

# Backward difference time discretization of parabolic differential equations on evolving surfaces

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A linear parabolic differential equation on a moving surface is discretized in space by evolving surface finite elements and in time by backward difference formulas (BDF). Using results from Dahlquist's G-stability theory and Nevanlinna & Odeh's multiplier technique together with properties of the spatial semi-discretization, stability of the full discretization is proven for the BDF methods up to order 5 and optimal-order convergence is shown. Numerical experiments illustrate the behaviour of the fully discrete method.

*Keywords:* parabolic PDE, evolving surface finite element method, backward difference formula, G-stability, multiplier technique, energy estimates

## 1. Introduction

This paper considers the time discretization by backward difference formulas (BDF) of the basic linear parabolic PDE on a closed moving surface,

$$\dot{u} + u \nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u = f \quad \text{on } \Gamma(t), \quad (1.1)$$

where the moving surface  $\Gamma(t)$  with velocity  $v(x, t)$  is given and the solution  $u(x, t)$  ( $x \in \Gamma(t), 0 \leq t \leq T$ ) is to be computed. Based on the weak formulation of the equation, Dziuk & Elliott [3, 4] developed and analysed a spatial discretization of (1.1) with piecewise linear finite elements on the evolving surface  $\Gamma(t)$ , which is approximated by a moving discrete surface  $\Gamma_h(t)$ . The semi-discretization in space of (1.1) with piecewise linear surface finite elements leads to a system of ordinary differential equations of the form

$$\frac{d}{dt} (\mathbf{M}(t) \mathbf{u}(t)) + \mathbf{A}(t) \mathbf{u}(t) = \mathbf{f}(t), \quad (1.2)$$

where  $\mathbf{M}(t)$  is the evolving mass matrix and  $\mathbf{A}(t)$  is the evolving stiffness matrix.  $\mathbf{u}(t)$  denotes the coefficient vector of the spatially discrete solution and  $\mathbf{f}(t)$  is the discrete right-hand side.

A full discretization with backward Euler time stepping was analyzed in [5]. In [6] higher-order implicit Runge–Kutta time discretizations were studied in a framework that will be used also here. As in that previous paper, the key is a stability estimate in the natural time-dependent norms for the time discretization. For the BDF methods, this is proved using

results from Dahlquist's G-stability theory [2] and Nevanlinna & Odeh's multiplier technique [9], both of which were originally devised for studying the stability of linear multistep methods for contractive nonlinear ordinary differential equations. Apart from first remarks in [9], it seems that these powerful techniques have so far not been used for the analysis of time discretizations of parabolic differential equations.

The paper is organized as follows: In Section 2 we recall the basic notation for PDEs on evolving surfaces and describe the spatial discretization of (1.1) using the evolving surface finite element method of [3]. This leads to the ODE system (1.2) for which we recall basic properties proved in [6]. In Section 3 this system is discretized in time by the BDF method. In Sections 4 and 5 we derive stability estimates and error bounds. Section 6 presents some numerical examples.

## 2. Discretization of parabolic equations on evolving surfaces

### 2.1 Basic notation

Let  $\Gamma(t)$ ,  $t \in [0, T]$ , be a smoothly evolving family of smooth  $d$ -dimensional compact hypersurfaces in  $\mathbb{R}^{d+1}$  without boundary, and let  $v(x, t)$ , for  $x \in \Gamma(t)$  and  $t \in [0, T]$ , denote the given velocity of the surface. The conservation of a scalar quantity  $u(x, t)$  with a linear diffusive flux on  $\Gamma(t)$  can be modelled by the linear parabolic partial differential equation (see [3])

$$\dot{u} + u \nabla_{\Gamma} \cdot v - \Delta_{\Gamma} u = f \text{ on } \Gamma \quad (2.1)$$

together with the initial condition  $u = u_0$  on  $\Gamma_0 = \Gamma(0)$ . By a dot we denote the material derivative

$$\dot{u} = \frac{\partial u}{\partial t} + v \cdot \nabla u, \quad (2.2)$$

where  $a \cdot b = \sum_{j=1}^{d+1} a_j b_j$  for vectors  $a$  and  $b$  in  $\mathbb{R}^{d+1}$ , and  $\nabla$  denotes the usual  $d+1$ -dimensional gradient. The material derivative  $\dot{u}$  only depends on the values of the function  $u$  on the space-time surface

$$G_T = \bigcup_{t \in (0, T)} \Gamma(t) \times \{t\}.$$

By  $\nabla_{\Gamma}$  we denote the surface or tangential gradient on the surface  $\Gamma$ . This gradient is the projection to the tangent space of the  $d+1$ -dimensional gradient. For a smooth function  $g$  defined in a neighbourhood of  $\Gamma$  we define

$$\nabla_{\Gamma} g = \nabla g - \nabla g \cdot n n,$$

where  $n$  is a normal vector field to  $\Gamma$ . The tangential gradient only depends on the values of  $g$  on the surface  $\Gamma$  and is independent of the extension. For a more detailed discussion we refer to [7] and [3]. The Laplace-Beltrami operator on  $\Gamma$  is the tangential divergence of the tangential gradient:

$$\Delta_{\Gamma} g = \nabla_{\Gamma} \cdot \nabla_{\Gamma} g = \sum_{j=1}^{d+1} (\nabla_{\Gamma})_j (\nabla_{\Gamma})_j g.$$

## 2.2 Weak formulation

A weak form of (2.1) reads

$$\frac{d}{dt} \int_{\Gamma} u \varphi + \int_{\Gamma} \nabla_{\Gamma} u \cdot \nabla_{\Gamma} \varphi = \int_{\Gamma} u \dot{\varphi} + \int_{\Gamma} f \varphi \quad (2.3)$$

for all smooth  $\varphi : G_T \rightarrow \mathbb{R}$ . This is obtained using the Leibniz formula or transport theorem on surfaces,

$$\frac{d}{dt} \int_{\Gamma} g = \int_{\Gamma} \dot{g} + g \nabla_{\Gamma} \cdot v.$$

## 2.3 The evolving surface finite element method [3]

The weak form (2.3) serves as the starting point for a spatial finite element discretization of the PDE after discretizing the evolving surface. The smooth surface  $\Gamma(t)$  is interpolated at nodes  $a_i(t) \in \Gamma(t)$  ( $i = 1, \dots, N$ ) by a discrete polygonal surface  $\Gamma_h(t)$ , where  $h$  denotes the grid size. These nodes move with velocity  $da_i(t)/dt = v(a_i(t), t)$ . The discrete surface

$$\Gamma_h(t) = \bigcup_{E(t) \in \mathcal{T}(t)} E(t)$$

is the union of  $d$ -dimensional simplices  $E(t)$  that is assumed to form an admissible triangulation  $\mathcal{T}(t)$ ; see [3] for details. The finite element space on the discrete surface  $\Gamma_h(t)$  is chosen as

$$S_h(t) = \{w_h \in C^0(\Gamma_h(t)) : w_h|_E \in \mathbb{P}_1 \text{ for all } E \in \mathcal{T}(t)\}.$$

Let  $\phi_j(\cdot, t)$  ( $j = 1, \dots, N$ ) be the nodal basis of  $S_h(t)$ , so that  $\phi_j(a_i(t), t) = \delta_{ji}$ . The discrete surface moves with the piecewise linear velocity

$$v_h(x, t) = \sum_{j=1}^N v(a_j(t), t) \phi_j(x, t).$$

The discrete material derivative on the discrete evolving surface then becomes

$$\dot{u}_h = \frac{\partial u_h}{\partial t} + v_h \cdot \nabla u_h. \quad (2.4)$$

We use the same dot notation as for the continuous material derivative, since it will be clear from the context which material derivative is meant. The construction is such that the (discrete) material derivative of the basis functions vanishes:

$$\dot{\phi}_j = 0. \quad (2.5)$$

The discrete surface gradient is defined piecewise as

$$\nabla_{\Gamma_h} g = \nabla g - \nabla g \cdot n_h n_h,$$

where  $n_h$  denotes the normal to the discrete surface.

The spatial semi-discretization of the parabolic PDE now reads as follows: For a given initial value  $u_h(\cdot, 0) = u_{h0} \in S_h(0)$ , find  $u_h(\cdot, t) \in S_h(t)$  such that

$$\frac{d}{dt} \int_{\Gamma_h} u_h \varphi_h + \int_{\Gamma_h} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \varphi_h = \int_{\Gamma_h} u_h \dot{\varphi}_h + \int_{\Gamma_h} f^{-l} \varphi_h \quad \forall \varphi_h \in S_h(t). \quad (2.6)$$

Here  $f^{-l} : \Gamma_h \rightarrow \mathbb{R}$  denotes the extension of the function  $f : \Gamma \rightarrow \mathbb{R}$  constantly in normal direction to  $\Gamma$ . For a function  $f_h : \Gamma_h \rightarrow \mathbb{R}$ , we let  $f_h^l : \Gamma \rightarrow \mathbb{R}$  be such that  $(f_h^l)^{-l} = f_h$ .

Under suitable regularity assumptions an error estimate between the continuous solution  $u$  and the spatially discrete solution  $u_h$  was proved in [3]:

$$\sup_{0 \leq t \leq T} \|u(\cdot, t) - u_h^l(\cdot, t)\|_{L_2(\Gamma(t))}^2 + \int_0^T \|\nabla_{\Gamma(t)}(u(\cdot, t) - u_h^l(\cdot, t))\|_{L_2(\Gamma(t))}^2 dt \leq ch^2.$$

An optimal error estimate in the  $L_2$ -norm is derived in [4]:

$$\sup_{0 \leq t \leq T} \|u(\cdot, t) - u_h^l(\cdot, t)\|_{L_2(\Gamma(t))} \leq ch^2.$$

While these error bounds for the spatial semi-discretization are of independent interest, they will not be used in our derivation of the error bounds for the fully discrete method including time discretization.

