Full discretization of wave equations on evolving surfaces

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Abstract

A linear wave equation on a moving surface is discretized in space by evolving surface finite elements and in time by the implicit midpoint rule. We study stability and convergence of the fully discrete scheme in the natural time-dependent norms. Under suitable assumptions we prove optimal-order error estimates.

Keywords. Wave equation, evolving surface finite elements, implicit midpoint rule, error analysis.

1 Introduction

The numerical study of partial differential equations on moving surfaces has attracted considerable attention over the last years.

In [1], the authors considered a wave equation on a moving surface, which is derived from Hamilton's principle, and presented a fully discrete variational integrator that is stable under a CFL condition. To overcome the time step restriction due to the CFL condition, we investigate in this paper the implicit midpoint rule for the time discretization. We prove the unconditional stability of the fully discrete scheme. Furthermore, under suitable regularity conditions, we show second order of the error measured in the L^2 norm over the time-dependent surface for displacements and their material derivatives, and first order for the L^2 norm of the error in the surface gradient of the displacements, uniformly on bounded time intervals.

2 The wave equation on evolving surfaces

Let $\Gamma(t)$, $t \in [0,T]$, be a smoothly evolving family of smooth m-dimensional compact closed hypersurfaces in $\mathbb{R}^{m+1}$ without boundary, with unit outward pointing normal ν. We let $v(x(t),t)$ denote the given *velocity* of the surface $\Gamma(t)$, i.e., $\dot{x}(t) = v(x(t),t)$.

We consider the linear wave equation on evolving surfaces (c.f [1])

$$\partial^\bullet \partial^\bullet u + \partial^\bullet u \; \nabla_\Gamma \cdot v - \Delta_\Gamma u = 0 \qquad (1)$$

with given initial data $u(0) \in H^2(\Gamma_0)$ and $\partial^\bullet u(0) \in H^1(\Gamma_0)$.

We let $\partial^\bullet u$ denote the *material derivative* $\partial^\bullet u = \frac{\partial u}{\partial t} + v \cdot \nabla u$. The tangential gradient is given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu \, \nu$. The Laplace-Beltrami operator is the tangential divergence of the tangential gradient $\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u = \sum_{j=1}^{d+1} (\nabla_\Gamma)_j (\nabla_\Gamma)_j u$.

2.1 Weak formulation

A weak form of (1) reads:

$$\frac{d}{dt} \int_\Gamma \partial^\bullet u \varphi + \int_\Gamma \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_\Gamma \partial^\bullet u \partial^\bullet \varphi \qquad (2)$$

for all smooth $\varphi : \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\} \to \mathbb{R}$.

2.2 The evolving surface finite element method

Following [2], the smooth surface $\Gamma(t)$ is interpolated at nodes $a_i(t) \in \Gamma(t)$ $(i = 1, \dots, m)$ by a discrete polygonal surface $\Gamma_h(t)$, where h denotes the grid size. These nodes move with velocity $da_i(t)/dt = v(a_i(t), t)$. The discrete surface $\Gamma_h(t) = \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t)$ is the union of d-dimensional simplices $E(t)$ that is assumed to form an admissible triangulation $\mathcal{T}_h(t)$; see [2] for details. We define for each $t \in [0,T]$ the finite element space $S_h(t) = \{\phi_h \in C^0(\Gamma_h(t)) \; : \; \phi_h|_E \text{ linear affine for each } E \in \mathcal{T}_h(t)\}$. The moving nodal basis $\{\chi_i\}_{i=1}^m$ of $S_h(t)$ are determined by $\chi_i(a_j(t),t) = \delta_{ij}$ for all j, so we have

