discretized and original surface (Lemma 7.1). We use directly Theorems 8.2 and 8.3 to bound the second difference. This proves our main result:

Theorem 10.1. *Let u be a sufficiently smooth solution of the wave equation* (2.4) *and assume that the discrete initial data satisfy*

$$\big\|u_h^0 - (\mathcal{P}_h u)(0)\big\|_{L^2(\Gamma(0))} + \big\|\nabla_{\Gamma(0)} u_h^0 - \nabla_{\Gamma(0)}(\mathcal{P}_h u)(0)\big\|_{L^2(\Gamma(0))} + \big\|\partial_h^\bullet u_h^0 - \partial_h^\bullet(\mathcal{P}_h u)(0)\big\|_{L^2(\Gamma(0))} \le C_0 h^2.$$

Then, there exist $h_0 > 0$ and $\tau_0 > 0$ such that for $h \le h_0$ and $\tau \le \tau_0$ satisfying the CFL condition (5.4)*, the following error bound holds for $0 \le t_n = n\tau \le T$:*

$$\|u_h^n - u(.,t_n)\|_{L^2(\Gamma(t_n))} + h\,\big\|\nabla_{\Gamma(t_n)} u_h^n - \nabla_{\Gamma(t_n)} u(.,t_n)\big\|_{L^2(\Gamma(t_n))} + \|\partial_h^\bullet u_h^n - \partial^\bullet u(.,t_n)\|_{L^2(\Gamma(t_n))} \le C\big(h^2 + \tau^2\big).$$

The constant C is independent of h, τ, and n subject to the stated conditions.

11. Numerical experiments

In this section, we present two numerical experiments to illustrate our theoretical results. The implementation is done using the DUNE-FEM module, which is based on the Distributed and Unified Numerics Environment (DUNE), see [3, 5, 6] for details. The implementation of the evolving surface finite element method is described in [8].

Example 1. We solve the wave equation

$$\partial^\bullet\partial^\bullet u + \partial^\bullet u\ \nabla_\Gamma \cdot v - \Delta_\Gamma u = f \quad \text{on } \Gamma(t), \tag{11.1}$$

where

$$\Gamma(t) = \left\{ x \in \mathbb{R}^3 \,:\, \frac{x_1^2}{1 + 0.25\sin(\pi \cdot t)} + x_2^2 + x_3^2 - 1 = 0 \right\}.$$

The right hand side f is calculated so that the exact solution is given by

$$u(x,t) = \sin(\sqrt{6}t)x_1 x_2.$$

Let $\{\mathcal{T}_h^i\}_{i=0}^k$ and $\{\tau_i\}_{i=0}^k$ be a sequence of meshes of a surface by uniform refinement and a sequence of time steps respectively. The uniform refinement is such that

$h_i \approx \frac{1}{2}h_{i-1}$, and since we have the same rate of convergence in τ and h, we choose $\tau_i = \frac{1}{2}\tau_{i-1}$. In order to satisfy the CFL condition, we start with $\tau_0 = 5 \times 10^{-2}$. Then the experimental order of convergence (EOC) is given by

$$EOC_i = \frac{\log \frac{E_{i-1}}{E_i}}{\log 2}, \qquad i = 1, \cdots k,$$

where E_i presents the error for the grid size h_i and the time step size τ_i. We solve on the time interval $0 \le t \le 1$. In Table 1 we give the errors and the corresponding EOCs in the following norms:

$$L^\infty\left(L^2\right): \quad \max_{0\le n\le N} \|u_h^n - u(t_n)\|_{L^2(\Gamma(t_n))},$$
$$L^\infty\left(H^1\right): \quad \max_{0\le n\le N} \left\|\nabla_{\Gamma(t_n)}u_h^n - \nabla_{\Gamma(t_n)}u(t_n)\right\|_{L^2(\Gamma(t_n))},$$
$$L^\infty\left(L^2\right)^\bullet: \quad \max_{0\le n\le N} \|\partial_h^\bullet u_h^n - \partial^\bullet u(t_n)\|_{L^2(\Gamma(t_n))}.$$

TABLE 1. Errors and observed orders of convergence for Example 1.

