AN INVERSE CURVATURE FLOW IN A SPACETIME WITH A FUTURE SINGULARITY

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Abstract. We consider an inverse curvature flow (ICF)

$$\dot{x} = -F^{-1}\nu \tag{0.1}$$

in a Lorentzian manifold N with a certain future singularity, which is still quite general. Here, F denotes a curvature function of class (K^*), which is homogeneous of degree one, e.g. the n-th root of the Gaussian curvature, and ν the past directed normal.

We prove longtime existence of the ICF and that the leaves of the ICF provide a foliation of the future of the initial flow hypersurface.

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1. Introduction and main result

In [6] it is shown that the inverse mean curvature flow in cosmological spacetimes satisfying some structural assumptions exists for all times and that the leaves of the flow provide a foliation of the future of the initial flow hypersurface.

There is a result of the author [11], where it is shown that the structural assumptions in [6] can be weakened.

Our aim is to extend the above mentioned result in [6] to a class of inverse curvature flows, for which the normal speed is given by the inverses of fairly general functions of the principal curvatures, i.e. we prove longtime existence for these flows and that the corresponding flow hypersurfaces provide a foliation of the future of the initial flow hypersurface.

Concerning the necessary structure of the ambient space N we let us inspire by [7], where the inverse mean curvature flow is considered in Lorentzian manifolds with a special future singularity, so-called ARW spaces, which are defined by several technical assumptions. For a definition of ARW spaces we refer to [10, Chapter 7] and especially concerning their relationship to the Einstein equations to [10, Section 7.9].

For our purposes we assume a spacetime as in Definition 1.1, which differs from ARW spaces mainly in the fact, that we don't need two assumptions concerning the asymptotical behaviour of higher order derivatives of the metric.

There is a paper in preparation by the author, in which it is shown that for our inverse curvature flows we get in ARW spaces – after rescaling – an analogous asymptotical behaviour as in [7] for the rescaled inverse mean curvature flow, especially the rescaled flows define a natural diffeomorphism through the singularity into a mirrored spacetime.

The present paper is based on parts of the author's PhD thesis [12] and the author would like to thank Claus Gerhardt under whose supervision the thesis was carried out.

Definition 1.1. (Definition of the ambient space N) Let $N = N^{n+1}$ be a globally hyperbolic spacetime within which a future end N_+ of N can be written as a product $[a, 0) \times S_0$, where S_0 is a compact Riemannian space and there exists a future directed time function $\tau = x^0$ so that the metric in N_+ can be written as

$$(\bar{g}_{\alpha\beta}) = e^{2\tilde{\psi}}\{-(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j\}, \tag{1.1}$$

where S_0 corresponds to

$$x^0 = a. \tag{1.2}$$

We assume that $\tilde{\psi}$ is of the form

$$\tilde{\psi}(x^0, x) = f(x^0) + \psi(x^0, x) \tag{1.3}$$

and that there exists a positive constant c_0 and a smooth Riemannian metric $\bar{\sigma}_{ij}$ on S_0 such that for $\tau \to 0$

$$e^{\psi(\tau,x)} \to c_0 \quad \wedge \quad \sigma_{ij}(\tau, x) \to \bar{\sigma}_{ij}(x) \quad \wedge \quad f(\tau) \to -\infty \quad \wedge \quad f^{'}(\tau) \to -\infty. \tag{1.4}$$

We assume that the derivatives of arbitrary order with respect to space and time of $e^{-2f}\breve{g}_{\alpha\beta}$ converge for $x^0 \to 0$ uniformly to the corresponding derivatives of the following metric

$$-(dx^0)^2 + \bar{\sigma}_{ij}(x)dx^i dx^j. \tag{1.5}$$

Furthermore there exists $\tilde{\gamma} > 0$ such that

$$\lim_{\tau\to 0} |f^{'}|e^{\tilde{\gamma} f} > 0 \quad \wedge \quad \lim_{\tau\to 0}(f^{''} + \tilde{\gamma}|f^{'}|^2) \text{ exists.} \tag{1.6}$$

If $\tilde{\gamma} < 1$ we assume that

$$\bar{\sigma}_{ij} \text{ has non-negative sectional curvature.} \tag{1.7}$$

Remark 1.2. The limit

$$\lim_{\tau\to 0} \frac{\tilde{\gamma} f^{'}\tau - 1}{\tau^2} \tag{1.8}$$

exists.

Proof. See [10, Lemma 7.3.4]. □

We can now state our main theorem, cf. also Section 2 for notations.

Theorem 1.3. *Let N be as in Definition 1.1. Let $F \in C^\infty(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ be a curvature function of class (K^*), cf. Definition (2.3), in the positive cone $\Gamma_+ \subset \mathbb{R}^n$, which is in addition positiv homogeneous of degree one and normalized such that*

$$F(1, ..., 1) = n. \tag{1.9}$$

Let M_0 be a smooth, closed, spacelike hypersurface in N which can be written as a graph over S_0 for which we furthermore assume that it is convex and that it satisfies

$$-\epsilon < \inf_{M_0} x^0 < 0, \tag{1.10}$$

where

$$\epsilon = \epsilon(N, \breve{g}_{\alpha\beta}) > 0. \tag{1.11}$$

Then the inverse curvature flow (ICF) given by the equation

$$\dot{x} = -\frac{1}{F}\nu \tag{1.12}$$

with initial surface $x(0) = M_0$ exists for all times and the flow hypersurfaces $M(t)$ provide a foliation of the future $I^+(M_0)$ of M_0. Here, ν denotes the past directed normal.

The flow hypersurfaces $M(t)$ can be expressed as graphs over S_0

$$M(t) = \operatorname{graph} u(t, \cdot), \tag{1.13}$$

and there exist constants $c_1, c_2 > 0$ with

$$a < -c_1 e^{-\frac{\tilde{\gamma}}{n}t} \le u \le -c_2 e^{-\frac{\tilde{\gamma}}{n}t} < 0. \tag{1.14}$$

Moreover there is a constant $c > 0$ such that

$$F \ge c e^{\frac{(\tilde{\gamma}+1)}{n}t} \tag{1.15}$$

for all $t \ge 0$.

2. Notations and definitions

In this section, where we want to introduce some general notations, we let N be as in Definition 1.1; now the special structure of $\tilde{\psi}$ and σ_{ij}, cf. (1.3) et seqq., is not necessarily needed.

Let $M \subset N$ be a connected and spacelike hypersurface with differentiable normal ν (which is then timelike). Geometric quantities in N are denoted by $(\bar{g}_{\alpha\beta})$, $\left(\bar{R}_{\alpha\beta\gamma\delta}\right)$ etc. and those in M by (g_{ij}), (R_{ijkl}) etc.. Greek indices range from 0 to n, Latin indices from 1 to n; summation convention is used. Coordinates in N and M are denoted by (x^α) and $\left(\xi^i\right)$ respectively. Covariant derivatives are written as indices, only in case of possibly confusion we precede them by a semicolon, i.e. for a function u the gradient is (u_α) and $(u_{\alpha\beta})$ the hessian, but for the covariant derivative of the Riemannian curvature tensor we write $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$.

