A NOTE ON INVERSE MEAN CURVATURE FLOW IN COSMOLOGICAL SPACETIMES

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ABSTRACT. In [8] Gerhardt proves longtime existence for the inverse mean curvature flow in globally hyperbolic Lorentzian manifolds with compact Cauchy hypersurface, which satisfy three main structural assumptions: a strong volume decay condition, a mean curvature barrier condition and the timelike convergence condition. Furthermore, it is shown in [8] that the leaves of the inverse mean curvature flow provide a foliation of the future of the initial hypersurface.

We show that this result persists, if we generalize the setting by leaving the mean curvature barrier assumption out. For initial hypersurfaces with sufficiently large mean curvature we can weaken the timelike convergence condition to a physically relevant energy condition.

CONTENTS

1. INTRODUCTION AND MAIN RESULT

The inverse mean curvature flow has been considered by Huisken and Ilmanen [13] to prove the Penrose inequality or by Gerhardt [4] in an Euclidean setting.

In [8] Gerhardt considers the inverse mean curvature flow in a globally hyperbolic Lorentzian manifold with compact Cauchy hypersurface, which satisfies the so-called timelike convergence condition, i.e. the Ricci tensor is nonnegative on the set of timelike unit vectors

$$(1.1) \qquad \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta \geq 0 \quad \forall \langle \nu, \nu \rangle = -1,$$

a mean curvature barrier condition with respect to the future and a strong volume decay condition with respect to the future. For definitions we refer to [8].

There it is shown, that the inverse mean curvature flow with connected, spacelike and closed initial surface with positive mean curvature exists for all times, that the

flow hypersurfaces provide a foliation of the future of the initial surface and that the evolution parameter t can be used as a new and physically meaningful time function.

We prove that all these results persist true, if we omit the assumption that a mean curvature barrier condition with respect to the future is fulfilled.

Furthermore we prove, that for initial hypersurfaces with sufficiently large mean curvature all above results stay true, if we relax the timelike convergence condition to the energy condition

$$(1.2) \qquad \bar{R}_{\alpha\beta}\nu^{\alpha}\nu^{\beta} \geq -\Lambda \quad \forall \langle \nu, \nu \rangle = -1$$

for some $\Lambda > 0$.

In the following we assume that $N = N^{n+1}$ is a connected, time-oriented and smooth Lorentzian manifold. Furthermore, all functions will be assumed to be smooth; quantities like the Ricci tensor $\bar{R}_{\alpha\beta}$, which refer to the ambient space (and not to hypersurfaces), are marked by a bar; greek indices range from 0 to n and latin indices from 1 to n. For a detailed introduction to our notations we refer to [8]. We formulate our result in two theorems.

Theorem 1.1. *Let N be globally hyperbolic with connected and compact Cauchy hypersurface S_0 und let N fulfill a strong volume decay condition with respect to the future. Furthermore we assume that*

$$(1.3) \qquad \bar{R}_{\alpha\beta}\nu^{\alpha}\nu^{\beta} \geq -\Lambda \quad \forall \langle \nu, \nu \rangle = -1$$

with a positive constant Λ. Let M_0 be a connected, compact and spacelike hypersurface with mean curvature

$$(1.4) \qquad H > \sqrt{n\Lambda}$$

with respect to the past directed normal. Then the inverse mean curvature flow (IMCF) with initial surface M_0 exists for all times, i.e. given a manifold $M = M^n$ and an embedding

$$(1.5) \qquad x_0 : M \longrightarrow N$$

with $x_0(M) = M_0$, there is a

$$(1.6) \qquad x : [0, \infty) \times M \longrightarrow N$$

with

- *$x(t, \cdot)$ embedding of a spacelike hypersurface $M(t) = x(t, \cdot)(M)$ with positive mean curvature*
- *$x(0, \cdot) = x_0$*
- *$\dot{x}(t, \xi) = -H^{-1}\nu$, where ν is the past directed normal of $M(t)$ in $x(t, \xi)$ and H the mean curvature of $M(t)$ in $x(t, \xi)$ with respect to ν.*

The hypersurfaces $M(t)$, $t > 0$, provide a foliation of the future $I^+(M_0)$ of M_0 and there holds

$$(1.7) \qquad |M(t)| = |M_0| e^{-t} \quad \forall \, t \geq 0.$$

Furthermore the evolution parameter t can be used as new time function in the future $I^+(M_0)$ of M_0.