#### 2.4 The ODE system

The discrete form (2.6) of the PDE (2.1) is a system of ODEs. The evolving mass matrix  $\mathbf{M}(t)$  and the stiffness matrix  $\mathbf{A}(t)$  are defined by

$$\mathbf{M}(t)_{ij} = \int_{\Gamma_h(t)} \phi_i(\cdot, t) \phi_j(\cdot, t), \quad \mathbf{A}(t)_{ij} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)} \phi_i(\cdot, t) \cdot \nabla_{\Gamma_h(t)} \phi_j(\cdot, t)$$

for  $i, j = 1, \dots, N$ . The mass matrix is symmetric and positive definite. The stiffness matrix is symmetric and only positive semidefinite, because we consider closed surfaces. We denote the discrete solution by

$$u_h(x, t) = \sum_{j=1}^N u_j(t) \phi_j(x, t)$$

and define  $\mathbf{u}(t) \in \mathbb{R}^N$  as the column vector with entries  $u_j(t)$ . Then (2.6) can be written as [3, 6]

$$\frac{d}{dt} (\mathbf{M}(t) \mathbf{u}(t)) + \mathbf{A}(t) \mathbf{u}(t) = \mathbf{f}(t), \quad (2.7)$$

where we let  $\mathbf{f} = (f_j) \in \mathbb{R}^N$  with  $f_j = \int_{\Gamma_h} f^{-l} \phi_j$ .

We work with the norm

$$\|\mathbf{w}\|_t^2 = \langle \mathbf{w} | \mathbf{M}(t) | \mathbf{w} \rangle = \mathbf{w}^T \mathbf{M}(t) \mathbf{w}, \quad \mathbf{w} \in \mathbb{R}^N,$$

and the semi-norm

$$\|\mathbf{w}\|_t^2 = \langle \mathbf{w} | \mathbf{A}(t) | \mathbf{w} \rangle = \mathbf{w}^T \mathbf{A}(t) \mathbf{w}, \quad \mathbf{w} \in \mathbb{R}^N.$$

Note that for a finite-element function  $w_h = \sum_{j=1}^N w_j \phi_j(\cdot, t) \in S_h(t)$  with the vector of nodal values  $\mathbf{w} = (w_j) \in \mathbb{R}^N$  we have

$$|\mathbf{w}|_t = \|w_h\|_{L_2(\Gamma_h(t))}, \quad \|\mathbf{w}\|_t = \|\nabla_{\Gamma_h(t)} w_h\|_{L_2(\Gamma_h(t))}. \quad (2.8)$$

The following result from [6] provides basic estimates.

LEMMA 2.1 There are constants  $\mu, \kappa$  (independent of the discretization parameter  $h$  and the length of the time interval  $T$ ) such that

$$\mathbf{w}^T (\mathbf{M}(s) - \mathbf{M}(t)) \mathbf{z} \leq (e^{\mu(s-t)} - 1) |\mathbf{w}|_t |\mathbf{z}|_t \quad (2.9)$$

$$\mathbf{w}^T (\mathbf{A}(s) - \mathbf{A}(t)) \mathbf{z} \leq (e^{\kappa(s-t)} - 1) \|\mathbf{w}\|_t \|\mathbf{z}\|_t \quad (2.10)$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$  and  $0 \leq t \leq s \leq T$ .

We will apply this lemma with  $s$  close to  $t$ . Note that then  $e^{\mu(s-t)} - 1 \leq 2\mu(s-t)$  and  $e^{\kappa(s-t)} - 1 \leq 2\kappa(s-t)$ .

Apart from the fact that  $\mathbf{M}(t)$  and  $\mathbf{A}(t)$  are symmetric positive semi-definite, the inequalities (2.9)–(2.10) are the only properties of the evolving-surface finite-element equations (2.6) that will be used in the stability analysis of their time discretizations. In the derivation of error bounds of the full discretization we will further use the analogous bound for the  $\mathbf{M}^{-1}$ -norm.

LEMMA 2.2 With  $\mu$  of Lemma 2.1,

$$\mathbf{w}^T (\mathbf{M}(s)^{-1} - \mathbf{M}(t)^{-1}) \mathbf{z} \leq (e^{\mu(s-t)} - 1) (\mathbf{w}^T \mathbf{M}(t)^{-1} \mathbf{w})^{1/2} (\mathbf{z}^T \mathbf{M}(t)^{-1} \mathbf{z})^{1/2} \quad (2.11)$$

for all  $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$  and  $0 \leq t \leq s \leq T$ .

*Proof.* We work with the dual basis of  $S_h(t)$  defined by

$$(\psi_i(\cdot, t))_{i=1}^N = \mathbf{M}(t)^{-1} (\phi_j(\cdot, t))_{j=1}^N,$$

which has the property that

$$\int_{\Gamma_h} \psi_i \phi_j = \delta_{ij} \quad \text{and} \quad \int_{\Gamma_h} \psi_i \psi_j = \mathbf{M}^{-1}|_{i,j}.$$

The Leibniz formula gives us

$$0 = \frac{d}{dt} \int_{\Gamma_h} \psi_i \phi_j = \int_{\Gamma_h} \dot{\psi}_i \phi_j + \psi_i \dot{\phi}_j + \psi_i \phi_j \nabla_{\Gamma} \cdot v_h,$$

and since  $\dot{\phi}_j = 0$ , it follows that

$$\int_{\Gamma_h} \dot{\psi}_i \phi_j = - \int_{\Gamma_h} \psi_i \phi_j \nabla_{\Gamma} \cdot v_h \quad \text{for all } i, j = 1, \dots, N.$$

This yields that for all  $z_h(\cdot, t) \in S_h(t)$  and  $w_h(x, t) = \sum_{i=1}^N w_i \psi_i(x, t)$  with time-independent coefficients  $w_i$  we have

$$\int_{\Gamma_h} \dot{w}_h z_h = - \int_{\Gamma_h} w_h z_h \nabla_{\Gamma} \cdot v_h. \quad (2.12)$$

For  $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$ , we define  $w_h(x, t) = \sum_{j=1}^N w_j \psi_j(x, t)$  and  $z_h(x, t) = \sum_{j=1}^N z_j \psi_j(x, t)$ . Using the Leibniz formula in the third equality and (2.12) in the fourth equality we obtain

$$\begin{aligned}
\mathbf{w}^T (\mathbf{M}(s)^{-1} - \mathbf{M}(t)^{-1}) \mathbf{z} &= \int_{\Gamma_h(s)} w_h(\cdot, s) z_h(\cdot, s) - \int_{\Gamma_h(t)} w_h(\cdot, t) z_h(\cdot, t) \\
&= \int_t^s \frac{d}{d\sigma} \int_{\Gamma_h(\sigma)} w_h(\cdot, \sigma) z_h(\cdot, \sigma) \, d\sigma \\
&= \int_t^s \int_{\Gamma_h(\sigma)} (\dot{w}_h z_h + w_h \dot{z}_h + w_h z_h \nabla_{\Gamma_h(\sigma)} \cdot v_h) \, d\sigma \\
&= \int_t^s \int_{\Gamma_h(\sigma)} -w_h z_h \nabla_{\Gamma_h(\sigma)} \cdot v_h \, d\sigma \\
&\leq \mu \int_t^s \|w_h\|_{L_2(\Gamma_h(\sigma))} \|z_h\|_{L_2(\Gamma_h(\sigma))} \, d\sigma \\
&= \mu \int_t^s |\mathbf{w}|_{\mathbf{M}(\sigma)^{-1}} |\mathbf{z}|_{\mathbf{M}(\sigma)^{-1}} \, d\sigma,
\end{aligned}$$

where we use that  $\max_{\sigma \in [t, s]} \|\nabla_{\Gamma_h(\sigma)} \cdot v_h\|_{L^\infty(\Gamma_h(\sigma))}$  is bounded by a constant  $\mu$  independent of  $h$  (the same constant  $\mu$  as appears in Lemma 2.1) and we denote  $|\mathbf{w}|_{\mathbf{M}(\sigma)^{-1}} = (\mathbf{w}^T \mathbf{M}(\sigma)^{-1} \mathbf{w})^{1/2}$ . With  $\mathbf{z} = \mathbf{w}$ , this inequality implies

$$|\mathbf{w}|_{\mathbf{M}(s)^{-1}}^2 \leq |\mathbf{w}|_{\mathbf{M}(t)^{-1}}^2 + \mu \int_t^s |\mathbf{w}|_{\mathbf{M}(\sigma)^{-1}}^2 \, d\sigma, \quad 0 \leq t \leq s \leq T,$$

and hence the Gronwall inequality yields

$$|\mathbf{w}|_{\mathbf{M}(s)^{-1}}^2 \leq e^{\mu(s-t)} |\mathbf{w}|_{\mathbf{M}(t)^{-1}}^2.$$

The result then follows by using this estimate for  $|\mathbf{w}|_{\mathbf{M}(\sigma)^{-1}}$  and  $|\mathbf{z}|_{\mathbf{M}(\sigma)^{-1}}$  in the integral of the last line of the inequality for  $\mathbf{w}^T (\mathbf{M}(s)^{-1} - \mathbf{M}(t)^{-1}) \mathbf{z}$ .  $\square$

### 3. BDF time discretization

#### 3.1 Formulation of the method

For the numerical integration of (2.7) we consider the  $k$ -step BDF method with step size  $\tau > 0$  given by

$$\frac{1}{\tau} \sum_{j=0}^k \delta_j \mathbf{M}(t_{n-j}) \mathbf{u}_{n-j} + \mathbf{A}(t_n) \mathbf{u}_n = \mathbf{f}(t_n), \quad n \geq k, \quad (3.1)$$

with given starting values  $\mathbf{u}_0, \dots, \mathbf{u}_{k-1}$ . The method coefficients  $\delta_j$  are determined from the relation

$$\delta(\zeta) = \sum_{j=0}^k \delta_j \zeta^k = \sum_{\ell=1}^k \frac{1}{\ell} (1 - \zeta)^\ell. \quad (3.2)$$

The method is known to have order  $k$  and to be 0-stable for  $k \leq 6$ .