In local coordinates, (x^α) in N and $\left(\xi^i\right)$ in M, the following four important equations hold; the Gauss formular

$$x^\alpha_{ij} = h_{ij}\nu^\alpha. \tag{2.1}$$

In this implicit definition (h_{ij}) is the second fundamental form of M with respect to ν. Here and in the following a covariant derivative is always a full tensor, i.e.

$$x^\alpha_{ij} = x^\alpha_{,ij} - \Gamma^k_{ij}x^\alpha_k + \bar{\Gamma}^\alpha_{\beta\gamma}x^\beta_i x^\gamma_j \tag{2.2}$$

and the comma denotes ordinary partial derivatives.

The second equation is the Weingarten equation

$$\nu^\alpha_i = h^k_i x^\alpha_k, \tag{2.3}$$

where ν^α_i is a full tensor. The third equation is the Codazzi equation

$$h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x^\beta_i x^\gamma_j x^\delta_k \tag{2.4}$$

and the fourth is the Gauß equation

$$R_{ijkl} = -\left\{h_{ik}h_{jl} - h_{il}h_{jk}\right\} + \bar{R}_{\alpha\beta\gamma\delta}x^\alpha_i x^\beta_j x^\gamma_k x^\delta_l. \tag{2.5}$$

As an example for the covariant derivative of a full tensor we give

$$\bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon}x^\epsilon_i, \tag{2.6}$$

where this identity follows by applying the chain rule from the definition of the covariant derivative of a full tensor; it can be generalized obviously to other quantities.

Let (x^α) be a future directed coordinate system in N, then the contravariant vector $(\xi^\alpha) = (1, 0, ..., 0)$ is future directed; as well its covariant version $(\xi_\alpha) = e^{2\psi}(-1, 0, ..., 0)$.

Now we want to express normal, metric and second fundamemtal form for spacelike hypersurfaces, which can be written as graphs over the Cauchyhypersurface. Let $M = \operatorname{graph} u|_{S_0}$ be a spacelike hypersurface in N, i.e.

$$M = \left\{\left(x^0, x\right) : x^0 = u(x),\ x \in S_0\right\}, \tag{2.7}$$

then the induced metric is given by

$$g_{ij} = e^{2\tilde{\psi}}\left\{-u_i u_j + \sigma_{ij}\right\}, \tag{2.8}$$

where σ_{ij} is evaluated at (u,x) and the inverse $\left(g^{ij}\right)=(g_{ij})^{-1}$ is given by

$$g^{ij}=e^{-2\tilde\psi}\left\{\sigma^{ij}+\frac{u^iu^j}{v^2}\right\}, \tag{2.9}$$

where $\left(\sigma^{ij}\right)=(\sigma_{ij})^{-1}$ and

$$\begin{aligned} u^i &= \sigma^{ij}u_j \\ v^2 &= 1-\sigma^{ij}u_iu_j\equiv 1-|Du|^2, \quad v>0. \end{aligned} \tag{2.10}$$

We define $\tilde v=v^{-1}$.

From (2.8) we conclude that $\operatorname{graph} u$ is spacelike if and only if $|Du|<1$.

The covariant version of the normal of a graph is

$$(\nu_\alpha)=\pm v^{-1}e^{\tilde\psi}\,(1,-u_i) \tag{2.11}$$

and the contravariant version

$$(\nu^\alpha)=\mp v^{-1}e^{-\tilde\psi}\left(1,u^i\right). \tag{2.12}$$

We have

Remark 2.1. Let M be a spacelike graph in a future directed coordinate system, then

$$(\nu^\alpha)=v^{-1}e^{-\tilde\psi}\left(1,u^i\right) \tag{2.13}$$

is the contravariant future directed normal and

$$(\nu^\alpha)=-v^{-1}e^{-\tilde\psi}\left(1,u^i\right) \tag{2.14}$$

the past directed.

In the following we choose ν always as the past directed normal.

Let us consider the component $\alpha=0$ in (2.1), so we have due to (2.14) that

$$e^{-\tilde\psi}v^{-1}h_{ij}=-u_{ij}-\bar\Gamma^0_{00}u_iu_j-\bar\Gamma^0_{0j}u_i-\bar\Gamma^0_{0i}u_j-\bar\Gamma^0_{ij}, \tag{2.15}$$

where u_{ij} are covariant derivatives with respect to M. Choosing $u\equiv\text{const}$, we deduce

$$e^{-\tilde\psi}\bar h_{ij}=-\bar\Gamma^0_{ij}, \tag{2.16}$$

where $\bar h_{ij}$ is the second fundamental form of the hypersurface $\left\{x^0=\text{const}\right\}$. An easy calculation shows

$$e^{-\tilde\psi}\bar h_{ij}=-\frac12\dot\sigma_{ij}-\dot\psi\sigma_{ij}, \tag{2.17}$$

where the dot indicates differentiation with respect to x^0.

Now we define the classes (K) and (K^*), which are special classes of curvature functions; for a more detailed treatment of these classes we refer to [10, Section 2.2].

For a curvature function F (i.e. symmetric in its variables) in the positive cone $\Gamma_+\subset\mathbb{R}^n$ we define

$$F(h_{ij})=F(\kappa_i), \tag{2.18}$$

where the κ_i are the eigenvalues of an arbitrary symmetric tensor (h_{ij}), whose eigenvalues are in Γ_+.

Definition 2.2. A symmetric curvature function $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$, positively homogeneous of degree $d_0 > 0$, is said to be of class (K), if

$$F_i = \frac{\partial F}{\partial \kappa^i} > 0 \quad \text{in } \Gamma_+, \tag{2.19}$$

$$F_{|\partial\Gamma_+} = 0, \tag{2.20}$$

and

$$F^{ij,kl}\eta_{ij}\eta_{kl} \le F^{-1}(F^{ij}\eta_{ij})^2 - F^{ik}\tilde{h}^{jl}\eta_{ij}\eta_{kl} \quad \forall\ \eta \in S, \tag{2.21}$$

where F is evaluated at an arbitrary symmetric tensor (h_{ij}), whose eigenvalues are in Γ_+ and S denotes the set of symmetric tensors. Here, F_i is a partial derivative of first order with respect to κ_i and $F^{ij,kl}$ are second partial derivatives with respect to (h_{ij}). Furthermore $(\tilde{h}^{ij})$ is the inverse of (h_{ij}).

In Theorem 1.3 the κ_i in (2.18) are the eigenvalues of the second fundamental form (h_{ij}) with respect to the metric (g_{ij}), i.e. the principal curvatures of the flow hypersurfaces.