Theorem 1.2. *Let t be the time function in $I^+(M_0)$ according to Theorem 1.1. If we define a second time function by setting*

$$\tau = 1 - e^{-\frac{t}{n}} \tag{1.8}$$

and use the obvious notation $M(t) = M(\tau)$ there holds

$$|M(\tau)| = |M_0|(1-\tau)^n \tag{1.9}$$

for $0 \leq \tau < 1$. Furthermore there is a $0 < \tau_0 = \tau_0(n, \Lambda, M_0) < 1$ and a constant $c = c(n, \Lambda, M_0) > 0$ such that the length $L(\gamma)$ of a future directed timelike curve starting at $M(\tau)$ is bounded from above

$$L(\gamma) \leq c(1-\tau) \quad \forall \tau_0 \leq \tau < 1. \tag{1.10}$$

Remark 1.3. Theorem 1.2 shows that the quantity $1 - \tau$ can be interpreted as the radius of the coordinate slices $\{\tau = const\}$ and as a measure for the remaining life span of the spacetime.

To prove the above theorems we use a small but effective modification of the proof in [8]. We will only point out the differences between our proof and the one in [8].

2. Proof of Theorem 1.1

We remark that under the assumptions of Theorem 1.1 the initial surface M_0 is a graph over S_0, cf. Section 4.

We need an evolution equation for H^{-1}.

Lemma 2.1. *We have*

$$\frac{d}{dt}\left(H^{-1}\right) - H^{-2}\Delta H^{-1} = -H^{-2}\left(\|A\|^2 + \bar{R}_{\alpha\beta}\nu^\alpha\nu^\beta\right)H^{-1}, \tag{2.1}$$

where

$$\|A\|^2 = h_{ij}h^{ij}. \tag{2.2}$$

Proof. See [8, Lemma 2.2]. $\qquad\qquad\square$

We show that the mean curvature of the leaves of the IMCF grows exponentially fast in time.

Lemma 2.2. *We assume that the IMCF exists on a maximal time interval $[0, T^*)$. Then there exists a constant $0 < \epsilon = \epsilon(n, \Lambda, M_0)$ such that*

$$H \geq e^{\epsilon t}\inf_{M_0} H \quad \forall 0 \leq t < T^*. \tag{2.3}$$

Proof. We define

$$f(t) = \frac{1}{n}\inf_{M(t)} H^2, \quad 0 \leq t < T^* \tag{2.4}$$

and choose $\Lambda < c < f(0)$. We show by contradiction that

$$f(t) > c \tag{2.5}$$

for all $0 \leq t < T^*$. Assume this is not the case. Due to the continuity of f there is a minimal $0 < t_0 < T^*$ with $f(t_0) = c$. For the function

$$\phi = H^{-1}e^{\epsilon t}, \quad \epsilon = \frac{1}{n}\left(1 - \frac{\Lambda}{c}\right), \tag{2.6}$$

we have for all $0 \leq t \leq t_0$

$$\dot{\phi} - H^{-2}\Delta\phi \leq -H^{-2}\|A\|^2\phi + H^{-2}\Lambda\phi + \epsilon\phi$$

(2.7)
$$\leq -\frac{1}{n}\phi + \frac{\Lambda}{nc}\phi + \epsilon\phi$$
$$= 0,$$

cf. Lemma 2.1. By applying the maximum principle we conclude

(2.8)
$$\phi(t,\cdot) \leq \sup_{M_0}\phi = \sup_{M_0}H^{-1}$$

for all $0 \leq t \leq t_0$ and hence

(2.9)
$$f(t_0) \geq \frac{1}{n}e^{2\epsilon t_0}\inf_{M_0}H^2$$
$$> c,$$

which is a contradiction.

With (2.5) also (2.7) and hence (2.8) is true for all $0 \leq t < T^*$ proving the lemma. $\qquad\square$

The lower bound for the mean curvature of the leaves of the IMCF in Lemma 2.2 can be improved.

Lemma 2.3. *We assume that the IMCF exists for all times. Then there exists a constant $c_0 = c_0(n, \Lambda, M_0) > 0$ such that*

(2.10)
$$H \geq c_0 e^{\frac{t}{n}}$$

during the evolution.