### 3.2 Defects and errors

The solution of (2.7) satisfies the BDF relation up to a defect  $\mathbf{d}_n$ , which is the error of numerical differentiation:

$$\frac{1}{\tau} \sum_{j=0}^k \delta_j \mathbf{M}(t_{n-j}) \mathbf{u}(t_{n-j}) + \mathbf{A}(t_n) \mathbf{u}(t_n) = \mathbf{f}(t_n) - \mathbf{d}_n. \quad (3.3)$$

For smooth solutions we have by Taylor expansion (in suitable norms!)  $\mathbf{d}_n = \mathcal{O}(\tau^k)$ . The error

$$\mathbf{e}_n = \mathbf{u}_n - \mathbf{u}(t_n) \quad (3.4)$$

then satisfies the following equation:

$$\frac{1}{\tau} \sum_{j=0}^k \delta_j \mathbf{M}(t_{n-j}) \mathbf{e}_{n-j} + \mathbf{A}(t_n) \mathbf{e}_n = \mathbf{d}_n, \quad n \geq k. \quad (3.5)$$

### 3.3 Basic results from Dahlquist (1978) and Nevanlinna & Odeh (1981)

We will use the following result from Dahlquist's G-stability theory.

LEMMA 3.1 (Dahlquist [2]; see also [1], [8, Sect.V.6]) Let  $\delta(\zeta)$  and  $\mu(\zeta)$  be polynomials of degree at most  $k$  (at least one of them of exact degree  $k$ ) that have no common divisor. Let  $\langle \cdot, \cdot \rangle$  be an inner product on  $\mathbb{R}^N$  with associated norm  $|\cdot|$ . If

$$\operatorname{Re} \frac{\delta(\zeta)}{\mu(\zeta)} > 0 \quad \text{for } |\zeta| < 1,$$

then there exists a symmetric positive definite matrix  $G = (g_{ij}) \in \mathbb{R}^{k \times k}$  and real  $\gamma_0, \dots, \gamma_k$  such that for all  $\mathbf{v}_0, \dots, \mathbf{v}_k \in \mathbb{R}^N$

$$\left\langle \sum_{i=0}^k \delta_i \mathbf{v}_{k-i}, \sum_{j=0}^k \mu_j \mathbf{v}_{k-j} \right\rangle = \sum_{i,j=1}^k g_{ij} \langle \mathbf{v}_i, \mathbf{v}_j \rangle - \sum_{i,j=1}^k g_{ij} \langle \mathbf{v}_{i-1}, \mathbf{v}_{j-1} \rangle + \left| \sum_{i=0}^k \gamma_i \mathbf{v}_i \right|^2.$$

In combination with the preceding result for  $\mu(\zeta) = 1 - \eta\zeta$ , the following property of BDF methods up to order 5 will play a key role in our stability analysis.

LEMMA 3.2 (Nevanlinna & Odeh [9]) If  $k \leq 5$ , then there exists  $0 \leq \eta < 1$  such that for  $\delta(\zeta) = \sum_{\ell=1}^k \frac{1}{\ell} (1 - \zeta)^\ell$ ,

$$\operatorname{Re} \frac{\delta(\zeta)}{1 - \eta\zeta} > 0 \quad \text{for } |\zeta| < 1.$$

The smallest possible value of  $\eta$  is found to be  $\eta = 0, 0, 0.0836, 0.2878, 0.8160$  for  $k = 1, \dots, 5$ , respectively.

## 4. Stability

We will show the following stability result.

LEMMA 4.1 For the  $k$ -step BDF method with  $k \leq 5$ , there exist  $\tau_0 > 0$  depending only on  $\mu$  and  $\kappa$  of Lemma 2.1 and  $C$  depending on  $\mu, \kappa, T$  such that for  $\tau \leq \tau_0$  and  $t_n \leq T$ , the errors  $\mathbf{e}_n$  given by (3.5) are bounded by

$$|\mathbf{e}_n|_{t_n}^2 + \tau \sum_{j=k}^n \|\mathbf{e}_j\|_{t_j}^2 \leq C \tau \sum_{j=k}^n \|\mathbf{d}_j\|_{*,t_j}^2 + C \max_{0 \leq i \leq k-1} |\mathbf{e}_i|_{t_i}^2$$

where  $\|\mathbf{w}\|_{*,t}^2 = \mathbf{w}^\top (\mathbf{A}(t) + \mathbf{M}(t))^{-1} \mathbf{w}$ . In particular,  $\tau_0$  and  $C$  are independent of the spatial grid size  $h$ .

*Proof.* For brevity, we write  $|\cdot|_n$  instead of  $|\cdot|_{t_n}$ , and  $\mathbf{A}_n = \mathbf{A}(t_n)$  and  $\mathbf{M}_n = \mathbf{M}(t_n)$ . We start from (3.5) and rewrite it as

$$\mathbf{M}_n \sum_{j=0}^k \delta_j \mathbf{e}_{n-j} + \tau \mathbf{A}_n \mathbf{e}_n = \tau \mathbf{d}_n + \sum_{j=1}^k \delta_j (\mathbf{M}_n - \mathbf{M}_{n-j}) \mathbf{e}_{n-j}.$$

We use a modified energy estimate. Instead of multiplying scalarly with  $\mathbf{e}_n$  as would be familiar with the implicit Euler method, we proceed similarly to the proof of Theorem 4.1 in [9] and take the Euclidean inner product with  $\mathbf{e}_n - \eta \mathbf{e}_{n-1}$ , for  $n \geq k+1$ . This gives

$$I_n + II_n = III_n + IV_n, \quad (4.1)$$

where

$$\begin{aligned} I_n &= \left\langle \sum_{j=0}^k \delta_j \mathbf{e}_{n-j} \mid \mathbf{M}_n \mid \mathbf{e}_n - \eta \mathbf{e}_{n-1} \right\rangle \\ II_n &= \tau \langle \mathbf{e}_n \mid \mathbf{A}_n \mid \mathbf{e}_n - \eta \mathbf{e}_{n-1} \rangle \\ III_n &= \tau \langle \mathbf{d}_n, \mathbf{e}_n - \eta \mathbf{e}_{n-1} \rangle \\ IV_n &= \sum_{j=1}^k \delta_j \langle \mathbf{e}_{n-j} \mid \mathbf{M}_n - \mathbf{M}_{n-j} \mid \mathbf{e}_n - \eta \mathbf{e}_{n-1} \rangle. \end{aligned}$$

To estimate the first term we introduce the following notation: for

$$\mathbf{E}_n = (\mathbf{e}_n, \dots, \mathbf{e}_{n-k+1})$$

we set

$$|\mathbf{E}_n|_{G,n}^2 = \sum_{i,j=1}^k g_{ij} \langle \mathbf{e}_{n-k+i} \mid \mathbf{M}_n \mid \mathbf{e}_{n-k+j} \rangle,$$

where  $G = (g_{ij})$  is the symmetric positive definite matrix of Lemma 3.1 for the BDF polynomial  $\delta(\zeta)$  of (3.2) and for  $\mu(\zeta) = 1 - \eta\zeta$  with  $\eta$  of Lemma 3.2. This defines a norm on  $\mathbb{R}^{kN}$  such that

$$c_0 \sum_{j=1}^k |\mathbf{e}_{n-k+j}|_n^2 \leq |\mathbf{E}_n|_{G,n}^2 \leq c_1 \sum_{j=1}^k |\mathbf{e}_{n-k+j}|_n^2,$$



where  $c_0$  and  $c_1$  denote the smallest and largest eigenvalue of  $G$ , respectively. Then we obtain by Lemmas 3.1 and 3.2 that

$$|\mathbf{E}_n|_{G,n}^2 - |\mathbf{E}_{n-1}|_{G,n}^2 \leq I_n, \quad n \geq k+1.$$

With (2.9) we have for sufficiently small  $\tau$  ( $\mu\tau \leq 1$ )

$$|\mathbf{E}_{n-1}|_{G,n}^2 - |\mathbf{E}_{n-1}|_{G,n-1}^2 \leq 2\mu\tau \sum_{i,j=1}^k |g_{ij}| |\mathbf{e}_{n-1-k+i}|_{n-1} |\mathbf{e}_{n-1-k+j}|_{n-1}.$$

We can choose  $\gamma > 0$  depending only on  $G$  such that

$$\sum_{i,j=1}^k |g_{ij}| |\mathbf{e}_{n-1-k+i}|_{n-1} |\mathbf{e}_{n-1-k+j}|_{n-1} \leq \gamma |\mathbf{E}_{n-1}|_{G,n-1}^2.$$

With (4.1), this yields the bound

$$|\mathbf{E}_n|_{G,n}^2 - |\mathbf{E}_{n-1}|_{G,n-1}^2 \leq 2\gamma\mu\tau |\mathbf{E}_{n-1}|_{G,n-1}^2 + III_n + IV_n - II_n, \quad n \geq k+1.$$