Definition 2.3. A curvature function $F \in (K)$ is said to be of class (K^*), if there exists $0 < \epsilon_0 = \epsilon_0(F)$ such that

$$\epsilon_0 FH \le F^{ij}h_{ik}h_j^k, \tag{2.22}$$

for any symmetric (h_{ij}) with all eigenvalues in Γ_+, where F is evaluated at (h_{ij}). H represents the mean curvature, i.e. the trace of (h_{ij}).

In the following a '+' sign attached to the symbol of a metric of the ambient space refers to the corresponding Riemannian background metric, if attached to an induced metric, it refers to the induced metric relative to the corresponding Riemannian background metric. Let us consider as an example the metrics $\breve{g}_{\alpha\beta}$ and g_{ij} introduced as above, then

$$\overset{+}{\breve{g}}_{\alpha\beta} = e^{2\tilde{\psi}}\{(dx^0)^2 + \sigma_{ij}(x^0,x)dx^i dx^j\}, \quad \overset{+}{g}_{ij} = \overset{+}{\breve{g}}_{\alpha\beta}\ x_i^\alpha x_j^\beta. \tag{2.23}$$

3. Lower order estimates

Let $M_\tau = \{x^0 = \tau\}$ denote the coordinate slices. Then

$$|M_\tau| = \int_{S_0} e^{n\tilde{\psi}(\tau,x)}\sqrt{|\det \sigma_{ij}(\tau,x)|}dx \longrightarrow 0, \quad \tau \to 0. \tag{3.1}$$

And for the second fundamental form $\bar{h}_{ij}$ of the M_τ we have

$$\bar{h}_j^i = -e^{-\tilde{\psi}}(\frac{1}{2}\sigma^{ik}\dot{\sigma}_{kj} + \dot{\tilde{\psi}}\delta_j^i), \tag{3.2}$$

hence there exists τ_0 such that M_τ is convex for all $\tau \ge \tau_0$.

From Lemma 4.2 (iii) and the homogeneity of F we conclude that for $m \in \mathbb{N}$ there is $a < \tau_0 < 0$ and $c_m > 0$ such that for all $\tau_0 \le \tau < 0$ we have

$$\begin{aligned} \varphi_1(\tau) := -nf' + c_m|\tau|^m \ge e^{\tilde{\psi}}F|_{M_\tau} = e^{\tilde{\psi}}F(\bar{h}_j^i) = F(-\frac{1}{2}\sigma^{ik}\dot{\sigma}_{kj} - \dot{\tilde{\psi}}\delta_j^i) \\ \ge -nf' - c_m|\tau|^m =: \varphi_2(\tau). \end{aligned} \tag{3.3}$$

We will show that the flow does not run into the future singularity within finite time.

Lemma 3.1. *There exists a time function $\tilde{x}^0 = \tilde{x}^0(x^0)$, so that the F-curvature $\bar{F}$ of the slices $\{\tilde{x}^0 = const\}$ satisfies*

$$e^{\tilde{\tilde{\psi}}}\bar{F} \ge 1. \tag{3.4}$$

$e^{\tilde{\tilde{\psi}}}$ is the conformal factor in the representation of the metric with respect to the coordinates $(\tilde{x}^0, x^i)$, i.e.

$$d\breve{s} = e^{2\tilde{\tilde{\psi}}}\{-(d\tilde{x}^0)^2 + \tilde{\sigma}_{ij}(\tilde{x}^0, x)dx^i dx^j\}. \tag{3.5}$$

Furthermore there holds

$$\tilde{x}^0(\{\tau_0 \le x^0 < 0\}) = [0, \infty) \tag{3.6}$$

and the future singularity corresponds to $\tilde{x}^0 = \infty$.

Proof. Define $\tilde{x}^0$ by

$$\tilde{x}^0 = \int_{\tau_0}^{\tau} \varphi_2(s)ds = -nf(\tau) - \frac{c_m}{m+1}|\tau|^{m+1} + nf(\tau_0) + \frac{c_m}{m+1}|\tau_0|^{m+1} \to \infty, \quad \tau \to 0, \tag{3.7}$$

where φ_2 is chosen as in (3.3). For the conformal factor in (3.5) we have

$$e^{2\tilde{\tilde{\psi}}} = e^{2\tilde{\psi}}\frac{\partial x^0}{\partial \tilde{x}^0}\frac{\partial x^0}{\partial \tilde{x}^0} = e^{2\tilde{\psi}}\varphi_2^{-2} \tag{3.8}$$

and therefore

$$e^{\tilde{\tilde{\psi}}}\bar{F} = e^{\tilde{\psi}}\bar{F}\varphi_2^{-1} \ge 1. \tag{3.9}$$

□

The evolution problem (1.12) is a parabolic problem, hence a solution of the ICF exists on a maximal time interval $[0, T^*)$, $0 < T^* \le \infty$.

Lemma 3.2. *We assume the situation of Theorem 1.3, $0 < T \le T^*$ finite and that for $0 \le t \le T$ the flow hyperfurfaces $M(t)$ of the ICF can be expressed as a graph of a function u over S_0*

$$M(t) = \{(x^0, x) : x^0 = u(t, x), x \in S_0\}. \tag{3.10}$$

Then the flow stays in a precompact set Ω_T for $0 \le t < T$.

Proof. We choose for x^0 the special time function in Lemma 3.1 with

$$e^{\tilde{\psi}}\bar{F} \ge 1 \tag{3.11}$$

for the coordinate slices $\{x^0 = const\}$. Let

$$M(t) = \operatorname{graph} u(t, \cdot) \tag{3.12}$$

be the flow hypersurfaces in this coordinate system and

$$\varphi(t) = \sup_{S_0} u(t, \cdot) = u(t, x_t) \tag{3.13}$$

with suitable $x_t \in S_0$. It is well-known that φ is Lipschitz continuous and that for a.e. $0 \le t < T$

$$\dot{\varphi}(t) = \frac{\partial}{\partial t}u(t, x_t). \tag{3.14}$$

From (2.15) we deduce in x_t the relation

$$h_{ij} \ge \bar{h}_{ij}, \tag{3.15}$$

hence

$$F \geq \bar{F}. \tag{3.16}$$

We look at the component $\alpha = 0$ in (1.12) and get

$$\dot{u} = \frac{\tilde{v}}{F e^{\tilde{\psi}}}, \tag{3.17}$$

where

$$\dot{u} = \frac{\partial u}{\partial t} + u_i \dot{x}^i \tag{3.18}$$

is a total derivative. This yields

$$\frac{\partial u}{\partial t} = e^{-\tilde{\psi}} v \frac{1}{F}, \tag{3.19}$$

so that we have in x_t

$$\frac{\partial u}{\partial t} = \frac{1}{e^{\tilde{\psi}} F} \leq \frac{1}{e^{\tilde{\psi}} \bar{F}} \leq 1. \tag{3.20}$$

With (3.14) we conclude

$$\varphi \leq \varphi(0) + t \quad \forall 0 \leq t < T^*, \tag{3.21}$$

which proves the lemma, since the future singularity corresponds to $x^0 = \infty$. □

Lemma 3.3. *We assume in the situation of Theorem 1.3 that the flow exists for all times , i.e. $T^* = \infty$, and that the flow hypersurfaces can be written as graphs over S_0 then the flow runs into the future singularity, i.e.*

$$\forall_{s \in [a,0)} \exists_{t_0 > 0} \forall_{t \geq t_0} \inf_{M(t)} x^0 \geq s. \tag{3.22}$$

Proof. Similar arguments as in the proofs of Lemmata 3.1 and 3.2 using the functions φ_1 and

$$\varphi(t) = \inf_{S_0} u(t, \cdot) \tag{3.23}$$

prove the lemma. □

We have actually a precise rate of convergence with respect to time.