Level	Dof	$L^\infty\left(L^2\right)$	EOC	$L^\infty\left(H^1\right)$	EOC	$L^\infty\left(L^2\right)^\bullet$	EOC
0	318	$2.05 \cdot 10^{-2}$	–	$1.77 \cdot 10^{-1}$	–	$2.26 \cdot 10^{-2}$	–
1	1266	$5.27 \cdot 10^{-3}$	1.958	$8.91 \cdot 10^{-2}$	0.996	$5.88 \cdot 10^{-3}$	1.942
2	5058	$1.34 \cdot 10^{-3}$	1.970	$4.27 \cdot 10^{-2}$	1.058	$1.47 \cdot 10^{-3}$	2.001
3	20226	$3.35 \cdot 10^{-4}$	2.003	$2.18 \cdot 10^{-2}$	0.968	$3.74 \cdot 10^{-4}$	1.971
4	80898	$8.35 \cdot 10^{-5}$	2.006	$1.11 \cdot 10^{-2}$	0.969	$9.50 \cdot 10^{-5}$	1.980
5	323586	$2.08 \cdot 10^{-5}$	1.999	$5.58 \cdot 10^{-3}$	0.999	$2.37 \cdot 10^{-5}$	1.999

Example 2. We choose a time-dependent surface of the form

$$\Gamma(t) := \left\{ x_1 + \max(0, x_1)t, \frac{g(x,t)x_2}{\sqrt{x_2^2 + x_3^2}}, \frac{g(x,t)x_3}{\sqrt{x_2^2 + x_3^2}} : x \in \Gamma(0) = S^2 \right\}, \tag{11.2}$$

$$g(x,t) = e^{-2t}\sqrt{x_2^2 + x_3^2} + (1 - e^{-2t})\left(\left(1 - x_1^2\right)\left(x_1^2 + 0.05\right) + x_1^2\sqrt{(1 - x_1^2)}\right).$$

We consider the wave equation (11.1) posed on the above surface on the time interval $[0,3]$, with right hand side $f = 0$ and initial data $u(x,0) = e^{-5|x-x_0|^2} + e^{-5|x-x_1|^2}$, where $x_0 = (1,0,0), x_1 = (-1,0,0)$, and $\partial^\bullet u(x,0) = 0$. The surface evolves from an initially spherical shape at $t = 0$ to a "baseball bat" like shape. Simultaneously we observe a wave traveling from the right to the left and another from the left to the right. They superimpose for a short time and cross paths without any dissipation. We choose the time step $\tau = 5 \times 10^{-4}$, in order to satisfy the CFL condition (5.4). Figure 1 shows snapshots of the discrete solution at time $t = 0, 0.8, 1.2, 1.8, 2.2, 3$ from the left to the right.

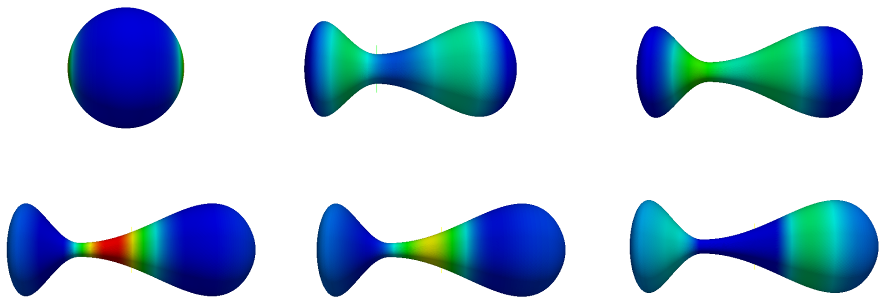

FIGURE 1. Snapshots of the discrete solution of the wave equation on a time-dependent surface of the form (11.2).

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