The term  $II_n/\tau$  is estimated using the Cauchy-Schwarz inequality, Young's inequality and (2.10):

$$\begin{aligned} \langle \mathbf{e}_n | \mathbf{A}_n | \mathbf{e}_n - \eta \mathbf{e}_{n-1} \rangle &= \|\mathbf{e}_n\|_n^2 - \eta \langle \mathbf{e}_n | \mathbf{A}_n | \mathbf{e}_{n-1} \rangle \\ &\geq \|\mathbf{e}_n\|_n^2 - \frac{1}{2}\eta \|\mathbf{e}_n\|_n^2 - \frac{1}{2}\eta \|\mathbf{e}_{n-1}\|_n^2 \\ &\geq \frac{2-\eta}{2} \|\mathbf{e}_n\|_n^2 - \frac{1}{2}\eta(1+2\kappa\tau) \|\mathbf{e}_{n-1}\|_{n-1}^2. \end{aligned}$$

For  $III_n/\tau$  we have, using (2.9) and (2.10) in the last step for sufficiently small  $\tau$ ,

$$\begin{aligned} \langle \mathbf{d}_n, \mathbf{e}_n - \eta \mathbf{e}_{n-1} \rangle &= \left\langle (\mathbf{A}_n + \mathbf{M}_n)^{-1/2} \mathbf{d}_n, (\mathbf{A}_n + \mathbf{M}_n)^{1/2} (\mathbf{e}_n - \eta \mathbf{e}_{n-1}) \right\rangle \\ &\leq \|\mathbf{d}_n\|_{*,n} (\|\mathbf{e}_n - \eta \mathbf{e}_{n-1}\|_n^2 + |\mathbf{e}_n - \eta \mathbf{e}_{n-1}|_n^2)^{1/2} \\ &\leq \frac{1}{1-\eta} \|\mathbf{d}_n\|_{*,n}^2 + \frac{1-\eta}{4} (\|\mathbf{e}_n - \eta \mathbf{e}_{n-1}\|_n^2 + |\mathbf{e}_n - \eta \mathbf{e}_{n-1}|_n^2) \\ &\leq \frac{1}{1-\eta} \|\mathbf{d}_n\|_{*,n}^2 + \frac{1-\eta}{2} \left( (\|\mathbf{e}_n\|_n^2 + |\mathbf{e}_n|_n^2) + \eta^2 (\|\mathbf{e}_{n-1}\|_n^2 + |\mathbf{e}_{n-1}|_n^2) \right) \\ &\leq \frac{1}{1-\eta} \|\mathbf{d}_n\|_{*,n}^2 + \frac{1-\eta}{2} (\|\mathbf{e}_n\|_n^2 + |\mathbf{e}_n|_n^2) \\ &\quad + \frac{1-\eta}{2} \eta^2 \left( (1+2\kappa\tau) \|\mathbf{e}_{n-1}\|_{n-1}^2 + (1+2\mu\tau) |\mathbf{e}_{n-1}|_{n-1}^2 \right). \end{aligned}$$

We estimate the term  $IV_n$  using the Cauchy-Schwarz inequality, Young's inequality and (2.9):

$$\begin{aligned} \langle \mathbf{e}_{n-j} | \mathbf{M}_n - \mathbf{M}_{n-j} | \mathbf{e}_n - \eta \mathbf{e}_{n-1} \rangle &= \langle \mathbf{e}_{n-j} | \mathbf{M}_n - \mathbf{M}_{n-j} | \mathbf{e}_n \rangle - \eta \langle \mathbf{e}_{n-j} | \mathbf{M}_n - \mathbf{M}_{n-j} | \mathbf{e}_{n-1} \rangle \\ &\leq 2\mu_j \tau |\mathbf{e}_{n-j}|_n |\mathbf{e}_n|_n + 2\eta \mu_j \tau |\mathbf{e}_{n-j}|_n |\mathbf{e}_{n-1}|_n \\ &\leq (1+\eta) \mu_j \tau |\mathbf{e}_{n-j}|_n^2 + \mu_j \tau |\mathbf{e}_n|_n^2 + \eta \mu_j \tau |\mathbf{e}_{n-1}|_n^2. \end{aligned}$$

Thus we get by the equivalence of norms

$$IV_n \leq C(\mu, \eta)\tau \left( |\mathbf{E}_n|_{G,n}^2 + |\mathbf{E}_{n-1}|_{G,n-1}^2 \right).$$

Combining the above inequalities and summing up gives, for sufficiently small  $\tau \leq \tau_0$  (which depends only on  $\kappa$  and  $\mu$ ) and for  $n \geq k+1$ ,

$$|\mathbf{E}_n|_{G,n}^2 + (1-\eta)\frac{\tau}{4} \sum_{j=k+1}^n \|\mathbf{e}_j\|_j^2 \leq C(\mu, \eta)\tau \sum_{j=k}^{n-1} |\mathbf{E}_j|_{G,j}^2 + C(\eta)\tau \sum_{j=k+1}^n \|\mathbf{d}_j\|_{*,j}^2 + C\eta^2\tau \|\mathbf{e}_k\|_k^2.$$

The discrete Gronwall inequality and the equivalence of norms thus yield the stated result with  $k+1$  instead of  $k$  and an extra term  $C(\mu, \eta)\tau c_1 |\mathbf{e}_k|_k^2 + C\eta^2\tau \|\mathbf{e}_k\|_k^2$ . To estimate  $|\mathbf{e}_k|_k^2 + \tau \|\mathbf{e}_k\|_k^2$ , we take the inner product of the error equation for  $n = k$  with  $\mathbf{e}_k$  to obtain

$$\delta_0 |\mathbf{e}_k|_k^2 + \tau \|\mathbf{e}_k\|_k^2 = \tau \langle \mathbf{d}_k, \mathbf{e}_k \rangle - \sum_{j=1}^k \delta_j \langle \mathbf{M}_{k-j} \mathbf{e}_{k-j}, \mathbf{e}_k \rangle.$$

Noting that  $\delta_0 > 0$  and estimating the terms on the right-hand side in the same way as above, in particular using  $\langle \mathbf{M}_{k-j} \mathbf{e}_{k-j}, \mathbf{e}_k \rangle \leq |\mathbf{e}_{k-j}|_{k-j} \cdot |\mathbf{e}_k|_{k-j}$  and  $|\mathbf{e}_k|_{k-j} \leq (1 + 2j\tau\mu) |\mathbf{e}_k|_k$ , we obtain

$$|\mathbf{e}_k|_k^2 + \tau \|\mathbf{e}_k\|_k^2 \leq C\tau \|\mathbf{d}_k\|_{*,k}^2 + C \max_{0 \leq i \leq k-1} |\mathbf{e}_i|_i^2.$$

Inserting this bound into the previous estimate completes the proof.  $\square$

## 5. Error bounds for the full discretization

We compare the numerical solution of the full discretization,

$$u_h^n = \sum_{i=1}^N u_{n,i} \phi_i(t_n),$$

which is a finite element function defined on the discretized surface  $\Gamma_h(t_n)$ , with a projection of the PDE solution  $u(t)$  to the finite element space  $S_h(t)$  at  $t = t_n$ :

$$P_h(t)u(t) = \sum_{i=1}^N \tilde{u}_i(t) \phi_i(t).$$

The projection  $P_h(t)$  could be the piecewise linear interpolation operator at the nodes or a Ritz projection. The finite element function  $P_h(t)u(t)$  on  $\Gamma_h(t)$  has a residual  $r_h(t) \in S_h(t)$  when inserted into the spatially discretized PDE (2.6):

$$\frac{d}{dt} \int_{\Gamma_h} P_h u \varphi_h + \int_{\Gamma_h} \nabla_{\Gamma_h} P_h u \cdot \nabla_{\Gamma_h} \varphi_h = \int_{\Gamma_h} P_h u \dot{\varphi}_h + \int_{\Gamma_h} f^l \varphi_h + \int_{\Gamma_h} r_h \varphi_h \quad \forall \varphi_h \in S_h(t). \quad (5.1)$$

Writing

$$r_h(t) = \sum_{i=1}^N r_i(t) \phi_i(t)$$

and denoting the coefficient vector by  $\mathbf{r}(t) = (r_i(t)) \in \mathbb{R}^N$ , we thus have for the vector  $\tilde{\mathbf{u}}(t) = (\tilde{u}_i(t)) \in \mathbb{R}^N$  of nodal values of  $P_h(t)u(t)$  that

$$\frac{d}{dt}(\mathbf{M}(t)\tilde{\mathbf{u}}(t)) + \mathbf{A}(t)\tilde{\mathbf{u}}(t) = \mathbf{f}(t) + \mathbf{M}(t)\mathbf{r}(t). \quad (5.2)$$

For the error

$$\mathbf{e}_n = \mathbf{u}_n - \tilde{\mathbf{u}}(t_n)$$

we thus obtain the error equation (3.5) with

$$\mathbf{d}_n = \mathbf{M}(t_n)\mathbf{r}(t_n) + \frac{d}{dt}(\mathbf{M}\tilde{\mathbf{u}})(t_n) - \frac{1}{\tau} \sum_{j=0}^k \delta_j(\mathbf{M}\tilde{\mathbf{u}})(t_{n-j}). \quad (5.3)$$

**THEOREM 5.1** Consider the space discretization of the parabolic equation (2.1) by the evolving surface finite element method and time discretization by the BDF method of order  $k \leq 5$ . Assume that the geometry and the solution of the parabolic equation are so regular that  $P_h u$  has continuous discrete material derivatives up to order  $k+1$ . Then, there exists  $\tau_0 > 0$  independent of  $h$  such that for  $\tau \leq \tau_0$ , the error  $e_h^n = u_h^n - P_h(t_n)u(t_n)$  is bounded for  $t_n = n\tau \leq T$  by

$$\begin{aligned} & \max_{k \leq j \leq n} \|e_h^j\|_{L_2(\Gamma_h(t_j))} + \left( \tau \sum_{j=k}^n \|\nabla_{\Gamma_h(t_j)} e_h^j\|_{L_2(\Gamma_h(t_j))}^2 \right)^{1/2} \\ & \leq C\beta_h \tau^k + \left( \tau \sum_{j=k}^n \|r_h(t_j)\|_{H_h^{-1}(\Gamma_h(t_j))}^2 \right)^{1/2} + C \max_{0 \leq i \leq k-1} \|e_h^i\|_{L_2(\Gamma_h(t_i))}. \end{aligned}$$

