

ASYMPTOTIC BEHAVIOUR OF AN INVERSE CURVATURE FLOW IN ARW SPACES

HEIKO KRÖNER

ABSTRACT. We consider an inverse curvature flow (ICF)

$$(0.1) \quad \dot{x} = -F^{-1}\nu$$

in so-called asymptotic Robertson-Walker spacetimes (ARW spaces). Here, ν denotes the past directed normal and F a curvature function of class (K^*) , which is homogeneous of degree one.

The most important example for F is the n -th root of the Gaussian curvature and the important case for an ARW space is a Lorentzian manifold with a quite general future singularity, in which the Einstein equations hold.

In a previous paper we proved existence of the ICF for all times and that the flow runs with a certain speed into the future singularity.

Here, we show convergence of the rescaled scalar solution in $C^\infty(S_0)$ to a smooth function. Using the rescaled ICF we maintain a transition from big crunch to big bang into a mirrored spacetime.

CONTENTS

1. Introduction	2
2. Notations and definitions	5
3. C^1 -estimates	7
4. C^2 -estimates	15
5. Appendix A	24
6. Appendix B	29
7. Appendix C	33
References	39

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UNIVERSITÄT TÜBINGEN, MATHEMATISCHES INSTITUT, AUF DER MORGENSTELLE 10, 72076
TÜBINGEN, GERMANY

E-mail address: kroener@na.uni-tuebingen.de

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1. INTRODUCTION

Asymptotic Robertson-Walker spaces, cf. [8, Definition 7.1.1], are Lorentzian manifolds with a future singularity satisfying some technical assumptions, which are still quite general, since they are satisfied under very mild assumptions on the singularity, if the Einstein equations hold, cf. [8, Section 7.9].

In [8, Chapter 7] the inverse mean curvature flow is considered in ARW spaces and it is shown, that the rescaled inverse mean curvature flow defines a natural C^3 -diffeomorphism through the singularity into a mirrored spacetime.

In [12] an inverse curvature flow of the type (0.1) is considered in spacetimes with a singularity, which is a little bit less restrictive than being ARW, and it is shown that the flow exists for all times and that the flow hypersurfaces provide a foliation of the future of the initial hypersurface. The precise setting for the ambient space is given in [12, Definition 1.1].

In the present paper we analyze the asymptotic behaviour of higher order derivatives of the flow (0.1) in ARW spaces N and show that the corresponding rescaled flow defines a natural C^3 -diffeomorphism through the singularity into a mirrored spacetime; this constitutes an analogous result to the result in [8, Chapter 7].

We define our ambient space N .

Definition 1.1. Let $N = N^{n+1}$ be a globally hyperbolic spacetime within which a future end N_+ of N can be written as a product $[a, 0) \times S_0$, where S_0 is a compact Riemannian space and there exists a future directed time function $\tau = x^0$ so that the metric in N_+ can be written as

$$(1.1) \quad (\tilde{g}_{\alpha\beta}) = e^{2\tilde{\psi}} \{-(dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j\},$$

where S_0 corresponds to

$$(1.2) \quad x^0 = a.$$

We assume that $\tilde{\psi}$ is of the form

$$(1.3) \quad \tilde{\psi}(x^0, x) = f(x^0) + \psi(x^0, x)$$

and that there exists a positive constant c_0 and a smooth Riemannian metric $\bar{\sigma}_{ij}$ on S_0 such that for $\tau \rightarrow 0$

$$(1.4) \quad e^{\psi(\tau, x)} \rightarrow c_0 \quad \wedge \quad \sigma_{ij}(\tau, x) \rightarrow \bar{\sigma}_{ij}(x) \quad \wedge \quad f(\tau) \rightarrow -\infty.$$

We assume that the derivatives of arbitrary order with respect to space and time of $e^{-2f} \tilde{g}_{\alpha\beta}$ converge for $x^0 \rightarrow 0$ uniformly to the corresponding derivatives of the following metric

$$(1.5) \quad -(dx^0)^2 + \bar{\sigma}_{ij}(x) dx^i dx^j.$$

We assume, that f satisfies the following six conditions

$$(1.6) \quad 0 < -f',$$

there exists $\omega \in \mathbb{R}$ such that

$$(1.7) \quad n + \omega - 2 > 0 \quad \wedge \quad \lim_{\tau \rightarrow 0} |f'|^2 e^{(n+\omega-2)f} = m > 0.$$

Set $\tilde{\gamma} = \frac{1}{2}(n + \omega - 2)$, then there exists the limit

$$(1.8) \quad \lim_{\tau \rightarrow 0} (f'' + \tilde{\gamma}|f'|^2)$$

and

$$(1.9) \quad |D_\tau^m(f'' + \tilde{\gamma}|f'|^2)| \leq c_m |f'|^m \quad \forall m \geq 1,$$

as well as

$$(1.10) \quad |D_\tau^m f| \leq c_m |f'|^m \quad \forall m \geq 1.$$

Furthermore, the following limit exists

$$(1.11) \quad \lim_{\tau \rightarrow 0} (f'' + \tilde{\gamma}|f'|^2)' \tau.$$

In case $\tilde{\gamma} < 1$ the limit metric $\bar{\sigma}_{ij}$ has non-negative sectional curvature.

The difference between Definition 1.1 and [12, Definition 1.1] is that we now assume (1.9), (1.10) and (1.11). Hence our previous result [12, Theorem 1.3] holds, i.e. we have

Theorem 1.2. *Let N be as in Definition 1.1. Let $F \in C^\infty(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$ be a curvature function of class (K^*) , cf. Definition 2.3, in the positive cone $\Gamma_+ \subset \mathbb{R}^n$, which is in addition positiv homogeneous of degree one and normalized such that*

$$(1.12) \quad F(1, \dots, 1) = n.$$

Let M_0 be a smooth, closed, spacelike hypersurface in N which can be written as a graph over S_0 for which we furthermore assume that it is convex and that it satisfies

$$(1.13) \quad -\epsilon < \inf_{M_0} x^0 < 0,$$

where

$$(1.14) \quad \epsilon = \epsilon(N, \check{g}_{\alpha\beta}) > 0.$$

Then the inverse curvature flow (ICF) given by the equation

$$(1.15) \quad \dot{x} = -\frac{1}{F}\nu$$

with initial surface $x(0) = M_0$ exists for all times and the flow hypersurfaces $M(t)$ provide a foliation of the future $I^+(M_0)$ of M_0 . Here, ν denotes the past directed normal.

The flow hypersurfaces $M(t)$ can be expressed as graphs over S_0

$$(1.16) \quad M(t) = \text{graph } u(t, \cdot),$$

and there exist constants $c_1, c_2 > 0$ with

$$(1.17) \quad a < -c_1 e^{-\frac{\tilde{\gamma}}{n}t} \leq u \leq -c_2 e^{-\frac{\tilde{\gamma}}{n}t} < 0.$$

Moreover there is a constant $c > 0$ such that

$$(1.18) \quad F \geq c e^{\frac{(\tilde{\gamma}+1)}{n}t}$$

for all $t \geq 0$.

We can now state our main theorem.

Theorem 1.3. *We consider the situation in Theorem 1.2.*

(i) If we set

$$(1.19) \quad \tilde{u} = u e^{\gamma t},$$

where $\gamma = \frac{1}{n}\tilde{\gamma}$, then \tilde{u} converges in $C^\infty(S_0)$ to a smooth negative function, if t goes to infinity.

(ii) Let (g_{ij}) be the induced metric of the leaves $M(t)$ of the inverse F -curvature flow, then the rescaled metric

$$(1.20) \quad e^{\frac{2}{n}t} g_{ij}$$

converges in $C^\infty(S_0)$ to

$$(1.21) \quad (\tilde{\gamma}^2 m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}} \bar{\sigma}_{ij},$$

where we are slightly ambiguous by using the same symbol to denote $\tilde{u}(t, \cdot)$ and $\lim \tilde{u}(t, \cdot)$.

(iii) The leaves $M(t)$ of the ICF get more umbilical, if t tends to infinity, namely

$$(1.22) \quad F^{-1} |h_i^j - \frac{1}{n} H \delta_i^j| \leq c e^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely

$$(1.23) \quad |h_i^j - \frac{1}{n} H \delta_i^j| \leq c e^{-\frac{1}{2n}(n+\omega-4)t}.$$

(iv) The rescaled ICF defines a natural C^3 -diffeomorphism through the singularity into a mirrored spacetime, cf. Section 7 for a precise formulation.

For later purposes we recall some facts from [8, Section 7.3].