Corollary 3.4. *In the situation of Lemma 3.3, especially the time function x^0 is assumed as in Theorem 1.3, there are $c_1, c_2 > 0$ such that*

$$-c_1 e^{-\frac{\tilde{\gamma}}{n} t} \leq x^0|_{M(t)} \leq -c_2 e^{-\frac{\tilde{\gamma}}{n} t} \tag{3.24}$$

for all $t \geq 0$.

Proof. A careful view on the proofs of the previous lemmas shows that there are $\tilde{c}_1, \tilde{c}_2 > 0$ such that for large t

$$f^{-1}(\tilde{c}_1 - \frac{t}{n}) \leq x^0|_{M(t)} \leq f^{-1}(\tilde{c}_2 - \frac{t}{n}). \tag{3.25}$$

Inserting in the monotone decreasing function $e^{\tilde{\gamma} f}$ yields

$$e^{\tilde{\gamma}(\tilde{c}_1 - \frac{t}{n})} \geq e^{\tilde{\gamma} f(x^0)}|_{M(t)} \geq e^{\tilde{\gamma}(\tilde{c}_2 - \frac{t}{n})}. \tag{3.26}$$

The claim follows in view of (1.8) and (1.6). □

4. Estimates for general spacelike graphs in N

Lemma 4.1. *Let N be as in Theorem 1.3. Let $\Omega \subset N$ be precompact and $M \subset \Omega$ a convex spacelike graph over S_0, then the quantity $\tilde{v}$ is bounded by a constant, which only depends on Ω.*

Proof. We use the convexity of M together with [10, Theorem 2.7.11]. □

In the following we generalize Lemma 4.1 by omitting the assumption that $M \subset \Omega$, Ω precompact. We will get a bound for $\tilde{v}$, which is related to the data of the ambient space.

We introduce a special convention for our notation, which is only used in this section. The metric of N, which is defined in (1.1), will be denoted by $\breve{g}_{\alpha\beta}$, deduced quantities like the induced metric of hypersurfaces, the second fundamental form, a unit normal or the curvature tensor by $\breve{g}_{ij}$, $\breve{h}_{ij}$, $\breve{\nu}$, $\breve{R}_{\alpha\beta\gamma\delta}$.

We can equip N with the conformal metric

$$\bar{g}_{\alpha\beta} = e^{-2\tilde{\psi}} \breve{g}_{\alpha\beta} = -(dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j, \tag{4.1}$$

and deduced quantities like the induced metric of hypersurfaces, the second fundamental form, a unit normal or the curvature tensor will be denoted by g_{ij}, h_{ij}, ν, $\bar{R}_{\alpha\beta\gamma\delta}$.

The second fundamental form with respect to $\bar{g}_{\alpha\beta}$ of a coordinate slice $\{x^0 = \textit{const}\}$ is denoted by $\bar{h}_{ij}$.

The second fundamental forms $\breve{h}^j_i$ and h^j_i are related by

$$e^{\tilde{\psi}} \breve{h}^j_i = h^j_i + \tilde{\psi}_\alpha \nu^\alpha \delta^j_i = h^j_i - \tilde{v} f' \delta^j_i + \psi_\alpha \nu^\alpha =_{def} \check{h}^j_i, \tag{4.2}$$

cf. [10, Proposition 1.1.11]. When we insert $\check{h}^j_i$ into F we will denote the result in accordance with our convention as $\breve{F}$. Due to a lack of convexity it would not make any sense to insert h^j_i into the curvature function F, so that we stipulate that the symbol F will stand for

$$F = e^{\tilde{\psi}} \breve{F} = F(h^j_i - \tilde{v} f' \delta^j_i + \psi_\alpha \nu^\alpha). \tag{4.3}$$

We prove a decay property of certain tensors.

Lemma 4.2. *In the following all covariant derivatives are taken with respect to the metric $\bar{g}_{\alpha\beta}$ and $|||\cdot|||$ will denote the norm with respect to*

$$\overset{+}{g}_{\alpha\beta} = (dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j. \tag{4.4}$$

(i) Let $\varphi \in C^\infty([a, 0))$, $a < 0$, and assume

$$\lim_{\tau \to 0} \varphi^{(k)}(\tau) = 0 \quad \forall k \in \mathbb{N}, \tag{4.5}$$

then for every $k \in \mathbb{N}$ there exists a $c_k > 0$ such that

$$|\varphi(\tau)| \le c_k |\tau|^k. \tag{4.6}$$

(ii) Let T be a tensor such that for all $k \in \mathbb{N}$

$$|||D^k T(x^0, x)||| \longrightarrow 0 \quad \textit{as} \quad x^0 \longrightarrow 0 \quad \textit{uniformly in } x \tag{4.7}$$

then

$$\forall_{k \in \mathbb{N}} \quad \exists_{c_k > 0} \quad \forall_{x \in S_0} \quad |||T(x^0, x)||| \le c_k |x^0|^k. \tag{4.8}$$

(iii) Let $(\eta_\alpha)=(-1,0,...,0)$ be a covariant vector field, then for its first covariant derivative $T=(\eta_{\alpha\beta})$ the relation (4.8) holds, analogously for $|||\eta_{\alpha\beta\gamma}|||$, $|||D\psi|||$, $|||\bar R_{\alpha\beta\gamma\delta}\eta^\alpha|||$, or more generally for any tensor that would vanish identically, if it would have been formed with respect to the product metric

$$-(dx^0)^2+\bar\sigma_{ij}dx^idx^j. \tag{4.9}$$

Proof. See [10, Lemma 7.4.4]. □

Lemma 4.3. *Let N be as in Theorem 1.3. Let $\epsilon>0$ be arbitrary, then there exists $\delta=\delta((N,\breve g_{\alpha\beta}),\epsilon)>0$ such that for every closed, spacelike, convex hypersurface M in the end $N_\delta^+=\{x^0>-\delta\}$ holds*