Here  $C$  is independent of  $h$  (but depends on  $T$ ), and

$$\beta_h^2 = \int_0^T \sum_{\ell=0}^{k+1} \|(P_h u)^{(\ell)}(t)\|_{L_2(\Gamma_h(t))}^2 dt,$$

where the superscript  $(\ell)$  denotes the  $\ell$ th discrete material derivative. The norm used for  $r_h$  is

$$\|r_h\|_{H_h^{-1}(\Gamma_h)} := \sup_{0 \neq \varphi_h \in S_h} \frac{(r_h, \varphi_h)_{L_2(\Gamma_h)}}{\|\varphi_h\|_{H^1(\Gamma_h)}}.$$

*Remarks.* (1) If  $P_h$  is the piecewise linear interpolation operator at the nodes, then  $\beta_h$  is clearly bounded uniformly in  $h$ . The same is expected for the Ritz projection of [5]. It can further be expected that  $\|r_h(t)\|_{H_h^{-1}(\Gamma_h(t))} = \mathcal{O}(h)$  or  $\mathcal{O}(h^2)$  when  $P_h$  is the piecewise linear interpolation operator or the Ritz projection, respectively. We prove the bound in the case of the Lagrange interpolant in Lemma 5.1 below. A detailed analysis of the purely spatial error with  $P_h$  taken to be the Ritz projection is outside the scope of this paper.

(2) We can also compare the fully discrete solution with the semi-discrete solution  $u_h$  of (2.6). For the corresponding error  $u_h^n - u_h(t_n)$  we obtain a similar bound where  $r_h$  does not appear and the factor in front of the  $\tau^k$  term is bounded in terms of higher-order discrete material derivatives of  $u_h$  instead of  $P_h u$ :

$$\begin{aligned} & \max_{k \leq j \leq n} \|u_h^j - u_h(t_j)\|_{L_2(\Gamma_h(t_j))} + \left( \tau \sum_{j=k}^n \|\nabla_{\Gamma_h(t_j)}(u_h^j - u_h(t_j))\|_{L_2(\Gamma_h(t_j))}^2 \right)^{1/2} \\ & \leq C\tilde{\beta}_h \tau^k + C \max_{0 \leq i \leq k-1} \|u_h^i - u_h(t_i)\|_{L_2(\Gamma_h(t_i))} \end{aligned}$$

with

$$\tilde{\beta}_h^2 = \int_0^T \sum_{\ell=0}^{k+1} \|u_h^{(\ell)}(t)\|_{L_2(\Gamma_h(t))}^2 dt.$$

We then need regularity results for the semi-discrete solution  $u_h$ , such as that of Theorem 9.1 in [6], which shows that

$$\sup_{(0,T)} \|u_h^{(m)}\|_{L_2(\Gamma_h)}^2 + \int_0^T \|\nabla_{\Gamma_h} u_h^{(m)}\|_{L_2(\Gamma_h)}^2 dt \leq c \sum_{\ell=0}^m \|u_h^{(\ell)}(\cdot, 0)\|_{L_2(\Gamma_{h0})}^2.$$

*Proof.* (of Theorem 5.1) We will use the stability lemma and translate back from the nodal value vectors to the corresponding finite element functions to prove the stated error bound. By Lemma 4.1 with  $\mathbf{d}_n$  of (5.3) and the norm identity (2.8) we obtain that (omitting the argument  $t$ )

$$\begin{aligned} \|\mathbf{M}\mathbf{r}\|_* &= (\mathbf{r}^T \mathbf{M}(\mathbf{A} + \mathbf{M})^{-1} \mathbf{M}\mathbf{r})^{1/2} = \|(\mathbf{A} + \mathbf{M})^{-1/2} \mathbf{M}\mathbf{r}\|_2 \\ &= \sup_{0 \neq \mathbf{w} \in \mathbb{R}^N} \frac{\mathbf{r}^T \mathbf{M}(\mathbf{A} + \mathbf{M})^{-1/2} \mathbf{w}}{(\mathbf{w}^T \mathbf{w})^{1/2}} = \sup_{0 \neq \mathbf{z} \in \mathbb{R}^N} \frac{\mathbf{r}^T \mathbf{M}\mathbf{z}}{(\mathbf{z}^T (\mathbf{A} + \mathbf{M})\mathbf{z})^{1/2}} \\ &= \sup_{0 \neq \varphi_h \in S_h} \frac{(r_h, \varphi_h)_{L_2(\Gamma_h)}}{\|\varphi_h\|_{H^1(\Gamma_h)}} = \|r_h\|_{H_h^{-1}(\Gamma_h)}. \end{aligned}$$

The backward differentiation error of a smooth function can be represented with a scalar Peano kernel  $\sigma(\theta)$ ,

$$g'(t) - \frac{1}{\tau} \sum_{j=0}^k \delta_j g(t - j\tau) = \tau^k \int_0^k \sigma(\theta) g^{(k+1)}(t - \theta\tau) d\theta.$$

We use this formula for  $g = \mathbf{M}\tilde{\mathbf{u}}$  and set  $\mathbf{w} = \mathbf{M}^{-1}(\mathbf{M}\tilde{\mathbf{u}})^{(k+1)}$ , so that

$$\frac{d}{dt}(\mathbf{M}\tilde{\mathbf{u}})(t_n) - \frac{1}{\tau} \sum_{j=0}^k \delta_j (\mathbf{M}\tilde{\mathbf{u}})(t_{n-j}) = \tau^k \int_0^k \sigma(\theta) (\mathbf{M}\mathbf{w})(t_n - \theta\tau) d\theta.$$

We note

$$\begin{aligned} \|\mathbf{M}(t)\mathbf{w}\|_{*,s}^2 &= \mathbf{w}^T \mathbf{M}(t) (\mathbf{A}(s) + \mathbf{M}(s))^{-1} \mathbf{M}(t)\mathbf{w} \\ &= \mathbf{w}^T \mathbf{M}(t) \mathbf{M}(s)^{-1/2} (\mathbf{M}(s)^{-1/2} \mathbf{A}(s) \mathbf{M}(s)^{-1/2} + I)^{-1} \mathbf{M}(s)^{-1/2} \mathbf{M}(t)\mathbf{w} \\ &\leq \|\mathbf{M}(s)^{-1/2} \mathbf{M}(t)\mathbf{w}\|_2^2 = \mathbf{w}^T \mathbf{M}(t) \mathbf{M}(s)^{-1} \mathbf{M}(t)\mathbf{w}. \end{aligned}$$

This is further estimated using Lemma 2.2:

$$\mathbf{w}^T \mathbf{M}(t) \mathbf{M}(s)^{-1} \mathbf{M}(t)\mathbf{w} = \mathbf{w}^T \mathbf{M}(t)\mathbf{w} + \mathbf{w}^T \mathbf{M}(t) (\mathbf{M}(s)^{-1} - \mathbf{M}(t)^{-1}) \mathbf{M}(t)\mathbf{w} \leq 2 \mathbf{w}^T \mathbf{M}(t)\mathbf{w},$$

provided that  $2\mu|t - s| \leq 1$ . For such  $t$  and  $s$  we have thus shown that

$$\|\mathbf{M}(t)\mathbf{w}\|_{*,s}^2 \leq 2 \|\mathbf{w}\|_t^2.$$

Lemma 9.2 of [6] shows that for  $\mathbf{w} = \mathbf{M}^{-1}(\mathbf{M}\tilde{\mathbf{u}})^{(k+1)}$  with  $\tilde{\mathbf{u}}$  the vector of nodal values of  $P_h u$ , we have

$$|\mathbf{w}(t)|_t^2 \leq C \sum_{l=0}^{k+1} \|(P_h u)^{(l)}(t)\|_{L_2(\Gamma_h(t))}^2.$$

Combining these estimates yields

$$\begin{aligned} & \left\| \frac{d}{dt}(\mathbf{M}\tilde{\mathbf{u}})(t_n) - \frac{1}{\tau} \sum_{j=0}^k \delta_j(\mathbf{M}\tilde{\mathbf{u}})(t_{n-j}) \right\|_{*,t_n}^2 \\ & \leq \tau^{2k} c \int_0^k \|\mathbf{M}\mathbf{w}(t_n - \theta\tau)\|_{*,t_n}^2 d\theta \\ & \leq \tau^{2k} 2c \int_0^k |\mathbf{w}(t_n - \theta\tau)|_{*,t_n - \theta\tau}^2 d\theta \\ & \leq \tau^{2k} 2cC \int_0^k \sum_{l=0}^{k+1} \|(P_h u)^{(l)}(t_n - \theta\tau)\|_{L_2(\Gamma_h(t_n - \theta\tau))}^2 d\theta. \end{aligned}$$

With Lemma 4.1 this completes the proof.  $\square$

We now proceed by proving an error estimate for the residual  $r_h(t)$  that appears in (5.1), for the case when the projection is the linear Lagrange interpolant. The proof is based on the results of [3], in which an error estimate for the semidiscrete scheme was proved. For the reader's convenience we recall some technical preliminaries from [3].