Proof. We define

(2.11)
$$w = H^{-1}e^{\frac{1}{n}t}$$

for all $t \geq 0$ and similar to inequality (2.7) we get

(2.12)
$$\dot{w} - H^{-2}\Delta w \leq H^{-2}\Lambda w.$$

The function

(2.13)
$$\varphi(t) = \sup w(t,\cdot)$$

is lipschitz continuous, cf. [8, Lemma 3.2], and for a.e. $t > 0$ there holds

(2.14)
$$\dot{\varphi}(t) = \dot{w}(t, x_t),$$

where x_t is an arbitrary point, in which the supremum is attained. Hence for a.e. $t > 0$ we have in view of (2.12) and Lemma 2.2

(2.15)
$$\dot{\varphi}(t) \leq H^{-2}\Delta w + H^{-2}\Lambda w$$
$$\leq H^{-2}\Lambda w$$
$$\leq ce^{-2\epsilon t}\varphi,$$

where the right side of the inequality is evaluated at (t, x_t). Integration of the last inequality yields

(2.16)
$$\varphi \leq c,$$

which proves the lemma.

$\qquad\square$

In [8] the assumption of barriers was used to show that the flow hypersurfaces of the IMCF run into the future singularity provided the flow exists for all times. In our proof we don't need barriers.

Lemma 2.4. *Let N be as in Theorem 1.1 and S_0, (x^α) the corresponding Gaussian coordinate system, especially*

$$N = (a, b) \times S_0, \quad a < 0 < b, \tag{2.17}$$

as a topological product, x^0 a global defined time function, (x^i) local coordinates for S_0 and $\{x^0 = 0\}$ corresponds to S_0.

If the IMCF exists for all times and if the leaves $M(t)$ of the IMCF are graphs, i.e.

$$M(t) = \left\{ (x^0, x) : x^0 = u(t, x), \ x \in S_0 \right\} \tag{2.18}$$

with a function $u \in C^\infty([0, \infty) \times S_0)$, then there holds

$$\lim_{t \to \infty} \inf_{S_0} u(t, \cdot) = b. \tag{2.19}$$

Proof. For every $s \in (a, b)$, there is a $t_0 > 0$ such that

$$\inf_{M(t)} H > \sup_{\{x^0 = s\}} H \tag{2.20}$$

for all $t \geq t_0$, cf. Lemma 2.3. Applying [11, Lemma 4.7.1] we conclude in view of Lemma 2.2 that

$$u(t, \cdot) > s \tag{2.21}$$

for all $t \geq t_0$.   □

The remaining part of Theorem 1.1 is proved exactly as in [8].

3. Proof of Theorem 1.2

We need the following generalization by Anderson-Galloway, cf. [1, Proposition 3.3], of a singularity theorem of Hawking, cf. [12, Theorem 4, p. 272], for globally hyperbolic Lorentzian manifolds satisfying the timelike convergence condition to the case of globally hyperbolic Lorentzian manifolds satisfying (1.2).

Theorem 3.1. *Let $N = N^{n+1}$ be globally hyperbolic and let the Ricci tensor satisfy (1.2). Let M be a compact, spacelike and achronal hypersurface with mean curvature (with respect to past directed normal)*

$$H \geq H_0 > \sqrt{n\Lambda}. \tag{3.1}$$

Then the length $L(\gamma)$ of an arbitrary future directed timelike curve γ starting on M is bounded from above

$$L(\gamma) \leq \frac{nH_0}{H_0^2 - n\Lambda}. \tag{3.2}$$

Proof. See [11, Theorem 1.9.23].   □

We need the evolution equation for the induced metric g_{ij} of the flow hypersurfaces $M(t)$.

Lemma 3.2. *There holds*

$$\dot{g}_{ij} = -2H^{-1}h_{ij}, \tag{3.3}$$

where h_{ij} denotes the second fundamental form of the flow hypersurfaces.