Lemma 1.4. (i)

$$(1.24) \quad \lim_{\tau \rightarrow 0} \frac{e^{\tilde{\gamma}f}}{\tau} = -\tilde{\gamma} \sqrt{m}.$$

(ii) There holds

$$(1.25) \quad f' e^{\tilde{\gamma}f} + \sqrt{m} \sim c\tau^2,$$

where c is a constant, and where the relation

$$(1.26) \quad \varphi \sim c\tau^2$$

means

$$(1.27) \quad \lim_{\tau \rightarrow 0} \frac{\varphi(\tau)}{\tau^2} = c.$$

Lemma 1.5. The asymptotic relation

$$(1.28) \quad \tilde{\gamma} f' \tau - 1 \sim c\tau^2$$

is valid.

The present paper is based on parts of the author's PhD thesis [10] and the author would like to thank Claus Gerhardt under whose supervision the thesis was carried out.

2. NOTATIONS AND DEFINITIONS

In this section, where we want to introduce some general notations, we let N be as in Definition 1.1; now the special structure of $\tilde{\psi}$ and σ_{ij} , cf. (1.3) et seqq., is not necessarily needed. In this section, where we want to introduce some general notations, we assume for N all properties listed from the beginning of Section 1 as far as equation (1.2) except for being ARW and we write ψ instead of $\tilde{\psi}$. Let $M \subset N$ be a connected and spacelike hypersurface with differentiable normal ν (which is then timelike). Geometric quantities in N are denoted by $(\bar{g}_{\alpha\beta})$, $(\bar{R}_{\alpha\beta\gamma\delta})$ etc. and those in M by (g_{ij}) , (R_{ijkl}) etc.. Greek indices range from 0 to n , Latin indices from 1 to n ; summation convention is used. Coordinates in N and M are denoted by (x^α) and (ξ^i) respectively. Covariant derivatives are written as indices, only in case of possibly confusion we precede them by a semicolon, i.e. for a function u the gradient is (u_α) and $(u_{\alpha\beta})$ the hessian, but for the covariant derivative of the Riemannian curvature tensor we write $\bar{R}_{\alpha\beta\gamma\delta;\epsilon}$.

In local coordinates, (x^α) in N and (ξ^i) in M , the following four important equations hold; the Gauss formular

$$(2.1) \quad x_{ij}^\alpha = h_{ij}\nu^\alpha.$$

In this implicit definition (h_{ij}) is the second fundamental form of M with respect to ν . Here and in the following a covariant derivative is always a full tensor, i.e.

$$(2.2) \quad x_{i,j}^\alpha = x_{,ij}^\alpha - \Gamma_{ij}^k x_k^\alpha + \bar{\Gamma}_{\beta\gamma}^\alpha x_i^\beta x_j^\gamma$$

and the comma denotes ordinary partial derivatives.

The second equation is the Weingarten equation

$$(2.3) \quad \nu_i^\alpha = h_i^k x_k^\alpha,$$

where ν_i^α is a full tensor. The third equation is the Codazzi equation

$$(2.4) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

and the fourth is the Gauß equation

$$(2.5) \quad R_{ijkl} = -\{h_{ik}h_{jl} - h_{il}h_{jk}\} + \bar{R}_{\alpha\beta\gamma\delta} x_i^\alpha x_j^\beta x_k^\gamma x_l^\delta.$$

As an example for the covariant derivative of a full tensor we give

$$(2.6) \quad \bar{R}_{\alpha\beta\gamma\delta;i} = \bar{R}_{\alpha\beta\gamma\delta;\epsilon} x_i^\epsilon,$$

where this identity follows by applying the chain rule from the definition of the covariant derivative of a full tensor; it can be generalized obviously to other quantities.

Let (x^α) be a future directed coordinate system in N , then the contravariant vector $(\xi^\alpha) = (1, 0, \dots, 0)$ is future directed; as well its covariant version $(\xi_\alpha) = e^{2\psi}(-1, 0, \dots, 0)$.

Now we want to express normal, metric and second fundamental form for spacelike hypersurfaces, which can be written as graphs over the Cauchyhypersurface. Let $M = \text{graph } u|_{S_0}$ be a spacelike hypersurface in N , i.e.

$$(2.7) \quad M = \{(x^0, x) : x^0 = u(x), x \in S_0\},$$

then the induced metric is given by

$$(2.8) \quad g_{ij} = e^{2\tilde{\psi}} \{-u_i u_j + \sigma_{ij}\},$$

where σ_{ij} is evaluated at (u, x) and the inverse $(g^{ij}) = (g_{ij})^{-1}$ is given by

$$(2.9) \quad g^{ij} = e^{-2\tilde{\psi}} \left\{ \sigma^{ij} + \frac{u^i u^j}{v^2} \right\},$$

where $(\sigma^{ij}) = (\sigma_{ij})^{-1}$ and

$$(2.10) \quad \begin{aligned} u^i &= \sigma^{ij} u_j \\ v^2 &= 1 - \sigma^{ij} u_i u_j \equiv 1 - |Du|^2, \quad v > 0. \end{aligned}$$

We define $\tilde{v} = v^{-1}$.

From (2.8) we conclude that graph u is spacelike if and only if $|Du| < 1$.

The covariant version of the normal of a graph is

$$(2.11) \quad (\nu_\alpha) = \pm v^{-1} e^{\tilde{\psi}} (1, -u_i)$$

and the contravariant version

$$(2.12) \quad (\nu^\alpha) = \mp v^{-1} e^{-\tilde{\psi}} (1, u^i).$$

We have

Remark 2.1. Let M be a spacelike graph in a future directed coordinate system, then

$$(2.13) \quad (\nu^\alpha) = v^{-1} e^{-\tilde{\psi}} (1, u^i)$$

is the contravariant future directed normal and

$$(2.14) \quad (\nu^\alpha) = -v^{-1} e^{-\tilde{\psi}} (1, u^i)$$

the past directed.

In the following we choose ν always as the past directed normal.

Let us consider the component $\alpha = 0$ in (2.1), so we have due to (2.14) that

$$(2.15) \quad e^{-\tilde{\psi}} v^{-1} h_{ij} = -u_{ij} - \bar{\Gamma}_{00}^0 u_i u_j - \bar{\Gamma}_{0j}^0 u_i - \bar{\Gamma}_{0i}^0 u_j - \bar{\Gamma}_{ij}^0,$$

where u_{ij} are covariant derivatives with respect to M . Choosing $u \equiv \text{const}$, we deduce

$$(2.16) \quad e^{-\tilde{\psi}} \bar{h}_{ij} = -\bar{\Gamma}_{ij}^0,$$

where \bar{h}_{ij} is the second fundamental form of the hypersurface $\{x^0 = \text{const}\}$. An easy calculation shows

$$(2.17) \quad e^{-\tilde{\psi}} \bar{h}_{ij} = -\frac{1}{2} \dot{\sigma}_{ij} - \dot{\psi} \sigma_{ij},$$

where the dot indicates differentiation with respect to x^0 .

Now we define the classes (K) and (K^*) , which are special classes of curvature functions; for a more detailed treatment of these classes we refer to [8, Section 2.2].

For a curvature function F (i.e. symmetric in its variables) in the positive cone $\Gamma_+ \subset \mathbb{R}^n$ we define

$$(2.18) \quad F(h_{ij}) = F(\kappa_i),$$

where the κ_i are the eigenvalues of an arbitrary symmetric tensor (h_{ij}) , whose eigenvalues are in Γ_+ .

Definition 2.2. A symmetric curvature function $F \in C^{2,\alpha}(\Gamma_+) \cap C^0(\bar{\Gamma}_+)$, positively homogeneous of degree $d_0 > 0$, is said to be of class (K) , if

$$(2.19) \quad F_i = \frac{\partial F}{\partial \kappa^i} > 0 \quad \text{in } \Gamma_+,$$

$$(2.20) \quad F|_{\partial\Gamma_+} = 0,$$

and

$$(2.21) \quad F^{ij,kl} \eta_{ij} \eta_{kl} \leq F^{-1} (F^{ij} \eta_{ij})^2 - F^{ik} \tilde{h}^{jl} \eta_{ij} \eta_{kl} \quad \forall \eta \in S,$$

where F is evaluated at an arbitrary symmetric tensor (h_{ij}) , whose eigenvalues are in Γ_+ and S denotes the set of symmetric tensors. Here, F_i is a partial derivative of first order with respect to κ_i and $F^{ij,kl}$ are second partial derivatives with respect to (h_{ij}) . Furthermore (\tilde{h}^{ij}) is the inverse of (h_{ij}) .

In Theorem 1.3 the κ_i in (2.18) are the eigenvalues of the second fundamental form (h_{ij}) with respect to the metric (g_{ij}) , i.e. the principal curvatures of the flow hypersurfaces.