We denote by  $d(x, t)$ ,  $x \in \mathbb{R}^{n+1}$ ,  $t \in [0, T]$  the *signed distance function* to the smooth closed surface  $\Gamma(t)$  and make the assumption that  $\mathcal{N}(t)$  is such that for every  $x \in \mathcal{N}(t)$  and  $t \in [0, T]$  there exists a unique  $a(x, t) \in \Gamma(t)$  such that

$$x = a(x, t) + d(x, t)n(a(x, t), t), \quad (5.4)$$

where  $n$  denotes the unit normal vector field to  $\Gamma$ . We assume  $\Gamma_h(t) \subset \mathcal{N}(t)$ . Thus for each triangle  $E(t)$  in  $\Gamma_h(t)$  there is a unique curved triangle  $a(E(t), t) \subset \Gamma(t)$ , and we assume a bijective correspondence between the triangles on  $\Gamma_h$  and the induced curvilinear triangles on  $\Gamma$ . Furthermore we assume  $\Gamma_h(t)$  consists of triangles  $E$  in  $\mathcal{T}_h(t)$  with inner radius bounded below by  $\sigma_h \geq ch$  for some  $c > 0$  (recalling that  $h$  denotes the mesh-size).

**LEMMA 5.1** Consider the residual  $r_h(t) \in S_h(t)$  that appears in (5.1). Assume the projection  $P_h(t)$  is the pointwise linear Lagrange interpolator. Assume sufficient regularity on the geometry and the solution to the continuous parabolic equation (2.1), and furthermore assume the discretized surfaces  $\Gamma_h(t)$  satisfy the assumptions made above. Then, for  $t \in [0, T]$ ,  $h$  sufficiently small and for some  $c > 0$ , the following bound holds:

$$\|r_h(t)\|_{H_h^{-1}(\Gamma_h(t))} \leq ch.$$

*Proof.* (a) We first recall the necessary geometric notation and estimates from [3]. We denote by  $\delta_h$  the quotient between the smooth and discrete surface measures which satisfies [3, Lemma 5.1]

$$\sup_{t \in (0, T)} \sup_{\Gamma_h(t)} |1 - \delta_h| \leq ch^2. \quad (5.5)$$

We introduce the space

$$S_h^l(t) = \{\eta^l \in C^0(\Gamma_h(t)) : \eta^l(a) = \eta(x(a)), \eta \in S_h(t) \text{ and } x(a) \text{ given by (5.4)}\},$$

where due to the above assumptions,  $x(a)$  (the solution to (5.4)) is unique. We shall make use of the following estimate for the lift of the material derivative from [3, (6.6)], for a sufficiently smooth function  $\eta$ ,

$$\dot{\eta}(x, t) = \dot{\eta}^l(a(x), t) + O(h^2 |\nabla_\Gamma \eta^l(a(x), t)|). \quad (5.6)$$

We also introduce  $\text{Pr}$  and  $\text{Pr}_h$  the projections onto the tangent planes of  $\Gamma$  and  $\Gamma_h$  respectively and the Weingarten map  $\mathcal{H}$  ( $\mathcal{H}_{ij} = \partial_{x_j} n_i$ ). Defining  $B_h = \text{Pr}_h(I - d\mathcal{H})$  and  $R_h = \frac{1}{\delta_h} \text{Pr}(I - d\mathcal{H}) \text{Pr}_h(I - d\mathcal{H})$  we have

$$\nabla_{\Gamma_h} \eta(x) = B_h \nabla_\Gamma \eta^l(a(x)), \quad x \in \Gamma_h, \quad (5.7)$$

$$\nabla_{\Gamma_h} \eta(x) \cdot \nabla_{\Gamma_h} \phi(x) = \delta_h R_h \nabla_\Gamma \eta^l(a(x)) \nabla_\Gamma \phi^l(a(x)) \quad x \in \Gamma_h. \quad (5.8)$$

From [3, Lemma 5.1] we have

$$\sup_{t \in (0, T)} \sup_{\Gamma_h(t)} |(I - R_h) \text{Pr}| \leq ch^2. \quad (5.9)$$

Furthermore, [3, Lemma 5.1] yields

$$(B_h - I) \text{Pr} = O(h^2) + \text{Pr} - \text{Pr}_h \text{Pr}.$$

A similar calculation to [3, Lemma 5.1, proof] gives for a unit vector  $z$

$$|(\text{Pr} - \text{Pr}_h \text{Pr})z| = |z \cdot n_h (n_h - (n_h \cdot n)n)|.$$

Applying the estimate of the term above contained in [3, Lemma 5.1, proof] we arrive at the estimate

$$\sup_{t \in (0, T)} \sup_{\Gamma_h(t)} |(I - B_h) \text{Pr}| \leq ch. \quad (5.10)$$

Finally we shall make use of the following interpolation estimates [3, Lemma 5.3]. For a given  $\eta \in H^2(\Gamma)$ ,

$$\|\eta - I_h \eta\|_{L_2(\Gamma)} + h \|\nabla_\Gamma (\eta - I_h \eta)\|_{L_2(\Gamma)} \leq ch^2 \left( \|\nabla_\Gamma^2 \eta\|_{L_2(\Gamma)} + h \|\nabla_\Gamma \eta\|_{L_2(\Gamma)} \right). \quad (5.11)$$

Here  $I_h \eta \in S_h^l$  is defined as the lift of the pointwise linear interpolation  $\tilde{I}_h \eta$ , i.e.,  $I_h \eta = (\tilde{I}_h \eta)^l$ .

(b) We start by defining a suitable lift of (5.1) onto the continuous surface  $\Gamma(t)$ , with  $P_h u$  taken as  $\tilde{I}_h u$ . In the interest of brevity, in the following, we omit the omnipresent argument  $t$ . We have

$$\begin{aligned} \int_{\Gamma_h} r_h \varphi_h &= \int_\Gamma (\tilde{I}_h u)^l \varphi_h^l \frac{1}{\delta_h^l} + \int_\Gamma I_h u \varphi_h^l \sum_i \sum_j (B_h^l)_{ij} (\nabla_\Gamma (v_h^l)_i)_j \frac{1}{\delta_h^l} \\ &+ \int_\Gamma R_h^l \nabla_\Gamma I_h u \cdot \nabla_\Gamma \varphi_h^l - \int_\Gamma f \varphi_h^l \frac{1}{\delta_h^l}. \end{aligned} \quad (5.12)$$

Here we have applied the (discrete) Leibniz formula to the first term of (5.1) and made use of (5.7) and (5.8) to obtain the second and third terms respectively. Subtracting (2.3) from (5.12) after applying the Leibniz formula to the first term and setting  $\varphi = \varphi_h^l$  in (2.3) we obtain

$$\begin{aligned} \int_{\Gamma_h} r_h \varphi_h &= \int_{\Gamma} \left( \frac{1}{\delta_h^l} \left( \tilde{I}_h u \right)^l - \dot{u} \right) \varphi_h^l + \int_{\Gamma} \left( I_h u \sum_i \sum_j (B_h^l)_{ij} (\nabla_{\Gamma} (v_h^l))_j \frac{1}{\delta_h^l} - u \nabla_{\Gamma} \cdot v \right) \varphi_h^l \\ &\quad + \int_{\Gamma} (R_h^l \nabla_{\Gamma} I_h u - \nabla_{\Gamma} u) \cdot \nabla_{\Gamma} \varphi_h^l + \int_{\Gamma} \left( f - \frac{1}{\delta_h^l} f \right) \varphi_h^l \\ &= I + II + III + IV, \end{aligned} \tag{5.13}$$

where  $I, II, III$  and  $IV$  are defined by the second equality.

The interpolation estimate (5.11), the bound (5.5) and (5.6) yield

$$|I| \leq ch^2 \|\varphi_h^l\|_{L_2(\Gamma)}.$$

Dealing with the second term, we proceed as follows, where (5.5) and (5.10) yield

$$\begin{aligned} |II| &\leq ch^2 \int_{\Gamma} \left| I_h u \sum_i \sum_j (B_h^l)_{ij} (\nabla_{\Gamma} (v_h^l))_j \varphi_h^l \right| + ch \int_{\Gamma} |I_h u \nabla_{\Gamma} \cdot v_h^l \varphi_h^l| \\ &\quad + \left( \int_{\Gamma} |(I_h u - u) \nabla_{\Gamma} \cdot v_h^l \varphi_h^l| + \int_{\Gamma} |u \nabla_{\Gamma} \cdot (v_h^l - v) \varphi_h^l| \right). \end{aligned}$$

Noting that the discrete material velocity is the interpolant of the continuous material velocity, the interpolation estimate (5.11) yields

$$|II| \leq ch \|\varphi_h^l\|_{L_2(\Gamma)}.$$

The interpolation estimate (5.11) and the bounds (5.5) and (5.9) yield

$$|III| \leq ch \|\varphi_h^l\|_{H^1(\Gamma)} \quad \text{and} \quad |IV| \leq ch^2 \|\varphi_h^l\|_{L_2(\Gamma)}.$$

Applying the above bounds in (5.13) and noting the equivalence of norms between the continuous and discrete surface completes the proof.  $\square$

Combining Theorem 5.1 and Lemma 5.1 we obtain our final error bound for the surface finite element / backward difference full discretization.