$$\tilde v\le \epsilon|f'|^{\frac{1}{\tilde\gamma}}. \tag{4.10}$$

Proof. Let $p>\tilde\gamma^{-1}$ and define

$$w=\tilde v\{e^f+|u|^p\} \tag{4.11}$$

and look at a point, where w attains its maximum, and infer

$$\begin{aligned} 0=w_i&=\tilde v_i\{e^f+|u|^p\}+\tilde v\{e^ff'-p|u|^{p-1}\}u_i\\ &=\{-h_{ik}u^k+\tilde v^{-1}\bar h_{ik}u^k\}\{e^f+|u|^p\}+\tilde v\{e^ff'-p|u|^{p-1}\}u_i\\ &=\{-\breve h_{ik}u^ke^{-\tilde\psi}-\tilde vf'u_i+\tilde\epsilon\tilde vu_i\}\{e^f+|u|^p\}+\tilde v\{e^ff'-p|u|^{p-1}\}u_i, \end{aligned} \tag{4.12}$$

where

$$|\tilde\epsilon|\le c_m|u|^m\quad \forall\ m\in\mathbb{N}. \tag{4.13}$$

Multiplying by u^i and assuming $Du\ne0$ we get the inequality

$$\begin{aligned} 0&\le(-f'+\tilde\epsilon)\{e^f+|u|^p\}+e^ff'-p|u|^{p-1}\\ &=-f'|u|^p+\tilde\epsilon\{e^f+|u|^p\}-p|u|^{p-1}<0, \end{aligned} \tag{4.14}$$

if $\delta>0$ small, since

$$f'u\le\tilde\gamma^{-1}+cu^2, \tag{4.15}$$

c.f. (1.8). This is a contradiction, hence $Du=0$.

Since

$$\varphi(\tau)=e^{f(\tau)}+|\tau|^p,\quad a\le\tau<0, \tag{4.16}$$

is monotone decreasing we conclude

$$\tilde v\le\frac{e^{f(u_{\min})}+|u_{\min}|^p}{e^{f(u)}+|u|^p}\le(e^{f(u_{\min})}+|u_{\min}|^p)e^{-f(u)}, \tag{4.17}$$

where $u_{\min}=\inf u$. Choosing δ appropiately small finishes the proof. □

Corollary 4.4. *Let $\delta>0$ be small and N_δ^+ and M be as in Lemma 4.3, then*

$$F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta\ge -c\delta F^{ij}g_{ij}, \tag{4.18}$$

if the limit metric $\bar\sigma_{ij}$ has non-negative sectional curvature.

Proof. We define

$$\bar R_{\alpha\beta\gamma\delta}(0,\cdot)=\lim_{\tau\uparrow 0}\bar R_{\alpha\beta\gamma\delta}(\tau,\cdot) \tag{4.19}$$

and have

$$\begin{aligned} F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta & \\ &=F^{ij}(\bar R_{\alpha\beta\gamma\delta}(0,\cdot)+\bar R_{\alpha\beta\gamma\delta}(u,\cdot)-\bar R_{\alpha\beta\gamma\delta}(0,\cdot))\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta\\ &\ge F^{ij}(\bar R_{\alpha\beta\gamma\delta}(u,\cdot)-\bar R_{\alpha\beta\gamma\delta}(0,\cdot))\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta\\ &\ge -|||F^{ij}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta|||\cdot|||\bar R_{\alpha\beta\gamma\delta}(u,\cdot)-\bar R_{\alpha\beta\gamma\delta}(0,\cdot)|||\\ &\ge -c_m|u|^m F^{ij}g_{ij}, \end{aligned} \tag{4.20}$$

for arbitrary $m\in\mathbb{N}$ and suitable c_m. Note that we used for the last inequality that

$$\bar R_{\alpha\beta\gamma\delta}(x^0,\cdot)-\bar R_{\alpha\beta\gamma\delta}(0,\cdot) \tag{4.21}$$

satisfies (4.7). □

We want to formulate the relation of the curvature tensors for conformal metrics.

Lemma 4.5. *The curvature tensors of the metrics $\breve g_{\alpha\beta},\bar g_{\alpha\beta}$ are related by*

$$\begin{aligned} e^{-2\tilde\psi}\breve R_{\alpha\beta\gamma\delta} =&\bar R_{\alpha\beta\gamma\delta}-\bar g_{\alpha\gamma}\tilde\psi_{\beta\delta}-\bar g_{\beta\delta}\tilde\psi_{\alpha\gamma}+\bar g_{\alpha\delta}\tilde\psi_{\beta\gamma}+\bar g_{\beta\gamma}\tilde\psi_{\alpha\delta}\\ &+\bar g_{\alpha\gamma}\tilde\psi_\beta\tilde\psi_\delta+\bar g_{\beta\delta}\tilde\psi_\alpha\tilde\psi_\gamma-\bar g_{\alpha\delta}\tilde\psi_\beta\tilde\psi_\gamma-\bar g_{\beta\gamma}\tilde\psi_\alpha\tilde\psi_\delta\\ &+\{\bar g_{\alpha\delta}\bar g_{\beta\gamma}-\bar g_{\alpha\gamma}\bar g_{\beta\delta}\}\|D\tilde\psi\|^2, \end{aligned} \tag{4.22}$$

where covariant derivatives and $\|\cdot\|$ are taken with respect to $\bar g_{\alpha\beta}$.

Proof. Straightforward calculation. □

Now we are able to prove the following lemma which is necessary for the C^2-estimates in the next section.

Lemma 4.6. *Let $\tilde c>0$, then there is $\delta=\delta(N,\breve g_{\alpha\beta})>0$ such that for N_δ^+ and M as in Lemma 4.3 holds*

$$\breve F^{ij}\breve R_{\alpha\beta\gamma\delta}\breve\nu^\alpha x_i^\beta\breve\nu^\gamma x_j^\delta\ge\tilde c|f'|^2e^{-2\tilde\psi}. \tag{4.23}$$

Here $\breve F^{ij}$ is evaluated at $\breve h_i^j$.