Definition 2.3. A curvature function $F \in (K)$ is said to be of class (K^*) , if there exists $0 < \epsilon_0 = \epsilon_0(F)$ such that

$$(2.22) \quad \epsilon_0 F H \leq F^{ij} h_{ik} h_j^k,$$

for any symmetric (h_{ij}) with all eigenvalues in Γ_+ , where F is evaluated at (h_{ij}) . H represents the mean curvature, i.e. the trace of (h_{ij}) .

In the following a '+' sign attached to the symbol of a metric of the ambient space refers to the corresponding Riemannian background metric, if attached to an induced metric, it refers to the induced metric relative to the corresponding Riemannian background metric. Let us consider as an example the metrics $\check{g}_{\alpha\beta}$ and g_{ij} introduced as above, then

$$(2.23) \quad \check{g}_{\alpha\beta}^+ = e^{2\tilde{\psi}} \{ (dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j \}, \quad \check{g}_{ij}^+ = \check{g}_{\alpha\beta}^+ x_i^\alpha x_j^\beta.$$

3. C^1 -ESTIMATES

We prove an estimate for \tilde{v} for the leaves of the ICF on the maximal existence interval $[0, T^*)$, cf. Lemma 3.4.

To prove this we consider the flow to be embedded in N with the conformal metric

$$(3.1) \quad \bar{g}_{\alpha\beta} = e^{-2\tilde{\psi}} \check{g}_{\alpha\beta} = -(dx^0)^2 + \sigma_{ij}(x^0, x) dx^i dx^j.$$

This point of view will be later on also a key ingredient in the proof of the convergence results for the flow. Though, formally we have a different ambient space we still denote it by the same symbol N and distinguish only the metrics $\check{g}_{\alpha\beta}$ resp. $\bar{g}_{\alpha\beta}$ and the corresponding quantities of the hypersurfaces \check{h}_{ij} , \check{g}_{ij} , $\check{\nu}$ resp. h_{ij} , g_{ij} , ν , etc., i.e., the standard notations now apply to the case when N is equipped with the metric (3.1).

The second fundamental forms \check{h}_i^j and h_i^j are related by

$$(3.2) \quad e^{\tilde{\psi}} \check{h}_i^j = h_i^j + \tilde{\psi}_\alpha \nu^\alpha \delta_i^j = h_i^j - \tilde{v} f' \delta_i^j + \psi_\alpha \nu^\alpha =_{def} \check{h}_i^j,$$

cf. [8, Proposition 1.1.11]. When we insert \check{h}_i^j into F we will denote the result in accordance with our convention as \check{F} . Due to a lack of convexity it would not make any sense to insert h_i^j into the curvature function F , so that we stipulate that the symbol F will stand for

$$(3.3) \quad F = e^{\check{\psi}} \check{F} = F(h_i^j - \check{v} f' \delta_i^j + \psi_\alpha \nu^\alpha),$$

which will be useful, cf. (3.5).

Quantities like \check{v} , that are not different if calculated with respect to $\check{g}_{\alpha\beta}$ or $\bar{g}_{\alpha\beta}$ are denoted in the usual way.

Due to

$$(3.4) \quad \check{\nu} = e^{-\check{\psi}} \nu$$

the evolution equation $\dot{x} = -\frac{1}{F} \check{\nu}$ can be written as

$$(3.5) \quad \dot{x} = -\frac{1}{F} \nu.$$

Lemma 3.1. (*Evolution of \check{v}*) Consider the flow (3.5). Then \check{v} satisfies the evolution equation

$$(3.6) \quad \begin{aligned} \dot{\check{v}} - F^{-2} F^{ij} \check{v}_{ij} &= -F^{-2} F^{ij} h_{kj} h_i^k \check{v} + F^{-2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ &\quad - F^{-2} F^{ij} h_{ij} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - F^{-1} \eta_{\alpha\beta} \nu^\alpha \nu^\beta \\ &\quad - F^{-2} (F^{ij} \eta_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_j^\gamma + 2F^{ij} \eta_{\alpha\beta} x_k^\alpha x_i^\beta h_j^k) \\ &\quad - F^{-2} (-\check{v} f'' \|Du\|^2 F^{ij} g_{ij} - \check{v}_k u^k f' F^{ij} g_{ij} \\ &\quad + \psi_{\alpha\beta} \nu^\alpha x_k^\beta u^k F^{ij} g_{ij} + \psi_\alpha x_l^\alpha h_k^l u^k F^{ij} g_{ij}), \end{aligned}$$

where $\eta = (\eta_\alpha) = (-1, 0, \dots, 0)$ is a covariant unit vectorfield.

Proof. We have

$$(3.7) \quad \check{v} = \eta_\alpha \nu^\alpha.$$

Let (ξ^i) be local coordinates for $M(t)$; differentiating \check{v} covariantly yields

$$(3.8) \quad \check{v}_i = \eta_{\alpha\beta} x_i^\beta \nu^\alpha + \eta_\alpha \nu_i^\alpha$$

and

$$(3.9) \quad \begin{aligned} \check{v}_{ij} &= \eta_{\alpha\beta\gamma} x_i^\beta x_j^\gamma \nu^\alpha + \eta_{\alpha\beta} \nu_j^\alpha x_i^\beta + \eta_{\alpha\beta} \nu^\alpha \nu^\beta h_{ij} + \eta_\alpha x_{rj}^\alpha h_i^r \\ &\quad + \eta_\alpha x_r^\alpha h_{ij}^r + \eta_{\alpha\beta} x_j^\beta x_r^\alpha h_i^r. \end{aligned}$$

As usual, cf. [8, Lemma 2.3.2], the evolution equation for the normal is

$$(3.10) \quad \dot{\nu}^\alpha = g^{ij} \left(-\frac{1}{F}\right)_i x_j^\alpha = \frac{1}{F^2} g^{ij} F_i x_j^\alpha$$

and for the time derivative of \check{v} we get

$$(3.11) \quad \begin{aligned} \dot{\check{v}} &= \eta_{\alpha\beta} \nu^\alpha \dot{x}^\beta + \eta_\alpha \dot{\nu}^\alpha \\ &= -\frac{1}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - \frac{1}{F^2} g^{ij} F_i u_j. \end{aligned}$$

Writing

$$(3.12) \quad \begin{aligned} F_k &= F^{ij} h_{ij;k} - \check{v}_k f' F^{ij} g_{ij} - \check{v} f'' u_k F^{ij} g_{ij} \\ &\quad + \psi_{\alpha\beta} \nu^\alpha x_k^\beta F^{ij} g_{ij} + \psi_\alpha x_r^\alpha h_k^r F^{ij} g_{ij} \end{aligned}$$

and using the Codazzi equation

$$(3.13) \quad h_{ij;k} - h_{ik;j} = \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_j^\gamma x_k^\delta$$

we deduce the desired evolution equation for \tilde{v} by putting together the above equations. \square

We have the following Lemma.

Lemma 3.2. *Let $\|\cdot\|$ denote the norm of a tensor with respect to the Riemannian metric $\overset{+}{g}_{\alpha\beta}$, cf Section 2, then*

(i)

$$(3.14) \quad \begin{aligned} |\eta_{\alpha\beta} \nu^\alpha \nu^\beta| &\leq c \tilde{v}^2 \|\eta_{\alpha\beta}\|, \\ |F^{ij} \eta_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_j^\gamma| &\leq c \tilde{v}^3 \|\eta_{\alpha\beta\gamma}\| \|F^{ij} g_{ij}\|, \\ |\psi_{\alpha\beta} \nu^\alpha x_k^\beta u^k| &\leq c \|\eta_{\alpha\beta}\| \tilde{v}^3. \end{aligned}$$

(ii) For any $\epsilon > 0$ we have

$$(3.15) \quad |F^{ij} \eta_{\alpha\beta} x_k^\alpha x_l^\beta h_j^k| \leq c \tilde{v} F^{ij} h_{kj} h_i^k \|\eta_{\alpha\beta}\| + c_\epsilon \tilde{v}^3 F^{ij} g_{ij} \|\eta_{\alpha\beta}\|.$$

(iii)

$$(3.16) \quad |F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l| \leq c \tilde{v}^3 F^{ij} g_{ij}.$$

(iv) Furthermore

$$(3.17) \quad |\psi_\alpha x_k^\alpha h_i^k u^i| \leq c \|D\psi\| \tilde{v}^3$$

in points where $\tilde{v}_i = 0$.

Remark 3.3. These are tensor estimates, i.e. not depending on the special local coordinates of the hypersurface and S_0 . But to prove these estimates we sometimes choose special coordinates such that in a fixed point $g_{ij} = \delta_{ij}$, $\overset{+}{g}_{ij} = \text{diagonal}$.

Proof of Lemma 3.2. We have $\|\nu^\alpha\| \leq 2\tilde{v}$,

$$(3.18) \quad \overset{+}{g}_{ij} \leq 2\sigma_{ij} \leq 2\tilde{v}^2 g_{ij} \quad \wedge \quad g^{ij} \leq c \tilde{v}^2 \sigma^{ij} \quad \wedge \quad u^i = \tilde{v}^2 \tilde{u}^i$$

and $\|Du\|^2 = \tilde{v}^2 |Du|^2$.