**THEOREM 5.2** Consider the space discretization of the parabolic equation (2.1) by the evolving surface finite element method and time discretization by the BDF method of order  $k \leq 5$ . Assume that the geometry and the solution  $u$  of the parabolic equation are so regular that  $u$  has continuous discrete material derivatives up to order  $k + 1$ , and that the discretized surfaces  $\Gamma_h(t)$  satisfy the regularity conditions of Lemma 5.1. Then, there exist  $h_0 > 0$  and  $\tau_0 > 0$  such that for  $h \leq h_0$  and  $\tau \leq \tau_0$  the following holds for the errors

$$e_h^n = u_h^n - (\tilde{I}_h u)(t_n)$$

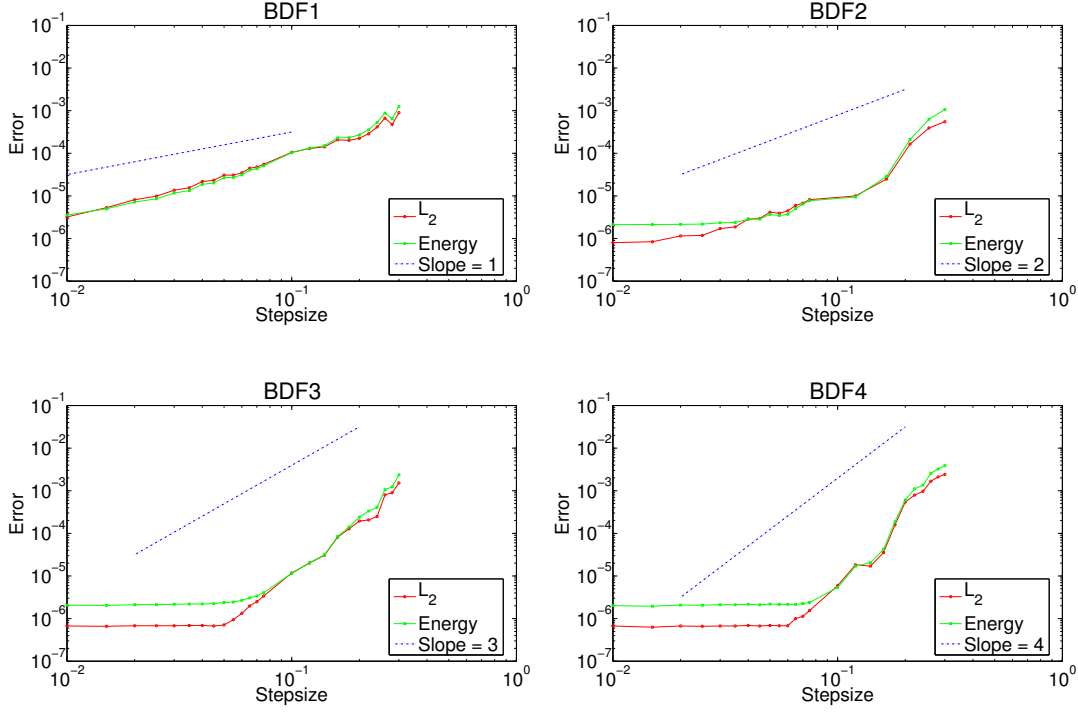


FIG. 1. Errors for BDF schemes in the  $\max_{k \leq j \leq n} \| \cdot \|_{L^2(\Gamma_h(t_j))}$  norm and  $(\tau \sum_{j=k}^n \| \nabla_{\Gamma_h(t_j)} \cdot \|_{L^2(\Gamma_h(t_j))}^2)^{1/2}$  seminorm over the time interval  $[2, 4]$  for a series of refinements of the timestep on a mesh with 65538 degrees of freedom.

between the fully discrete numerical solution  $u_h^n$  and the piecewise linear Lagrange interpolant  $(\tilde{I}_h u)(t_n)$  of the exact solution on the discrete surface  $\Gamma_h(t_n)$  for  $t_n = n\tau \leq T$ : whenever the errors  $e_h^i$  of the starting values are bounded by  $c\tau^k + ch$  in the  $L^2(\Gamma_h(t_i))$  norm for  $i = 0, \dots, k-1$ , then the errors are bounded by

$$\max_{k \leq j \leq n} \|e_h^j\|_{L^2(\Gamma_h(t_j))} + \left( \tau \sum_{j=k}^n \| \nabla_{\Gamma_h(t_j)} e_h^j \|_{L^2(\Gamma_h(t_j))}^2 \right)^{1/2} \leq C\tau^k + Ch,$$

where  $C$  is independent of  $h$  and  $\tau$ .

## 6. Numerical experiments

In this section we illustrate our theoretical results with numerical simulations. The details of the implementation of the evolving surface finite element method are described elsewhere [3].

*Example 1.* We consider the numerical example from [3, Example 7.3] and [6], which is a PDE posed on an ellipsoid with time-dependent axis: the surface is given as the level set

$$\Gamma(t) := \{x \in \mathbb{R}^3 : d(x, t) = 0\} \quad \text{with} \quad d(x, t) = \frac{x_1^2}{a(t)} + x_2^2 + x_3^2 - 1. \quad (6.1)$$



We set  $a(t) = 1 + 0.25 \sin(t)$ . We consider (1.1) posed on  $\Gamma(t)$ ,  $t \in [0, 4]$ , and construct a suitable right hand side  $f(x, t)$  such that the exact solution is  $u(x, t) = e^{-6t} x_1 x_2$ .

We consider the BDF schemes (3.1) of order  $k = 1, \dots, 5$ . For a given timestep  $\tau$ , the starting values  $\mathbf{u}_0, \dots, \mathbf{u}_{k-1}$  are taken to be the exact solution values at the nodes, i.e., for  $j = 0, \dots, k-1$ , we set  $(\mathbf{u}_j)_i = u(a_i(t_j), t_j)$  for  $i = 1, \dots, N$ , with  $t_j = j\tau$ . We construct a vector  $\mathbf{e}_n \in \mathbb{R}^N$  consisting of the error at each of the nodes of the triangulation, such that  $(\mathbf{e}_n)_i := (\mathbf{u}_n)_i - u(a_i(t_n), t_n)$  for  $i = 1, \dots, N$ , and we denote by  $e_h^n \in S_h(t_n)$  the piecewise linear interpolant on  $\Gamma_h(t_n)$ . We consider the norm and seminorm that appear in Theorem 5.2, see also (2.8).

In Figure 1 we plot the error for the BDF methods up to order 4 in these norms over the time interval  $[2, 4]$  (chosen to ensure a sufficient number of points within the interval for the higher order BDF schemes) versus the timestep size. In the regime where the error due to the time discretisation is dominant we clearly observe the theoretical orders of temporal convergence as the timestep is refined.

As a second experiment we repeat the experiment conducted in [6] with the BDF1, BDF4 and BDF5 schemes. As in the above example, we investigate equation (2.1) on a time-dependent surface of the form (6.1) with suitable right hand side such that the exact solution is  $u(x, t) = e^{-6t} x_1 x_2$ . In Figure 2 we plot the errors in the discrete  $L_2$  norm and the discrete energy seminorm at  $t = 1$  versus the stepsize for a series of spatial refinements of the triangulation. We observe analogous results to [6] with optimal convergence in the regime where the temporal error is dominant and error independence of the spatial refinement level in this region. In the region where the spatial error is dominant (only applicable to the BDF4 and BDF5 schemes) we observe faster convergence in the  $L_2$  norm than in the  $H^1$  seminorm.

To illustrate the performance of the scheme with an exact solution that has a more challenging time-dependence, we consider equation (2.1) on a time-dependent surface of the form (6.1) with suitable right hand side such that the exact solution is  $u(x, t) = \cos(\pi t) x_1 x_2$ . In Figure 3 we plot the error in norm and seminorm that appear in Theorem 5.1, see also (2.8), over the time interval  $[5, 7]$  (chosen to ensure a sufficient number of points within the interval for the higher order BDF schemes) versus the timestep size. In the regime where the error due to the time discretisation is dominant we clearly observe the theoretical orders of temporal convergence as the timestep is refined.

*Example 2.* We choose a time-dependent surface of the form

$$\Gamma(t) := \left\{ x_1 + \max(0, x_1)t, \frac{g(x, t)x_2}{\sqrt{x_2^2 + x_3^2}}, \frac{g(x, t)x_3}{\sqrt{x_2^2 + x_3^2}} : x \in \Gamma(0) = S^2 \right\}, \quad (6.2)$$

$$g(x, t) = e^{-2t} \sqrt{x_2^2 + x_3^2} + (1 - e^{-2t}) \left( (1 - x_1^2) (x_1^2 + 0.05) + x_1^2 \sqrt{(1 - x_1^2)} \right).$$

We consider equation (2.1) posed on the above surface on the time interval  $[0, 1]$ , with right hand side  $f = 0$  and initial data  $u(x, 0) = x_1 x_2$ . The surface evolves from a initially spherical shape at  $t = 0$  to a ‘‘baseball bat’’ at  $t = 1$ . We present results of 4 different numerical experiments, firstly we employ the implicit Euler scheme (BDF1) with timestep  $\tau = 10^{-4}$  and a mesh with 4098 degrees of freedom, next we consider the implicit Euler, BDF2 and BDF4 schemes with  $\tau = 5 \times 10^{-2}$  for  $t \in [0.2, 1.0]$ . In all three cases the starting values are determined by the implicit Euler scheme with timestep  $\tau = 10^{-4}$  for  $t \in [0, 0.15]$ .