Proof. In view of the homogeneity of F we have

$$F_j^i=\breve F_j^i, \tag{4.24}$$

hence

$$F^{ij}=e^{2\tilde\psi}\breve F^{ij}. \tag{4.25}$$

We have due to Lemma 4.5

$$\begin{aligned} e^{2\tilde\psi}\breve F^{ij}\breve R_{\alpha\beta\gamma\delta}\breve\nu^\alpha x_i^\beta\breve\nu^\gamma x_j^\delta & \\ &=F^{ij}\bar R_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta+F^{ij}x_i^\beta x_j^\delta\tilde\psi_{\beta\delta}-F^{ij}g_{ij}\tilde\psi_{\alpha\gamma}\nu^\alpha\nu^\gamma\\ &\quad -F^{ij}x_i^\beta x_j^\delta\tilde\psi_\beta\tilde\psi_\delta+F^{ij}g_{ij}\tilde\psi_\alpha\tilde\psi_\gamma\nu^\alpha\nu^\gamma+F^{ij}g_{ij}\|D\tilde\psi\|^2. \end{aligned} \tag{4.26}$$

We have

$$\overset{+}{g}_{ij}\le 2\sigma_{ij}\le 2\tilde v^2 g_{ij}. \tag{4.27}$$

Now we estimate each summand in (4.26) separately with the help of the Riemmanian background metric $\overset{+}{g}_{\alpha\beta}$, namely

$$|F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_j^\delta| \le c\tilde{v}^2(F^{ij}F^{\bar{i}\bar{j}}\overset{+}{g}_{i\bar{i}}\overset{+}{g}_{j\bar{j}})^{\frac{1}{2}} \le c\tilde{v}^2 F^{ij}\sigma_{ij} \le c\tilde{v}^4 F^{ij}g_{ij}, \tag{4.28}$$

$$F^{ij}x_i^\beta x_j^\delta \tilde{\psi}_{\beta\delta} = F^{ij}u_iu_jf^{''} + F^{ij}x_i^\beta x_j^\delta \psi_{\beta\delta} \ge F^{ij}u_iu_jf^{''} - c\tilde{v}^2F^{ij}g_{ij}, \tag{4.29}$$

$$\begin{aligned} -F^{ij}g_{ij}\tilde{\psi}_{\alpha\gamma}\nu^\alpha\nu^\gamma &= -\tilde{v}^2F^{ij}g_{ij}f^{''} - F^{ij}g_{ij}\psi_{\alpha\gamma}\nu^\alpha\nu^\gamma \\ &\ge -\tilde{v}^2F^{ij}g_{ij}f^{''} - c\tilde{v}^2F^{ij}g_{ij}, \end{aligned} \tag{4.30}$$

$$\begin{aligned} -F^{ij}x_i^\beta x_j^\delta\tilde{\psi}_\beta\tilde{\psi}_\delta &= -F^{ij}u_iu_j(\psi_0+f^{'})^2 - F^{ij}\psi_i\psi_j - 2F^{ij}u_j\psi_i(\psi_0+f^{'}) \\ &\ge -F^{ij}u_iu_j(\psi_0+f^{'})^2 - c(1+|f^{'}||Du|)F^{ij}\sigma_{ij}|D\psi| \\ &\ge -F^{ij}u_iu_j(\psi_0+f^{'})^2 - c\tilde{v}^2(1+|f^{'}||Du|)F^{ij}g_{ij}|D\psi| \\ &\ge -F^{ij}u_iu_j(\psi_0+f^{'})^2 - c|f^{'}|\tilde{v}^2F^{ij}g_{ij}, \end{aligned} \tag{4.31}$$

where $|D\psi|^2 = \sigma^{ij}\psi_i\psi_j$,

$$F^{ij}g_{ij}\tilde{\psi}_\alpha\tilde{\psi}_\gamma\nu^\alpha\nu^\gamma \ge \tilde{v}^2(\psi_0+f^{'})^2F^{ij}g_{ij} - c\tilde{v}^2|f^{'}|F^{ij}g_{ij}, \tag{4.32}$$

$$\begin{aligned} F^{ij}g_{ij}\|D\tilde{\psi}\|^2 &= -(f^{'}+\psi_0)^2F^{ij}g_{ij} + \sigma^{ij}\psi_i\psi_jF^{ij}g_{ij} \\ &\ge -(f^{'}+\psi_0)^2F^{ij}g_{ij} - cF^{ij}g_{ij}. \end{aligned} \tag{4.33}$$

Thus we conclude (using $u_iu_j \le (\tilde{v}^2-1)g_{ij}$)

$$\begin{aligned} e^{2\tilde{\psi}}\breve{F}^{ij}\breve{R}_{\alpha\beta\gamma\delta}\breve{\nu}^\alpha x_i^\beta\breve{\nu}^\gamma x_j^\delta \ge &- c\tilde{v}^4F^{ij}g_{ij} + F^{ij}u_iu_jf^{''} - \tilde{v}^2f^{''}F^{ij}g_{ij} \\ &- c\tilde{v}^2|f^{'}|F^{ij}g_{ij} \\ &+ (\psi_0+f^{'})^2F^{ij}(\tilde{v}^2g_{ij} - u_iu_j - g_{ij}) \\ \ge &- c\tilde{v}^4F^{ij}g_{ij} - \tilde{v}^2f^{''}F^{ij}g_{ij} - c|f^{'}|\tilde{v}^2F^{ij}g_{ij}. \end{aligned} \tag{4.34}$$

Now, the claim follows with Lemma 4.3 in case $\tilde{\gamma} \ge 1$.

Let us now consider the case $\tilde{\gamma} < 1$. Due to assumption the limit metric $\bar{\sigma}_{ij}$ has non-negative sectional curvature. Now we use Corollary 4.4 to bound the first summand of the right side of (4.26) from below by the term $-cF^{ij}g_{ij}$, one easily checks that this term replaces the summand with $\tilde{v}^4$ in (4.34) completing the proof. □

5. C^2-estimates

In this section we consider N to be equipped only with the metric in (1.1) and apply standard notation to this case, i.e. the notation conventions in Section 2 are used.

Lemma 5.1. *The following evolution equation holds*

$$\frac{d}{dt}\left(\frac{1}{F}\right) - \frac{1}{F^2}F^{ij}\left(\frac{1}{F}\right)_{ij} = -\frac{1}{F^3}F^{ij}h_{ik}h_j^k - \frac{1}{F^3}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta\nu^\gamma x_j^\delta. \tag{5.1}$$

Proof. cf. [10, Lemma 2.3.4]. □

Lemma 5.2. *In the situation of Theorem 1.3 we assume that the flow exists on a maximal time interval* $[0, T^*)$, $0 < T^* \le \infty$, *and that the flow hypersurfaces can be written as graphs over* S_0. *Furthermore let* ϵ *in (1.11) be so, that (4.23) holds for all flow hypersurfaces. Then*

$$F \ge \inf_{M_0} F \tag{5.2}$$

for $0 \le t < T^*$. *If* $T^* = \infty$ *we have for a constant* $c > 0$

$$F \ge c e^{\frac{(\tilde\gamma+1)}{n} t} \tag{5.3}$$

for all times.