Proof. See [8, Lemma 2.1]. □

For $g = \det(g_{ij})$ we conclude from Lemma 3.2 that

$$\frac{d}{dt}\sqrt{g} = -\sqrt{g} \tag{3.4}$$

and therefore

$$|M(t)| = |M_0|e^{-t} = |M_0|(1-\tau)^n. \tag{3.5}$$

To prove (1.10) we deduce from Lemma 2.3 that

$$H|_{M(\tau)} = H|_{M(t)} \geq e^{\frac{t}{n}}c_0 = (1-\tau)^{-1}c_0 > \sqrt{n\Lambda} \tag{3.6}$$

for all $\tau_0 \leq \tau < 1$ with $0 < \tau_0 < 1$ suitable, and apply Theorem 3.1.

4. Appendix

Lemma 4.1. *Let N, (x^α) be as in Lemma 2.4 and $M \subset N$ a compact, connected and spacelike C^m-hypersurface. Then there exists $u \in C^m(S_0)$ so that*

$$M = \operatorname{graph} u|_{S_0}. \tag{4.1}$$

Proof. We combine the techniques in the proofs of [6, Proposition 2.5] and [11, Proposition 3.2.5].

(i) The projection

$$\operatorname{pr} : M \longrightarrow S_0, \quad p = \left(x^0(p), x(p)\right) \mapsto (0, x(p)) \tag{4.2}$$

is continuous and injective, the latter will be shown in (ii). Hence

$$G \equiv \operatorname{pr}(M) \subset S_0 \tag{4.3}$$

is closed and $M = \operatorname{graph} u|_G$ with a suitable $u \in C^0(G)$. We will show that $G \subset S_0$ is also open ($\Rightarrow G = S_0$) and that $u \in C^m(S_0)$. For this let $z_0 \in G$ be arbitrary, $p = (u(z_0), z_0) \in M$ and U a neighbourhood of p in N, so that there is a $\varphi \in C^m(U)$ with

$$U \cap M = \left\{\left(x^0, x\right) \in U : \varphi\left(x^0, x\right) = 0\right\}. \tag{4.4}$$

Since M is spacelike, $D\varphi$ is timelike and hence

$$\frac{\partial \varphi}{\partial x^0} = \left\langle D\varphi, \frac{\partial}{\partial x^0}\right\rangle \neq 0. \tag{4.5}$$

Due to the implicit function theorem there are neighbourhoods V of z_0 in S_0 and $\tilde{U}$ of p in U with

$$\tilde{U} \cap M = \operatorname{graph} \psi|_V, \quad \psi \in C^m(V). \tag{4.6}$$

This proves the lemma in view of $\psi = u|_V$.

(ii) We show that pr is injective. Let $\bar{x} \in M$ be so that

$$x^0(\bar{x}) = \inf_M x^0, \tag{4.7}$$

where we assume w.l.o.g. that this infimum is strictly larger than 0.

For a point $p \in M$ we define

$$(4.8) \qquad z(p) = \left\{ (t, \mathrm{pr}(p)) : 0 \le t < x^0(p) \right\}.$$

The set

$$(4.9) \qquad \Lambda = \{ p \in M : z(p) \cap M = \emptyset \}$$

is nonempty, since $\bar{x} \in \Lambda$ and closed and open as will be proven. Hence we deduce $\Lambda = M$ and the injectivity of pr.

(a) For proving that Λ is closed let $p_k \in \Lambda$ be a sequence with $p_k \to p_0 \in M$. If $p_0 \notin \Lambda$, then there exists $\tilde{p}_0 \in z(p_0) \cap M$ and by implicit function theorem neighborhoods U of $\mathrm{pr}(p_0)$ in S_0 and V of $\tilde{p}_0$ in M as well as $u \in C^m(U)$ with

$$(4.10) \qquad V = \mathrm{graph} \; u|_U \quad \wedge \quad u < x^0(p_0).$$

This implies $p_k \notin \Lambda$ for a.e. k, a contradiction.

(b) Suppose that Λ is not open, then there exists $p_0 \in \Lambda$, a sequence $p_k \in M \backslash \Lambda$ with $p_k \to p_0$ and a sequence $q_k \in z(p_k) \cap M$.

Due to inverse function theorem there is a neighbourhood U of p_0 in N, so that $U \cap M$ is a C^m-graph over a suitable subset of S_0, which implies

$$(4.11) \qquad \overline{\lim}_{k \to \infty} x^0(q_k) < x^0(p_0).$$

Hence a subsequence of the q_k converges to a point in $z(p_0) \cap M$, a contradiction. $\square$

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