Proof of (i): Using these properties together with Schwarz inequality proves (i).

Proof of (ii):

$$(3.19) \quad \begin{aligned} \||F^{ij} x_k^\alpha x_l^\beta h_j^k|\|^2 &= F^{ij} F^{\bar{i}\bar{j}} h_j^k h_{\bar{j}}^{\bar{k}} \overset{+}{g}_{k\bar{k}} \overset{+}{g}_{i\bar{i}}, \quad g_{ij} = \delta_{ij}, \overset{+}{g}_{ij} = \text{diagonal} \\ &\leq c \tilde{v}^4 F^{ij} F^{\bar{i}\bar{j}} h_j^k h_{\bar{j}}^{\bar{k}} g_{k\bar{k}} g_{i\bar{i}}, \quad g_{ij} = \delta_{ij}, h_{ij} = \kappa_i \delta_{ij}, F^{ij} = \text{diagonal} \\ &\leq c \tilde{v}^4 \sum_i (F^{ii})^2 (h_{ii})^2 \\ &\leq c \tilde{v}^4 \left(\sum_i F^{ii} |h_{ii}| \right)^2 \\ &\leq c \tilde{v}^4 \left(\sum_i F^{ii} \left(\frac{\epsilon}{\tilde{v}} h_{ii}^2 + c_\epsilon \tilde{v} g_{ii} \right) \right)^2, \end{aligned}$$

taking the square root yields the result.

Proof of (iii): See [8, Lemma 5.4.5].

Proof of (iv): Differentiating the equation

$$(3.20) \quad \tilde{v}^2 = 1 + \|Du\|^2$$

with respect to i yields

$$(3.21) \quad 0 = 2\tilde{v}\tilde{v}_i = 2u_{ij}u^j$$

which implies in view of

$$(3.22) \quad \tilde{v}h_{ij} = -u_{ij} + \bar{h}_{ij},$$

cf. Section 2, that

$$(3.23) \quad h_{ij}u^j = \tilde{v}\bar{h}_{ij}\tilde{u}^j$$

hence

$$(3.24) \quad \begin{aligned} \psi_\alpha x_k^\alpha h_i^k u^i &= \psi_\alpha x_k^\alpha g^{kl} h_{li} u^i = \tilde{v} \psi_\alpha x_k^\alpha g^{kl} \bar{h}_{li} \tilde{u}^i \\ &= \tilde{v} \psi_\alpha x_k^\alpha (\sigma^{kl} + \tilde{v}^2 \check{u}^k \check{u}^l) \bar{h}_{li} \tilde{u}^i. \end{aligned}$$

Applying Schwarz inequality finishes the proof. \square

Lemma 3.4. \tilde{v} is uniformly bounded on $[0, T^*)$ namely

$$(3.25) \quad \sup_{[0, T^*)} \tilde{v} \leq c = c(\sup_{M_0} \tilde{v}, (N, \check{g}_{\alpha\beta})).$$

Proof. We have (1.13) in mind. For $0 < T < T^*$ assume that there are $0 < t_0 \leq T$ and $x_0 \in S_0$ such that

$$(3.26) \quad \sup_{[0, T]} \sup_{M(t)} \tilde{v} = \tilde{v}(t_0, x_0) \geq 2.$$

In (t_0, x_0) we have $\|Du\|^2 \geq \frac{1}{4}\tilde{v}^2$,

$$(3.27) \quad 0 \leq \dot{\tilde{v}} - F^{-2} F^{ij} \tilde{v}_{ij},$$

and after multiplying this inequality by F^2 we get if $\epsilon > 0$ sufficiently small that

$$(3.28) \quad \begin{aligned} 0 &\leq -F^{ij} h_{kj} h_i^k \tilde{v} + F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l - F^{ij} h_{ij} \eta_{\alpha\beta} \nu^\alpha \nu^\beta \\ &\quad - F \eta_{\alpha\beta} \nu^\alpha \nu^\beta - F^{ij} \eta_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_j^\gamma + 2F^{ij} \eta_{\alpha\beta} x_k^\alpha x_i^\beta h_j^k \\ &\quad + \tilde{v} f'' \|Du\|^2 F^{ij} g_{ij} + \tilde{v}_k u^k f' F^{ij} g_{ij} - \psi_{\alpha\beta} \nu^\alpha x_k^\beta u^k F^{ij} g_{ij} \\ &\quad - \psi_\alpha x_l^\alpha h_k^l u^k F^{ij} g_{ij} \\ &\leq -\frac{1}{2} F^{ij} h_{kj} h_i^k \tilde{v} + c\tilde{v}^3 |f'| F^{ij} g_{ij} + \tilde{v} f'' \|Du\|^2 F^{ij} g_{ij}, \end{aligned}$$

which is a contradiction if $\epsilon > 0$ very small.

Hence

$$(3.29) \quad \tilde{v}(t_0, x_0) \leq \max(\sup_{M_0} \tilde{v}, 2).$$

\square

We recall a decay property of certain tensors.

Lemma 3.5. (i) Let $\varphi \in C^\infty([a, 0))$, $a < 0$, and assume

$$(3.30) \quad \lim_{\tau \rightarrow 0} \varphi^{(k)}(\tau) = 0 \quad \forall k \in \mathbb{N},$$

then for every $k \in \mathbb{N}$ there exists a $c_k > 0$ such that

$$(3.31) \quad |\varphi(\tau)| \leq c_k |\tau|^k.$$

(ii) Let T be a tensor such that for all $k \in \mathbb{N}$

$$(3.32) \quad \||D^k T(x^0, x)\|| \rightarrow 0 \quad \text{as } x^0 \rightarrow 0 \quad \text{uniformly in } x$$

then

$$(3.33) \quad \forall k \in \mathbb{N} \quad \exists c_k > 0 \quad \forall x \in S_0 \quad \||T(x^0, x)\|| \leq c_k |x^0|^k$$

(iii) For $T = (\eta_{\alpha\beta})$ the relation (3.33) is true, analogously for $\||\eta_{\alpha\beta\gamma}\||$, $\||D\psi\||$, $\||\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha\||$, or more generally for any tensor that would vanish identically, if it would have been formed with respect to the product metric

$$(3.34) \quad -(dx^0)^2 + \bar{\sigma}_{ij} dx^i dx^j.$$

Proof. See [8, Lemma 7.4.4]. □

Lemma 3.6. For any $k \in \mathbb{N}^*$ there exists $c_k > 0$ such that

$$(3.35) \quad |f^{(k)}| \leq c_k e^{k\gamma t},$$

where $f^{(k)}$ is evaluated at u .

Proof. In view of the assumption (1.10) there holds

$$(3.36) \quad |f^{(k)}| \leq c_k |f'|^k = c_k |f'|^k u^k \tilde{u}^{-k} e^{k\gamma t}.$$

Then use (1.28) and (1.17). □

In Lemma 3.4 we proved that \tilde{v} is uniformly bounded for all times. We recall that

$$(3.37) \quad \tilde{u} = u e^{\gamma t}.$$

Our final goal is to show that $\|D\tilde{u}\|^2$ is uniformly bounded, but this estimate has to be deferred to Section 4. At the moment we only prove an exponential decay for any $0 < \lambda < \gamma$, i.e., we shall estimate $\|Du\|e^{\lambda t}$.

We remember that we have

$$(3.38) \quad F = F(\check{h}_i^j) = F(e^{\check{\psi}} \check{h}_i^j) = F(h_i^j - \tilde{v} f' \delta_i^j + \psi_\alpha \nu^\alpha \delta_i^j).$$

We need in the following a slightly different estimate from the one in (3.16).

Lemma 3.7.

$$(3.39) \quad \begin{aligned} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l &= -\tilde{v} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \eta^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ &\quad - \tilde{v} F^{ij} \bar{R}_{r\beta\gamma\delta} \check{u}^r x_i^\beta x_l^\gamma x_j^\delta u^l. \end{aligned}$$

With the help of the boundedness of \tilde{v} , cf. Lemma 3.4, we prove the following estimate.

Lemma 3.8. There exists $\epsilon > 0$ and a constant c_ϵ such that

$$(3.40) \quad \|Du\|e^{\epsilon t} \leq c_\epsilon.$$

Proof. We have

$$(3.41) \quad \tilde{v}^2 = 1 + \|Du\|^2.$$

Taking the log yields since \tilde{v} is bounded

$$(3.42) \quad \|Du\|^2(1 - c_1\|Du\|^2) \leq 2\log \tilde{v} = \log(1 + \|Du\|^2) \leq \|Du\|^2(1 + c_1\|Du\|^2),$$

where c_1 is a positive constant, i.e., it is sufficient to prove that $\log \tilde{v}e^{2\epsilon t}$ is uniformly bounded.