Figure 4 shows snapshots of the discrete solution for the four different experiments. In

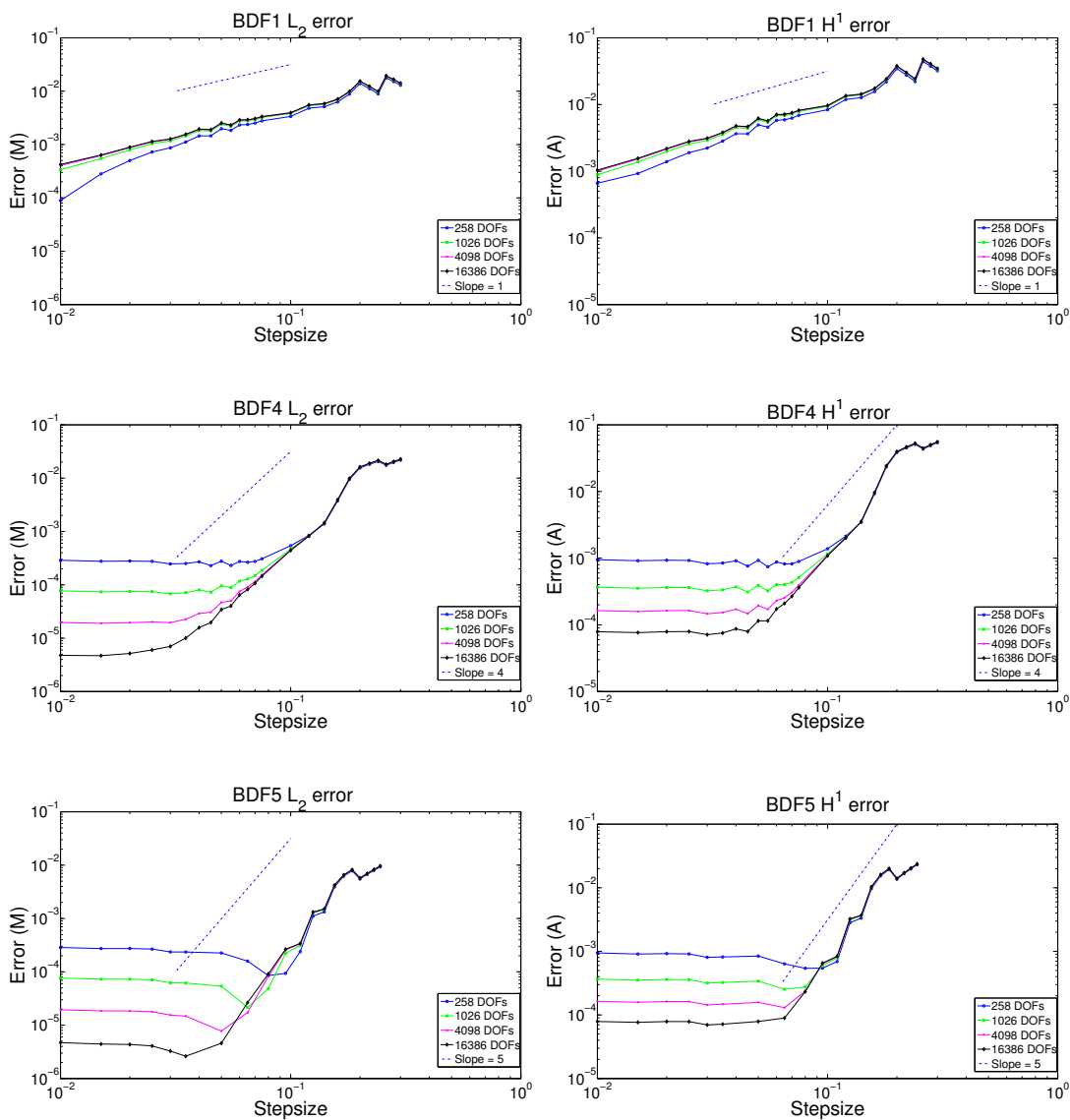


FIG. 2. Errors of the BDF1 (top), BDF4 (middle) and BDF5 (bottom) schemes in the  $\|\cdot\|_{L_2(\Gamma_h)}$  norm (left) and the  $\|\cdot\|_{H^1(\Gamma_h)}$  seminorm (right) vs. timestep size for four spatial refinements at  $t = 1$ .

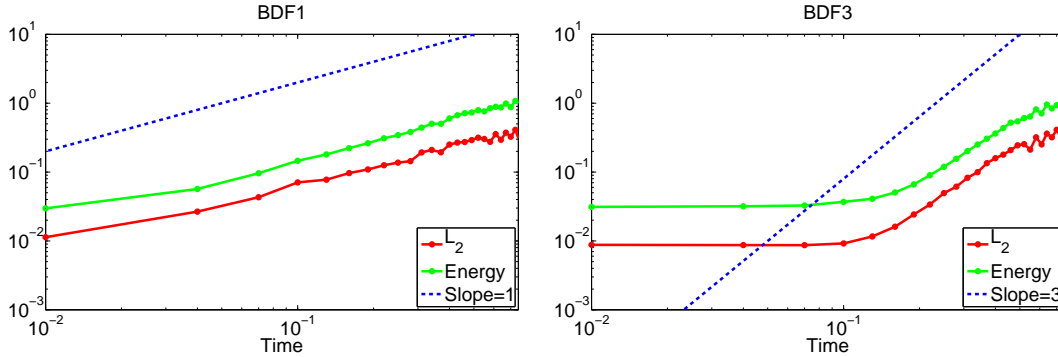


FIG. 3. Errors for BDF schemes, with an oscillatory in time exact solution, in the  $\max_{k \leq j \leq n} \|\cdot\|_{L_2(\Gamma_h(t_j))}$  norm and  $\left(\tau \sum_{j=k}^n \|\nabla_{\Gamma_h(t_j)} \cdot\|_{L_2(\Gamma_h(t_j))}^2\right)^{1/2}$  seminorm over the time interval  $[5, 7]$  for a series of refinements of the timestep on a mesh with 65538 degrees of freedom.

accordance with the theory, as the order of the scheme is increased we observe less discrepancy from the implicit Euler scheme with the uniformly small timestep (top row of each of the subfigures in Figure 4). The computational time for the implicit Euler scheme with the refined timestep is 264 seconds while the schemes with the larger steps all had computational times of approximately 3 seconds.

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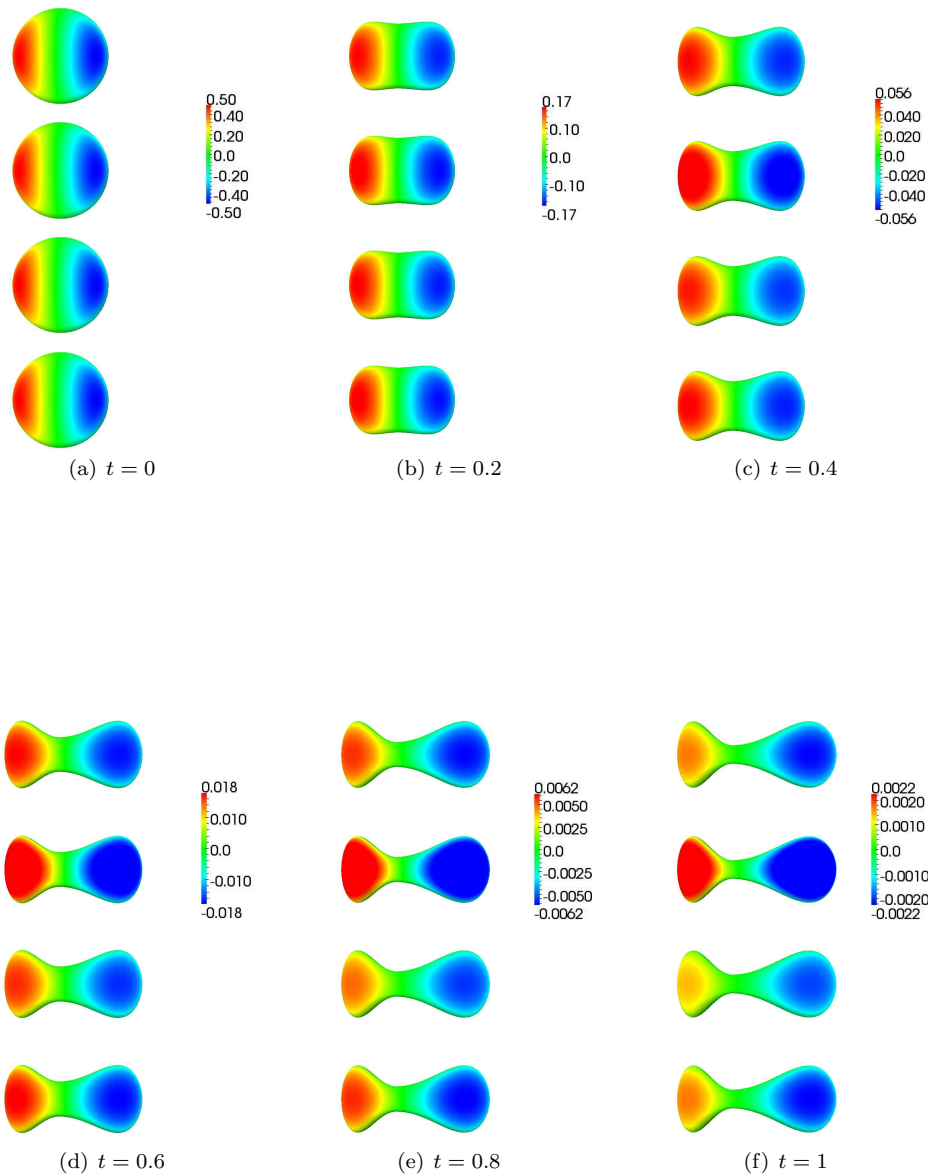


FIG. 4. Snapshots of the discrete solution of equation (2.1) on a time-dependent surface of the form (6.2). Reading from top to bottom, each subfigure shows results of the BDF1 scheme with the uniformly small timestep ( $\tau = 10^{-4}$ ) and the BDF1, BDF2 and BDF4 schemes with the larger timestep ( $\tau = 5 \times 10^{-2}$ ). For the schemes with the larger timestep the first value computed is at  $t = 0.2$  with the starting value(s) determined by the BDF1 scheme with  $\tau = 10^{-4}$  on the time interval  $[0.0, 0.15]$ .