Proof. We define

$$\varphi(t) = \inf_{M(t)} F \tag{5.4}$$

and infer from Lemma 5.1

$$\begin{aligned} \frac{d}{dt} F - F^{-2} F^{ij} F_{ij} =& \frac{1}{F} F^{ij} h_{ik} h_j^k + \frac{1}{F} F^{ij} \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ & - \frac{2}{F^3} F^{ij} F_i F_j, \end{aligned} \tag{5.5}$$

hence using Lemma 4.6, (2.22) and the well-known relation $F \le H$, cf. [10, Lemma 2.2.20], we deduce

$$\dot\varphi(t) \ge \tilde c \frac{|f'|^2}{\varphi} e^{-2f} + \epsilon_0 \varphi, \tag{5.6}$$

especially $\dot\varphi(t) \ge 0$ for a.e. $0 < t < T^*$. If $T^* = \infty$ we use (3.24), (1.8) and (1.6) to conclude from (5.6) that there are positive constants c_3, c_4 such that

$$\dot\varphi(t) \ge c_3 e^{c_4 + \frac{2(\tilde\gamma+1)}{n} t} \varphi^{-1} + \epsilon_0 \varphi, \tag{5.7}$$

hence

$$\varphi \ge c_5 e^{\frac{(\tilde\gamma+1)}{n} t}, c_5 > 0. \tag{5.8}$$

□

Remark 5.3. Due to [10, Lemma 1.8.3], and the remark at the beginning of Section 3, especially inequality (3.2), for every relative compact subset Ω of N lying sufficiently far in the future of N, i.e. $|\inf_\Omega x^0|$ close to 0, there exists a strictly convex function $\chi \in C^2(\bar\Omega)$, this means

$$\chi_{\alpha\beta} \ge c_0 \bar g_{\alpha\beta} \tag{5.9}$$

with a constant $c_0 > 0$.

Lemma 5.4. *The following evolution equation holds*

$$\dot\chi - \frac{1}{F^2} F^{ij} \chi_{ij} = -\frac{2}{F} \chi_\alpha \nu^\alpha - \frac{1}{F^2} F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta \tag{5.10}$$

Proof. Direct calculation. □

Lemma 5.5. *The following evolution equation holds*

$$\begin{aligned} (\log F)' - \frac{1}{F^2} F^{ij} (\log F)_{ij} =& \frac{1}{F^2} F^{ij} h_{ik} h_j^k + \frac{1}{F^2} F^{ij} \bar R_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ & - \frac{1}{F^4} F^{ij} F_i F_j \end{aligned} \tag{5.11}$$

Proof. Use Lemma 5.1. □

Lemma 5.6. *The following evolution equation holds*

$$\dot{\tilde{v}} - \frac{1}{F^2} F^{ij} \tilde{v}_{ij} = - \frac{1}{F^2} F^{ij} h_{ik} h_j^k \tilde{v} - \frac{2}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - \frac{2}{F^2} F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} - \frac{1}{F^2} F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha - \frac{1}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl}, \tag{5.12}$$

where $(\eta_\alpha) = e^{\tilde{\psi}}(-1, 0, ..., 0)$.

Proof. cf. [10, Lemma 2.4.4]. □

Lemma 5.7. *Let* $\Omega \subset N$ *be precompact and assume that the flow stays in* Ω *for* $0 \le t \le T < T^*$*, then the* F*-curvature of the flow hypersurfaces is bounded from above,*

$$0 < F < c(\Omega). \tag{5.13}$$

Proof. Consider the function

$$w = \log F + \lambda \tilde{v} + \mu \chi, \tag{5.14}$$

where $\lambda, \mu > 0$ will be chosen later appropiately. Assume

$$w(t_0, x_0) = \sup_{[0,T]} \sup_{M(t)} w \tag{5.15}$$

with $0 < t_0 \le T$, then we have in (t_0, x_0)

$$\begin{aligned} 0 \le& \dot{w} - \frac{1}{F^2} F^{ij} w_{ij} \\ =& \frac{1}{F^2} F^{ij} h_{ik} h_j^k + \frac{1}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta - \frac{1}{F^4} F^{ij} F_i F_j \\ &- \frac{\lambda}{F^2} F^{ij} h_{ik} h_j^k \tilde{v} - \frac{2\lambda}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - \frac{2\lambda}{F^2} F^{ij} h_j^k x_i^\alpha x_k^\beta \eta_{\alpha\beta} \\ &- \frac{\lambda}{F^2} F^{ij} \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha - \frac{\lambda}{F^2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_k^\gamma x_j^\delta \eta_\epsilon x_l^\epsilon g^{kl} \\ &- \frac{2\mu}{F} \chi_\alpha \nu^\alpha - \frac{\mu}{F^2} F^{ij} \chi_{\alpha\beta} x_i^\alpha x_j^\beta \\ \le& - \epsilon_0 (\frac{\lambda}{2} - 1) \tilde{v} + \frac{c\lambda}{F^2} F^{ij} g_{ij} + c(\mu + \lambda) \frac{1}{F} - c_0 \frac{\mu}{F^2} F^{ij} g_{ij}. \end{aligned} \tag{5.16}$$

Now we choose $\lambda > 2$ arbitrary and $\mu >> 1$ large and we deduce that F is a priori bounded from above in (t_0, x_0) from which we conclude the Lemma. □

Let $\Omega \subset N$ be precompact and assume that the flow stays in Ω for $0 \le t \le T < T^*$, then there exist–as we have just proved–constants $0 < c_1(\Omega) < c_2(\Omega)$ such that

$$c_1(\Omega) < F < c_2(\Omega). \tag{5.17}$$

It remains to prove that there also holds an estimate for the principal curvatures from above

$$\kappa_i \le c_3(\Omega), \tag{5.18}$$

yielding

$$0 < c_4(\Omega) \le \kappa_i \le c_3(\Omega) \tag{5.19}$$

due to the convexity of the flow hypersurfaces and (5.17).

Lemma 5.8. *The mixed tensor h_i^j satisfies the parabolic equation*

$$\begin{aligned}\dot{h}_i^j - \frac{1}{F^2}F^{kl}h_{i;kl}^j &= -F^{-2}F^{kl}h_{rk}h_l^r h_i^j + \frac{1}{F}h_{ri}h^{rj} + \frac{1}{F}h_i^k h_k^j \\ &+ \frac{1}{F^2}F^{kl,rs}h_{kl;i}h_{rs}^{;j} - \frac{2}{F^3}F_iF^j + \frac{2}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_i^\beta x_k^\gamma x_r^\delta h_l^m g^{rj} \\ &- \frac{1}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_r^\gamma x_l^\delta h_i^m g^{rj} - \frac{1}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}x_m^\alpha x_k^\beta x_i^\gamma x_l^\delta h^{mj} \\ &- \frac{1}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_k^\beta \nu^\gamma x_l^\delta h_i^j + \frac{2}{F}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta \nu^\gamma x_m^\delta g^{mj} \\ &+ \frac{1}{F^2}F^{kl}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\left\{\nu^\alpha x_k^\beta x_l^\gamma x_i^\delta x_m^\epsilon g^{mj} + \nu^\alpha x_i^\beta x_k^\gamma x_m^\delta x_l^\epsilon g^{mj}\right\}.\end{aligned} \tag{5.20}$$

Proof. cf. [10, Lemma 2.4.1]. □

Lemma 5.9. *Let $\Omega \subset N$ be precompact and assume that the flow stays in Ω for $0 \le t < T^*$, then there exists $c_3(\Omega)$ such that*

$$\kappa_i \le c_3(\Omega). \tag{5.21}$$

Proof. The proof is more or less word by word as in [4, Lemma 5.1], for reasons of completeness we present it here.