Let $\epsilon > 0$ be small and set

$$(3.43) \quad \varphi = \log \tilde{v}e^{2\epsilon t},$$

then φ satisfies

$$(3.44) \quad \dot{\varphi} - F^{-2}F^{ij}\varphi_{ij} = \frac{1}{\tilde{v}}(\dot{\tilde{v}} - F^{-2}F^{ij}\tilde{v}_{ij})e^{2\epsilon t} + F^{-2}\frac{1}{\tilde{v}^2}F^{ij}\tilde{v}_i\tilde{v}_je^{2\epsilon t} + 2\epsilon\varphi$$

hence (cf. Lemma 3.1)

$$(3.45) \quad \begin{aligned} F^2e^{-2\epsilon t}(\dot{\varphi} - F^{-2}F^{ij}\varphi_{ij}) &= -F^{ij}h_{kj}h_i^k + \frac{1}{\tilde{v}}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ &\quad - \frac{1}{\tilde{v}}F^{ij}h_{ij}\eta_{\alpha\beta}\nu^\alpha\nu^\beta - \frac{1}{\tilde{v}}F\eta_{\alpha\beta}\nu^\alpha\nu^\beta \\ &\quad - \frac{1}{\tilde{v}}F^{ij}\eta_{\alpha\beta\gamma}\nu^\alpha x_i^\beta x_j^\gamma - \frac{2}{\tilde{v}}F^{ij}\eta_{\alpha\beta}x_k^\alpha x_i^\beta h_j^k \\ &\quad + f''\|Du\|^2F^{ij}g_{ij} + \frac{1}{\tilde{v}}\tilde{v}_k u^k f'F^{ij}g_{ij} \\ &\quad - \psi_{\alpha\beta}\nu^\alpha x_k^\beta u^k \frac{1}{\tilde{v}}F^{ij}g_{ij} - \frac{1}{\tilde{v}}\psi_\alpha x_l^\alpha h_k^l u^k F^{ij}g_{ij} \\ &\quad + F^{ij}\tilde{v}_i\tilde{v}_j \frac{1}{\tilde{v}^2} + 2\epsilon F^2 \log \tilde{v}. \end{aligned}$$

For T , $0 < T < \infty$, assume that

$$(3.46) \quad \sup_{[0, T]} \sup_{M(t)} \varphi = \varphi(t_0, x_0),$$

where $0 < t_0 \leq T$ large, $x_0 \in S_0$.

Applying the maximum principle we deduce in (t_0, x_0) using Lemma 3.2, Lemma 3.5 and Lemma 3.7 that (note that $\tilde{u} = ue^{\gamma t}$ is bounded) for t_0 large and $\epsilon > 0$ small.

$$(3.47) \quad \begin{aligned} 0 &\leq -\frac{1}{2}F^{ij}h_{kj}h_i^k + cu^2F^{ij}g_{ij} + c|u|\|Du\|F^{ij}g_{ij} \\ &\quad + c\|Du\|^2F^{ij}g_{ij} + f''\|Du\|^2F^{ij}g_{ij} + c\epsilon|f'|^2 \log \tilde{v}F^{ij}g_{ij}, \end{aligned}$$

here we used that we have

$$(3.48) \quad F^2 \leq c(F^{ij}h_{ik}h_j^k + |f'|^2F^{ij}g_{ij})$$

due to $\epsilon_0 F^2 \leq F^{ij}\check{h}_{kj}\check{h}_i^k$, cf. Definition 2.3.

The $\log \tilde{v}$ in (3.47) can be estimated by $c\|Du\|^2$ yielding

$$(3.49) \quad 0 \leq -\frac{1}{2}F^{ij}h_{kj}h_i^k + cu^2F^{ij}g_{ij} + \frac{1}{2}f''\|Du\|^2F^{ij}g_{ij},$$

where we have chosen $\epsilon > 0$ small and assumed that $t_0 > 0$ large.

Hence in (t_0, x_0)

$$(3.50) \quad \varphi = \log \tilde{v} e^{2\epsilon t} \leq c \|Du\|^2 e^{2\epsilon t} \leq \frac{cu^2}{|f''|} e^{2\epsilon t} \leq c.$$

□

Lemma 3.9. (*Evolution of u*)

$$\dot{u} - F^{-2} F^{ij} u_{ij} = 2F^{-1} \tilde{v} + F^{-2} \tilde{v}^2 f' F^{ij} g_{ij} - F^{-2} \tilde{v} \psi_\alpha \nu^\alpha F^{ij} g_{ij} - F^{-2} F^{ij} \bar{h}_{ij}$$

Proof. The claim follows from the three identities

$$(3.51) \quad \begin{aligned} \dot{u} &= \frac{\tilde{v}}{F} \\ u_{ij} &= -\tilde{v} h_{ij} + \bar{h}_{ij} \\ -F^{ij} h_{ij} &= -F - \tilde{v} f' F^{ij} g_{ij} + \psi_\alpha \nu^\alpha F^{ij} g_{ij}. \end{aligned}$$

□

Lemma 3.10. *For any $0 < \lambda < \gamma$, there exists c_λ such that*

$$(3.52) \quad \|Du\| e^{\lambda t} \leq c_\lambda.$$

Proof. Define

$$(3.53) \quad \varphi = \log \tilde{v} - \frac{\mu}{2} |u|^{2-\epsilon},$$

with $0 < \epsilon < 1$ arbitrary and $\mu \gg 1$ chosen appropriately later. The interesting case is, when ϵ is close to 0.

Then φ satisfies the following evolution equation, cf. Lemma 3.1 and Lemma 3.9,

$$(3.54) \quad \begin{aligned} \dot{\varphi} - F^{-2} F^{ij} \varphi_{ij} &= \frac{1}{\tilde{v}} (\dot{\tilde{v}} - F^{-2} F^{ij} \tilde{v}_{ij}) + \frac{2-\epsilon}{2} \mu |u|^{1-\epsilon} (\dot{u} - F^{-2} F^{ij} u_{ij}) \\ &\quad + F^{-2} \frac{1}{\tilde{v}^2} F^{ij} \tilde{v}_i \tilde{v}_j + \frac{2-\epsilon}{2} \mu (1-\epsilon) |u|^{-\epsilon} F^{-2} F^{ij} u_i u_j \\ &= -F^{-2} F^{ij} h_{kj} h_i^k + \frac{1}{\tilde{v}} F^{-2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta x_l^\gamma x_j^\delta u^l \\ &\quad - \frac{1}{\tilde{v}} F^{-2} F^{ij} h_{ij} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - \frac{1}{\tilde{v}} F^{-1} \eta_{\alpha\beta} \nu^\alpha \nu^\beta \\ &\quad - \frac{1}{\tilde{v}} F^{-2} F^{ij} \eta_{\alpha\beta\gamma} \nu^\alpha x_i^\beta x_j^\gamma + \frac{2}{\tilde{v}} F^{-2} F^{ij} \eta_{\alpha\beta} x_k^\alpha x_l^\beta h_j^k \\ &\quad + F^{-2} f'' \|Du\|^2 F^{ij} g_{ij} + \frac{1}{\tilde{v}} F^{-2} \tilde{v}_k u^k f' F^{ij} g_{ij} \\ &\quad - F^{-2} \psi_{\alpha\beta} \nu^\alpha x_k^\beta u^k \frac{1}{\tilde{v}} F^{ij} g_{ij} - \frac{1}{\tilde{v}} F^{-2} \psi_\alpha x_l^\alpha h_k^l u^k F^{ij} g_{ij} \\ &\quad + (2-\epsilon) \mu |u|^{(1-\epsilon)} \frac{\tilde{v}}{F} + (1-\frac{\epsilon}{2}) \mu |u|^{1-\epsilon} F^{-2} \tilde{v}^2 f' F^{ij} g_{ij} \\ &\quad - (1-\frac{\epsilon}{2}) \mu |u|^{1-\epsilon} F^{-2} \tilde{v} \psi_\alpha \nu^\alpha F^{ij} g_{ij} \\ &\quad - (1-\frac{\epsilon}{2}) \mu |u|^{1-\epsilon} F^{-2} F^{ij} \bar{h}_{ij} \\ &\quad + F^{-2} F^{ij} \tilde{v}_i \tilde{v}_j \frac{1}{\tilde{v}^2} + (1-\frac{\epsilon}{2}) (1-\epsilon) \mu |u|^{-\epsilon} F^{-2} F^{ij} u_i u_j \\ &= RHS. \end{aligned}$$

□

We will show

$$(3.55) \quad \varphi < 0 \quad \forall t \geq 0.$$

Assume that this is not the case. Let $t_0 > 0$ be minimal such that

$$(3.56) \quad \sup_{S_0} \varphi(t_0, \cdot) = 0$$

and $x_0 \in S_0$ such that

$$(3.57) \quad \varphi(t_0, x_0) = 0,$$

which implies that in (t_0, x_0) the RHS in (3.54) is ≥ 0 ,

$$(3.58) \quad \frac{1}{2} \|Du\|^2 \geq \log \tilde{v} = \frac{\mu}{2} |u|^{2-\epsilon}$$

for $\mu > 0$ large (which implies t_0 large) and

$$(3.59) \quad \tilde{v}_i = -(1 - \frac{\epsilon}{2}) \mu \tilde{v} |u|^{1-\epsilon} u_i.$$

We now show that RHS in (3.54) is negative, if t_0 is sufficiently large, which can be guaranteed by increasing μ accordingly.