$$S_h(t) = \text{span}\{\chi_1(\cdot,t), \dots, \chi_m(\cdot,t)\}.$$

The discrete velocity V_h of the discrete surface $\Gamma_h(t)$ is the piecewise linear interpolant of v: $V_h(x,t) = \sum_{j=1}^N v(a_j(t),t)\chi_j(x,t), \quad x \in \Gamma_h(t)$. Then the discrete material derivative on $\Gamma_h(t)$ is given by $\partial_h^\bullet \phi_h = \frac{\partial \phi_h}{\partial t} + V_h \cdot \nabla \phi_h$. The construction is such that

$$\partial_h^\bullet \chi_j = 0. \qquad (3)$$

The discrete surface gradient is defined piecewise as $\nabla_{\Gamma_h} g = \nabla g - \nabla g \cdot \nu_h \nu_h$, where ν_h denotes the normal to the discrete surface.

2.3 The spatial semi-discretization

The spatial semi-discretization of the wave equation reads as follows: Find $u_h(\cdot,t) \in S_h(t)$ such that for all temporally smooth ϕ_h with $\phi_h(\cdot,t) \in S_h(t)$ and for all $t \in [0,T]$,

$$\frac{d}{dt} \int_{\Gamma_h} \partial_h^\bullet U_h \, \phi_h + \int_{\Gamma_h} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h = \int_{\Gamma_h} \partial_h^\bullet U_h \partial_h^\bullet \phi_h. \qquad (4)$$

2.4 The Hamiltonian ODE system

We denote the discrete solution $U_h(\cdot,t) = \sum_{j=1}^m q_j(t)\chi_j(\cdot,t) \ \in S_h(t)$ and define $q(t) \in \mathbb{R}^m$ as the nodal vector with entries $q_j(t) = U_h(a_j(t),t)$. Then by the transport property (3), we have $\partial_h^\bullet U_h(\cdot,t) = \sum_{j=1}^m \dot{q}_j(t)\chi_j(\cdot,t) \ \in S_h(t)$. The evolving mass matrix $M(t)$ and the stiffness matrix $A(t)$ are defined by $M(t)_{ij} = \int_{\Gamma_h(t)} \chi_i(t)\chi_j(t)$, $\quad A(t)_{ij} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\chi_i(t)\cdot\nabla_{\Gamma_h(t)}\chi_j(t)$. The mass matrix is symmetric and positive definite. The stiffness matrix is symmetric and only positive semidefinite. Then (4) can be written as

$$\frac{d}{dt}\left(M(t)\dot{q}(t)\right) + A(t)q(t) = 0. \tag{5}$$

By introducing the conjugate *momenta* $p(t) = M(t)\dot{q}(t)$, we reformulate (5) in the variable $y(t) = (p(t),q(t))^{\mathrm{T}}$ as *Hamilton's equations* ($\dot{} = \frac{d}{dt}$)

$$\dot{y}(t) = J^{-1}H(t)y(t), \tag{6}$$

with

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad H(t) = \begin{pmatrix} M(t)^{-1} & 0 \\ 0 & A(t) \end{pmatrix}.$$

3 The implicit midpoint rule

For the numerical integration of the above Hamilton's equations (6) we consider the implicit midpoint rule with time step size $\tau > 0$ given by

$$Y_{n+\frac{1}{2}} = y_n + \frac{\tau}{2}J^{-1}H_{n+\frac{1}{2}}Y_{n+\frac{1}{2}} \tag{7a}$$

$$y_{n+1} = y_n + \tau J^{-1}H_{n+\frac{1}{2}}Y_{n+\frac{1}{2}} \tag{7b}$$

3.1 Defects and errors

Let $\widetilde{y}_n$ and $\widetilde{Y}_{n+\frac{1}{2}}$ be reference values that we want to compare with y_n and $Y_{n+\frac{1}{2}}$ respectively. Inserted into (7) they yield defects in

$$\widetilde{Y}_{n+\frac{1}{2}} = \widetilde{y}_n + \frac{\tau}{2}J^{-1}H_{n+\frac{1}{2}}\widetilde{Y}_{n+\frac{1}{2}} + \Delta_{n+\frac{1}{2}} \tag{8a}$$

$$\widetilde{y}_{n+1} = \widetilde{y}_n + \tau J^{-1}H_{n+\frac{1}{2}}\widetilde{Y}_{n+\frac{1}{2}} + \delta_{n+1} \tag{8b}$$