Let φ and w be defined respectively by

$$\begin{aligned}\varphi &= \sup\{h_{ij}\eta^i\eta^j : \|\eta\| = 1\}, \\ w &= \log\varphi + \lambda\tilde{v} + \mu\chi,\end{aligned} \tag{5.22}$$

where λ, μ are large positive parameters to be specified later. We claim that w is bounded for a suitable choice of λ, μ.

Let $0 < T < T^*$, and $x_0 = x_0(t_0)$, with $0 < t_0 \le T$, be a point in $M(t_0)$ such that

$$\sup_{M_0} w < \sup\{\sup_{M(t)} w : 0 < t \le T\} = w(x_0). \tag{5.23}$$

We then introduce a Riemannian normal coordinate system (ξ^i) at $x_0 \in M(t_0)$ such that at $x_0 = x(t_0, \xi_0)$ we have

$$g_{ij} = \delta_{ij} \quad \textit{and} \quad \varphi = h_n^n. \tag{5.24}$$

Let $\tilde{\eta} = (\tilde{\eta}^i)$ be the contravariant vector field defined by

$$\tilde{\eta} = (0, ..., 0, 1), \tag{5.25}$$

and set

$$\tilde{\varphi} = \frac{h_{ij}\tilde{\eta}^i\tilde{\eta}^j}{g_{ij}\tilde{\eta}^i\tilde{\eta}^j}. \tag{5.26}$$

$\tilde{\varphi}$ is well defined in a neighbourhood of (t_0, ξ_0).

Now, define $\tilde{w}$ by replacing φ by $\tilde{\varphi}$ in (5.22); then $\tilde{w}$ assumes its maximum at (t_0, ξ_0). Moreover, at (t_0, ξ_0) we have

$$\dot{\tilde{\varphi}} = \dot{h}_n^n, \tag{5.27}$$

and the spatial derivatives do also coincide; in short, at (t_0, ξ_0) $\tilde{\varphi}$ satisfies the same differential equation (5.20) as h_n^n. For the sake of greater clarity, let us therefore treat h_n^n like a scalar and pretend that w is defined by

$$w = \log h_n^n + \lambda\tilde{v} + \mu\chi. \tag{5.28}$$

W.l.o.g. we assume that μ, λ and h^n_n are larger than 1.

At (t_0, ξ_0) we have $\dot{w} \geq 0$ and in view of the maximum principle, we deduce from (5.20), (5.12), (5.10) and (5.17)

$$\begin{aligned} 0 \leq & c h^n_n + c\lambda F^{ij} g_{ij} - \frac{\lambda}{2}\epsilon_0 \tilde{v}\frac{H}{F} + \mu c - c_0 \frac{\mu}{F^2} F^{ij} g_{ij} \\ & + \frac{1}{F^2} F^{ij} (\log h^n_n)_i (\log h^n_n)_j - \frac{2}{h^n_n F^3} F^n F_n + \frac{1}{h^n_n F^2} F^{kl,rs} h_{kl;n} h_{rs;i} g^{ni}. \end{aligned} \tag{5.29}$$

Because of [10, Lemma 2.2.6] we have

$$F^{kl,rs} h_{kl;n} h_{rs;n} \leq F^{-1} (F^{ij} h_{ij;n})^2 - \frac{1}{h^n_n} F^{ij} h_{in;n} h_{jn;n} \tag{5.30}$$

so that we can estimate the last two summands of (5.29) from above by

$$-\frac{1}{(h^n_n)^2} \frac{1}{F^2} F^{ij} (h^n_{n;i} + \bar{R}_i)(h^n_{n;j} + \bar{R}_j); \tag{5.31}$$

here

$$\bar{R}_i = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x^\beta_n x^\gamma_i x^\delta_n = h_{in;n} - h_{nn;i} \tag{5.32}$$

denotes the correction term which comes from the Codazzi equation when changing the indices from $h_{in;n}$ to $h_{nn;i}$.

Thus the terms in (5.29) containing derivatives of h^n_n are estimated from above by

$$-2\frac{1}{(h^n_n)^2 F^2} F^{ij} h^n_{n;i} \bar{R}_j = -2\frac{1}{h^n_n F^2} F^{ij} (\log h^n_n)_i \bar{R}_j. \tag{5.33}$$

Moreover Dw vanishes at ξ_0, i.e.,

$$\begin{aligned} (\log h^n_n)_i &= -\lambda \tilde{v}_i - \mu \chi_i \\ &= -\lambda \eta_{\alpha\beta} x^\beta_i \nu^\alpha - \lambda \eta_\alpha x^\alpha_k h^k_i - \mu \chi_\alpha x^\alpha_i. \end{aligned} \tag{5.34}$$

Hence we conclude from (5.29) that

$$\begin{aligned} 0 \leq & c h^n_n + c\lambda F^{ij} g_{ij} - \frac{\lambda}{2}\epsilon_0 \tilde{v}\frac{H}{F} + \mu c + \mu \frac{c}{h^n_n} F^{ij} g_{ij} - c_0 \frac{\mu}{F^2} F^{ij} g_{ij} \\ \leq & c_1 h^n_n + c_2 \lambda F^{ij} g_{ij} - \lambda c_3 h^n_n + \mu c_4 + \mu \frac{c_5}{h^n_n} F^{ij} g_{ij} - c_0 \mu F^{ij} g_{ij}, \end{aligned} \tag{5.35}$$

where c_i, $i = 0, ..., 5$, are positive constants and the value of c_0 changed. We note that we used the estimate

$$F^{ij} \bar{R}_j \eta_\alpha x^\alpha_k h^k_i \leq cF, \tag{5.36}$$

which can be immediately proved.

Now suppose h^n_n to be so large that

$$\frac{c_5}{h^n_n} < \frac{1}{2} c_0, \tag{5.37}$$

and choose λ, μ such that

$$\frac{\lambda}{2} c_3 > c_1 \quad \textit{and} \quad \frac{1}{4} c_0 \mu > c_2 \lambda \tag{5.38}$$

yielding that estimating the right side of (5.35) yields

$$0 \leq -\frac{\lambda}{2} c_3 h^n_n - \frac{c_0}{4} \mu F^{ij} g_{ij} + \mu c_4, \tag{5.39}$$

hence h_n^n is a priori bounded at (t_0, ξ_0). □

Remark 5.10. Now all necessary a priori estimates are proved so that we can deduce existence of the flow for all times. This is standard theory and can be found more detailed in [10, Section 6.6]. Furthermore the flow provides a foliation of the future of M_0 as can be seen literally as in [10, Section 6.7].

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