We use

$$(3.60) \quad F \leq |u|^{1-\beta} \delta F^{ij} h_{ik} h_j^k + \frac{c(\delta)}{|u|^{1-\beta}} F^{ij} g_{ij} + \tilde{v} |f'| F^{ij} g_{ij}$$

where $\beta > 0$ is chosen according to Lemma 3.8 such that

$$(3.61) \quad \log \tilde{v} \leq c|u|^\beta$$

and $\delta > 0$ is small, $c(\delta)$ also depends on the upper bound of $|\psi_\alpha \nu^\alpha|$.

We find

$$(3.62) \quad \begin{aligned} 0 &\leq F^2(RHS) \\ &\leq -\frac{1}{2} F^{ij} h_{kj} h_i^k + c \|Du\|^2 F^{ij} g_{ij} + c\mu |u| F^{ij} g_{ij} + f'' \mu |u|^{2-\epsilon} F^{ij} g_{ij} \\ &\quad + (1 - \frac{\epsilon}{2}) \mu |u|^{-\epsilon} \|Du\|^2 (\frac{1}{\tilde{\gamma}} + cu^2) F^{ij} g_{ij} \\ &\quad + (2 - \epsilon) \mu |u|^{1-\epsilon} \tilde{v} (|u|^{1-\beta} \delta F^{ij} h_{ik} h_j^k + \frac{c(\delta)}{|u|^{1-\beta}} F^{ij} g_{ij} + \tilde{v} |f'| F^{ij} g_{ij}) \\ &\quad + (1 - \frac{\epsilon}{2}) \mu |u|^{1-\epsilon} \tilde{v}^2 f' F^{ij} g_{ij} + 2(1 - \frac{\epsilon}{2})^2 \mu \log \tilde{v} |u|^{-\epsilon} |Du|^2 F^{ij} g_{ij} \\ &\quad + (1 - \frac{\epsilon}{2}) \mu (1 - \epsilon) |u|^{-\epsilon} |Du|^2 F^{ij} g_{ij} \\ &\leq \frac{\mu}{\tilde{\gamma}} |u|^{-\epsilon} (-1 + c \|Du\| + (1 - \frac{\epsilon}{2}) \tilde{v}^2 + c\tilde{\gamma} |u|^\epsilon + c|u|^\beta) F^{ij} g_{ij} \\ &< 0, \end{aligned}$$

where we have chosen μ large ($\Rightarrow t_0$ large). Here we used

$$(3.63) \quad |f' u - \frac{1}{\tilde{\gamma}}| \leq cu^2, \quad ||f''| u^2 - \frac{1}{\tilde{\gamma}}| \leq cu^2$$

and

$$(3.64) \quad \mu = \frac{2 \log \tilde{v}}{|u|^{2-\epsilon}} \leq 2|u|^{\beta+\epsilon-2}.$$

4. C^2 -ESTIMATES

F grows exponentially fast in time, more precisely we have the following

Theorem 4.1. *The estimate*

$$(4.1) \quad F \geq ce^{\gamma t}$$

is valid, where $c > 0$ depends on M_0 .

Proof. Use (1.18), note that we used a different notation there and (3.3). \square

For later purposes we obtain an evolution equation for F .

We have (we remark that in our case the evolution equations are the same as in [8, Lemma 2.3.2, Lemma 2.3.3], see also (3.51))

$$(4.2) \quad \begin{aligned} \dot{h}_i^j &= \left(-\frac{1}{F}\right)_i^j + \frac{1}{F} h_i^k h_k^j + \frac{1}{F} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_k^\delta g^{kj} \\ \dot{\nu}^\alpha &= g^{ij} \frac{F_i}{F^2} x_j^\alpha \\ \dot{v} &= -\frac{1}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta - g^{ij} \frac{F_i}{F^2} u_j \\ \dot{u} &= \frac{\tilde{v}}{F} \\ \dot{g}_{ij} &= -\frac{2}{F} h_{ij} \end{aligned}$$

and, furthermore, since

$$(4.3) \quad F = F(\check{h}_i^j) = F(h_i^j - \tilde{v} f' \delta_i^j + \psi_\alpha \nu^\alpha \delta_i^j)$$

we infer

$$(4.4) \quad \dot{F} = F_j^i \dot{\check{h}}_i^j,$$

and finally

Lemma 4.2.

$$(4.5) \quad \begin{aligned} \dot{F} - \frac{1}{F^2} F^{ij} F_{ij} &= -\frac{2}{F^3} F^{ij} F_i F_j + \frac{1}{F} F^{ij} h_i^k h_{kj} + \frac{1}{F} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ &+ \frac{1}{F} \eta_{\alpha\beta} \nu^\alpha \nu^\beta f' F^{ij} g_{ij} - \frac{1}{F} \tilde{v}^2 f'' F^{ij} g_{ij} + \frac{1}{F^2} f' F_k u^k F^{ij} g_{ij} \\ &- \frac{1}{F} \psi_{\alpha\beta} \nu^\alpha \nu^\beta F^{ij} g_{ij} + \frac{1}{F^2} \psi_\alpha x_k^\alpha F^k F^{ij} g_{ij}. \end{aligned}$$

In the following lemma we prove the important evolution equation for the second fundamental form (h_i^k) .

Lemma 4.3.

$$\begin{aligned}
(4.6) \quad & \dot{h}_l^k - F^{-2} F^{ij} h_{l;ij}^k = -2F^{-3} F^k F_l + F^{-1} h^{kr} h_{rl} + F^{-1} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_l^\beta \nu^\gamma x_r^\delta g^{rk} \\
& - F^{-2} F^{ij} h_{aj} h_i^a h_l^k + F^{-2} F^{ij} h_{ij} h_{al} h^{ak} + 2F^{-2} g^{pk} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} x_r^\alpha x_p^\beta x_i^\gamma x_l^\delta h_j^r \\
& - F^{-2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} x_a^\alpha x_i^\beta x_l^\gamma x_j^\delta h^{ak} - F^{-2} g^{pk} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} x_r^\alpha x_i^\beta x_p^\gamma x_j^\delta h_l^r \\
& - F^{-2} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h_l^k + F^{-2} g^{pk} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_p^\beta \nu^\gamma x_l^\delta h_{ij} \\
& + F^{-2} g^{pk} F^{ij} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x_p^\beta x_i^\gamma x_l^\delta x_j^\epsilon + F^{-2} g^{pk} F^{ij} \bar{R}_{\alpha\beta\gamma\delta;\epsilon} \nu^\alpha x_i^\beta x_j^\gamma x_p^\delta x_l^\epsilon \\
& + F^{-2} g^{pk} F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;l} \\
& + F^{-2} F^{ij} g_{ij} (-u_l u^k \check{v} f''' + g^{pk} \psi_{\alpha\beta\gamma} \nu^\alpha x_p^\beta x_l^\gamma + \psi_{\alpha\beta} \nu^\alpha \nu^\beta h_l^k \\
& \quad + g^{pk} \psi_{\alpha\beta} x_r^\alpha x_p^\beta h_l^r + \psi_{\alpha\beta} x_r^\alpha x_l^\beta h^{rk} + \psi_\alpha \nu^\alpha h_{lr} h^{rk} + \psi_\alpha x_r^\alpha h^{rk}{}_l) \\
& + F^{-2} F^{ij} g_{ij} (-g^{pk} f' \eta_{\alpha\beta\gamma} \nu^\alpha x_p^\beta x_l^\gamma - g^{pk} f' \eta_{\alpha\beta} x_r^\alpha x_p^\beta h_l^r - f' \eta_{\alpha\beta} \nu^\alpha \nu^\beta h_l^k \\
& \quad - f' \eta_{\alpha\beta} x_r^\alpha x_l^\beta h^{rk} - f' h_{rl} h^{rk} \check{v} + f' u^r h_{l;r}^k + f' u^r g^{kp} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_p^\beta x_r^\gamma x_l^\delta \\
& \quad - f'' (\check{v}^k u_l + \check{v}_l u^k) + f'' \check{v}^2 h_l^k + f'' \check{v} \eta_{\alpha\beta} x_l^\alpha x_r^\beta g^{rk}).
\end{aligned}$$