3.2 Stability

We define the symmetric positive definite matrix $\hat{H}(t)$ as

$$\hat{H}(t) = \begin{pmatrix} M(t)^{-1} & 0 \\ 0 & A(t)+M(t) \end{pmatrix},$$

and therewith the time-dependent energy norm:

$$\|y\|_t^2 = \left\langle y \,\middle|\, \hat{H}(t) \,\middle|\, y \right\rangle = y^{\mathrm{T}}\hat{H}(t)y. \tag{9}$$

Lemma 3.1 *The error is bounded for $0 \le t_n \le T$ by*

$$\|y_n - \widetilde{y}_n\|_{t_n} \le C\left\|\Delta_{\frac{1}{2}}\right\|_{t_0} + C\left\|\delta_n - \Delta_{n-\frac{1}{2}}\right\|_{t_n} + C\sum_{j=1}^{n-1}\left\|\delta_j + \Delta_{j+\frac{1}{2}} - \Delta_{j-\frac{1}{2}}\right\|_{t_j}.$$

The constant C is independent of h, τ and n.

4 Error bound for the full discretization

For $U_h : \Gamma_h \to \mathbb{R}$ we define the extension or the lift onto Γ by $U_h^l(a(x)) = U_h(x)$, where $a(x) \in \Gamma$ is the orthogonal projection of $x \in \Gamma_h$. We consider the lifts of the fully discrete numerical solution and its numerical material derivative given by $u_h^n := (U_h^n)^l = \sum_{j=1}^m q_j^n \chi_j^l(t_n)$, $\partial_h^\bullet u_h^n := (\partial_h^\bullet U_h^n)^l = \sum_{j=1}^m \left(M(t_n)^{-1}p_n\right)_j \chi_j^l(t_n)$, which are lifted finite element functions defined on the surface $\Gamma(t_n)$. This will be compared with the solution $u(t_n)$ of the wave equation (1) and its material derivative $\partial^\bullet u(t_n)$.

We rewrite the error by subtracting and adding the Ritz map applied to the exact solution,

$$u_h^n - u(t_n) = u_h^n - \mathcal{P}_h(t_n)u(t_n) + \mathcal{P}_h(t_n)u(t_n) - u(t_n),$$

where $\mathcal{P}_h(t)$ is the Ritz map defined in [1]. Then we are able to prove our main result:

Theorem 4.1 *Let u be a sufficiently smooth solution of the wave equation* (1) *and assume that the discrete initial data satisfy*

$$\left\|u_h^0 - (\mathcal{P}_h u)(0)\right\|_{L^2(\Gamma_0)} + \left\|\nabla_{\Gamma_0}u_h^0 - \nabla_{\Gamma_0}(\mathcal{P}_h u)(0)\right\|_{L^2(\Gamma_0)} + \left\|\partial_h^\bullet u_h^0 - \partial_h^\bullet(\mathcal{P}_h u)(0)\right\|_{L^2(\Gamma_0)} \le C_0 h^2.$$

Then, there exist $h_0 > 0$ and $\tau_0 > 0$ such that for $h \le h_0$ and $\tau \le \tau_0$, the following error bound holds for $0 \le t_n = n\tau \le T$:

$$\|u_h^n - u(t_n)\|_{L^2(\Gamma_n)} + h\,\|\nabla_{\Gamma_n}u_h^n - \nabla_{\Gamma_n}u(t_n)\|_{L^2(\Gamma_n)} + \|\partial_h^\bullet u_h^n - \partial^\bullet u(t_n)\|_{L^2(\Gamma_n)} \le C\big(h^2 + \tau^2\big).$$

The constant C is independent of h, τ, and n subject to the stated conditions.

References

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