Proof. The starting point of the proof is the equation for \dot{h}_i^j given in (4.2), which contains the summand

$$(4.7) \quad \left(-\frac{1}{F}\right)_i^j = \frac{1}{F^2} F_i^j - \frac{2}{F^3} F_i F^j.$$

To finish the proof, we only have to calculate the covariant derivative F_i^j in detail. Deriving the purely covariant version of this tensor we first get

$$(4.8) \quad F_{kl} = F^{ij} \check{h}_{ij;kl} + F^{ij,rs} \check{h}_{ij;k} \check{h}_{rs;l},$$

then $\check{h}_{ij;kl}$ will be expressed as

$$(4.9) \quad \check{h}_{ij;kl} = h_{ij;kl} + \text{additional terms}$$

and interchanging indices in the usual way (which is technical using the Codazzi equations and the Ricci identities, cf. the proof of [8, Lemma 2.4.1]) leads to the representation

$$(4.10) \quad \check{h}_{ij;kl} = h_{kl;ij} + \text{additional terms},$$

with different *additional terms*. \square

We already know the estimate

$$(4.11) \quad -c|f'| \leq \kappa_i,$$

$c > 0$, because of the fact that the $\check{\kappa}_i$ are positive, remember $\check{\kappa}_i = \kappa_i - \check{v} f' + \psi_\alpha \nu^\alpha$. Now we prove an estimate from above.

Theorem 4.4. *We have*

$$(4.12) \quad \kappa_i \leq c.$$

Proof. Let φ be defined by

$$(4.13) \quad \varphi = \sup\{h_{ij} \eta^i \eta^j : \|\eta\| = 1\}.$$

We shall prove that

$$(4.14) \quad w = \log \varphi + \lambda \check{v}$$

is uniformly bounded from above, if λ is large enough.

The proof is divided into two steps:

(i) There is a $\mu > 0$ such that if a maximum of $w|_{[0,T]}$ (where $0 < T < \infty$ arbitrary but fixed) is attained in (t_0, x_0) , $0 < t_0 \leq T$, $x_0 \in S_0$, then there holds in (t_0, x_0)

$$(4.15) \quad h_n^n \leq \mu |f'|$$

(h_n^n denotes as usual the largest principal curvature).

(ii) Secondly we prove that

$$(4.16) \quad h_n^n \leq c$$

in (t_0, x_0) , where, without loss of generality, we may assume that t_0 is large.

Now we prove (i) by contradiction. Introducing Riemannian normal coordinates around (t_0, x_0) and arguing as usual, i.e. second derivatives of φ with respect to space and the first derivative with respect to time coincide with the corresponding ones of h_n^n , furthermore $g_{ij} = \delta_{ij}$ and h_i^j is diagonal, we may assume that w is defined by

$$(4.17) \quad w = \log h_n^n + \lambda \tilde{v}.$$

Moreover, we assume $h_n^n > \mu |f'|$ in (t_0, x_0) , where μ is large and will be chosen later. Applying the maximum principle we obtain

$$(4.18) \quad 0 \leq \dot{w} - \frac{1}{F^2} F^{ij} w_{ij}.$$

in (t_0, x_0) .

Using $F = F^{ij} \check{h}_{ij}$ and $F \in (K^*)$ we have, cf. Definition 2.3,

$$(4.19) \quad \begin{aligned} \epsilon_0 F \check{H} &\leq F^{ij} \check{h}_i^k \check{h}_{kj} = F^{ii} (\check{h}_{ii})^2 \leq F^{ii} (h_{ii} + \tilde{v} |f'| g_{ii} + \psi_\alpha \nu^\alpha g_{ii})^2 \\ &\leq (1 + \epsilon) F^{ii} h_{ii}^2 + 2\tilde{v} |f'| F^{ij} h_{ij} + \tilde{v}^2 |f'|^2 F^{ij} g_{ij} + c_\epsilon |u| F^{ij} g_{ij} \\ &\leq (1 + \epsilon) F^{ii} h_{ii}^2 + 2\tilde{v} |f'| F, \end{aligned}$$

where $\epsilon > 0$. In view of

$$(4.20) \quad \check{h}_n^n = h_n^n - \tilde{v} |f'| + \psi_\alpha \nu^\alpha.$$

we infer

$$(4.21) \quad \begin{aligned} -(1 + \epsilon) F^{ii} h_{ii}^2 &\leq -\epsilon_0 F \check{H} + 2\tilde{v} |f'| F \\ &\leq -\frac{\epsilon_0}{2} F h_n^n - \frac{\epsilon_0}{2} F h_n^n + 2\tilde{v} |f'| F \\ &\leq -\frac{\epsilon_0}{2} F h_n^n - \frac{\epsilon_0}{2} \mu F |f'| + 2\tilde{v} |f'| F \\ &\leq -\frac{\epsilon_0}{2} F h_n^n, \end{aligned}$$

where we assume that μ is large; hence there is $\delta_0 > 0$ such that

$$(4.22) \quad -F^{ij} h_{ik} h_j^k \leq -\delta_0 F h_n^n$$

in (t_0, x_0) .

In (t_0, x_0) we have

$$(4.23) \quad h_{n;i}^n = -\lambda \tilde{v}_i h_n^n$$

and in view of (4.18)

$$(4.24) \quad 0 \leq \frac{1}{h_n^n} (\dot{h}_n^n - F^{-2} F^{ij} h_{n;ij}^n) + \lambda (\dot{\tilde{v}} - F^{-2} F^{ij} \tilde{v}_{ij}) + \frac{\lambda^2}{F^2} F^{ij} \tilde{v}_i \tilde{v}_j.$$

Multiplying this inequality by F^2 , inserting the evolution equations for h_n^n and \tilde{v} , cf. Lemma 4.3 and Lemma 3.1, as well as some trivial estimates yield (no summation with respect to n)

$$(4.25) \quad \begin{aligned} 0 \leq & -2 \frac{1}{h_n^n} F^{-1} F^n F_n + 2F h_n^n + \frac{c}{h_n^n} F + \frac{1}{h_n^n} F^{ij,rs} \check{h}_{ij;n} \check{h}_{rs;n} \\ & + c |f'|^{\frac{3}{2}} F^{ij} g_{ij} + \tilde{v}^2 f'' F^{ij} g_{ij} + \lambda |u| F^{ij} g_{ij} \\ & - \frac{\lambda}{2} \tilde{v} F^{ii} h_{ii}^2 + \lambda^2 F^{ij} \tilde{v}_i \tilde{v}_j. \end{aligned}$$

We remark that we have estimated the term arising from the second term in the second line of equation (4.6) together with two other terms arising from (4.6) by employing the homogeneity of F , namely, $F = F^{ij} h_{ij} - \tilde{v} f' F^{ij} g_{ij} + \psi_\alpha \nu^\alpha F^{ij} g_{ij}$.

Terms arising from the two terms in (4.6) depending linearly on the derivatives of the second fundamental form are first rewritten with the help of the Codazzi equation (the correction terms can be estimated very easily) such that we obtain the derivative of h_n^n . The resulting terms can be estimated as follows:

$$(4.26) \quad \frac{1}{h_n^n} \psi_\alpha x_r^\alpha h_n^{n;r} F^{ij} g_{ij} \leq \lambda c |u| F^{ij} g_{ij} + \lambda \psi_\alpha x_r^\alpha u_s h^{sr} F^{ij} g_{ij}$$

for the first term, where we used

$$(4.27) \quad \tilde{v}_i = \eta_{\alpha\beta} \nu^\alpha x_i^\beta - u_r h_i^r,$$

with (η_α) as in Lemma 3.1, cf. also Lemma 3.5, and

$$(4.28) \quad \frac{1}{h_n^n} f' u^r h_{n;r}^n F^{ij} g_{ij} = -\lambda f' u^r \tilde{v}_r F^{ij} g_{ij}$$

for the second one. Both last summands in the previous inequalities appear among the terms coming from the evolution equation of \tilde{v} with opposite sign.

Since $F \in (K)$ and homogenous of degree 1 we deduce from [8, Lemma 2.2.14] that F is concave, hence [8, Proposition 2.1.23] implies

$$(4.29) \quad F^{ij,rs} (\check{h}_{ij}) \check{h}_{ij;n} \check{h}_{rs;n} \leq 0.$$

Together with

$$(4.30) \quad F^{ij} \tilde{v}_i \tilde{v}_j \leq c |u| F^{ij} g_{ij} + c \|Du\|^2 F^{ij} h_{ik} h_j^k$$

(which follows by using $\|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$, here $\|\cdot\|$ is the norm induced by the quadratic form F^{ij}) we conclude

$$(4.31) \quad \begin{aligned} 0 \leq & 2F h_n^n + \frac{c}{h_n^n} F + c |f'|^{\frac{3}{2}} F^{ij} g_{ij} + \tilde{v}^2 f'' F^{ij} g_{ij} + c \lambda^2 |u| F^{ij} g_{ij} \\ & - \frac{\lambda}{4} \tilde{v} \delta_0 F h_n^n. \end{aligned}$$

For $\lambda > 0$ large we get a contradiction, which finishes the proof of (i).

We now prove (ii). From (i) we deduce that the largest principal curvature of $M(t)$ is bounded by $ce^{\gamma t}$ for all $t > 0$. Combining this with Lemma 4.1, namely,

$$(4.32) \quad 0 < c_0 \leq F(e^{-\gamma t} \check{h}_j^i) = F(e^{-\gamma t} (h_j^i - \tilde{v} f' \delta_j^i + \psi_\alpha \nu^\alpha \delta_j^i)),$$

we infer that $e^{-\gamma t} \check{h}_j^i$ lies in a compact subset of Γ_+ for all $t > 0$. Hence we have constants $c, \check{c}_1, \check{c}_2, \bar{c}_1, \bar{c}_2 > 0$ (not depending on t_0 or T), such that for all times and especially in (t_0, x_0)

$$(4.33) \quad -ce^{\gamma t} \leq \kappa_i \leq ce^{\gamma t} \quad \wedge \quad \check{c}_1 e^{\gamma t} \leq F \leq \check{c}_2 e^{\gamma t} \quad \wedge \quad 0 < \bar{c}_1 g_{ij} \leq F^{ij} \leq \bar{c}_2 g_{ij}.$$

We again look at (4.24) multiplied by F^2 in (t_0, x_0) . We assume that h_n^n is large and will show that it is a priori bounded. We have

$$(4.34) \quad -F^{ii} h_{ii}^2 \leq -\bar{c}_1 (h_n^n)^2$$

and furthermore

$$(4.35) \quad \begin{aligned} 0 &\leq 2F h_n^n + \frac{c}{h_n^n} F + \lambda |f'|^{\frac{3}{2}} F^{ij} g_{ij} + f'' \tilde{v}^2 F^{ij} g_{ij} - \frac{\lambda}{2} F^{ij} h_{ik} h_j^k \tilde{v} \\ &\leq \epsilon F^2 + c_\epsilon (h_n^n)^2 + \lambda |f'|^{\frac{3}{2}} F^{ij} g_{ij} + f'' \tilde{v}^2 F^{ij} g_{ij} - \frac{\lambda}{2} \bar{c}_1 \tilde{v} (h_n^n)^2, \end{aligned}$$

$\epsilon > 0$ small; we remember

$$(4.36) \quad f'' \leq -ce^{2\gamma t}.$$

If λ is sufficiently large and t_0 sufficiently large we get a contradiction. \square

Lemma 4.5.

$$(4.37) \quad \sup_{M(t)} \max_i |\kappa_i u| \rightarrow 0 \quad t \rightarrow \infty.$$

Proof. We remember that

$$(4.38) \quad F = F(\check{h}_i^j) = F(h_i^j - \tilde{v} f' \delta_i^j + \psi_\alpha \nu^\alpha \delta_i^j) = F(\kappa_i - \tilde{v} f' + \psi_\alpha \nu^\alpha),$$

where the κ_i are the eigenvalues of h_i^j , now numbered such that κ_n is the smallest one.

The function $\varphi = -uF$ satisfies the following parabolic equation, cf. Lemma 4.2 and Lemma 3.9,

$$(4.39) \quad \begin{aligned} \dot{\varphi} - F^{-2} F^{ij} \varphi_{ij} &= \frac{2u}{F^3} F^{ij} F_i F_j - \frac{u}{F} F^{ij} h_i^k h_{kj} - \frac{u}{F} F^{ij} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta \\ &\quad - \frac{u}{F} f' \eta_{\alpha\beta} \nu^\alpha \nu^\beta F^{ij} g_{ij} - \frac{u}{F^2} f' F^k u_k F^{ij} g_{ij} + \frac{u}{F} \tilde{v}^2 f'' F^{ij} g_{ij} \\ &\quad + \frac{u}{F} \psi_{\alpha\beta} \nu^\alpha \nu^\beta F^{ij} g_{ij} - \frac{u}{F^2} F^k \psi_\alpha x_k^\alpha F^{ij} g_{ij} - 2\tilde{v} \\ &\quad - \frac{\tilde{v}^2}{F} f' F^{ij} g_{ij} + \frac{\tilde{v}}{F} \psi_\alpha \nu^\alpha F^{ij} g_{ij} + \frac{1}{F} F^{ij} \bar{h}_{ij} + \frac{2}{F^2} F^{ij} u_i F_j. \end{aligned}$$

For $t > 0$ we define $\tilde{\varphi}(t) = \inf_{S_0} \varphi(t, \cdot)$ and choose $x_t \in S_0$ such that

$$(4.40) \quad \tilde{\varphi}(t) = \varphi(t, x_t),$$

then $\tilde{\varphi}$ is differentiable a.e. and we have

$$(4.41) \quad \dot{\tilde{\varphi}}(t) = \dot{\varphi}(t, x_t)$$

for a.e. $t > 0$.

Let $t_0 > 0$ be sufficiently large, then combining (4.39) and (4.41) and using $\varphi_i = 0$ yields

$$(4.42) \quad \begin{aligned} \dot{\tilde{\varphi}}(t) &\geq -\frac{u}{F} F^{ij} h_i^k h_{kj} + u \frac{\tilde{v}^2}{F} f'' F^{ij} g_{ij} - \frac{\tilde{v}^2}{F} f' F^{ij} g_{ij} \\ &\quad - 2\tilde{v} - \frac{c_0}{F} F^{ij} g_{ij} \end{aligned}$$

for a.e. $t > t_0$, where $c_0 = c_0(t_0)$ and the right side is evaluated at (t, x_t) . Due to the assumptions on f we may furthermore assume that for all $t > t_0$ the following inequality holds in (t, x_t)

$$(4.43) \quad \frac{u}{F} \tilde{v}^2 f'' F^{ij} g_{ij} - \frac{\tilde{v}^2}{F} f' F^{ij} g_{ij} \geq 2\tilde{v} - \frac{c_0}{F} F^{ij} g_{ij},$$

which leads to

$$(4.44) \quad \dot{\tilde{\varphi}}(t) \geq -\frac{u}{F} F^{ij} h_i^k h_{kj} - 2\frac{c_0}{F} F^{ij} g_{ij}$$

for a.e. $t > t_0$ in view of (4.42); again the right side is evaluated at (t, x_t) .

We assume that (4.37) is not true, then there are sequences $0 < t_k \rightarrow \infty$, $x_k \in S_0$ and a constant $c_1 > 0$ such that

$$(4.45) \quad \sup_{M(t_k)} \max_i \kappa_i u = \kappa_n u|_{(t_k, x_k)} \rightarrow c_1,$$

which implies

$$(4.46) \quad \begin{aligned} \limsup_{k \rightarrow \infty} \tilde{\varphi}(t_k) &< F\left(-\frac{c_1}{2} + \tilde{\gamma}^{-1}, \tilde{\gamma}^{-1}, \dots, \tilde{\gamma}^{-1}\right) \\ &< F(\tilde{\gamma}^{-1} - r, \dots, \tilde{\gamma}^{-1} - r) \\ &=: c(r), \end{aligned}$$

for $r > 0$ sufficiently small and fixed from now on.

Next, we will show that, after increasing t_0 if necessary, there exists $\delta > 0$ such that the following implication holds for a.e. $t > t_0$

$$(4.47) \quad \tilde{\varphi}(t) \leq c(r) \Rightarrow \dot{\tilde{\varphi}}(t) \geq \delta$$

in contradiction to (4.46).

For that purpose assume t_0 to be sufficiently large. Let $t > t_0$ be such that $\tilde{\varphi}$ is differentiable in t and $\tilde{\varphi}(t) \leq c(r)$, then it follows from (4.38) that we have in (t, x_t)

$$(4.48) \quad |u| \kappa_n + \tilde{v} |f' u| + |u| \psi_\alpha \nu^\alpha \leq -r + \tilde{\gamma}^{-1},$$

i.e.

$$(4.49) \quad \kappa_n \leq -\frac{r}{2|u|}.$$

Hence, we infer from (4.44)

$$(4.50) \quad \dot{\tilde{\varphi}}(t) \geq \frac{r^2}{4F|u|} - 2\frac{c_0}{F} F^{ij} g_{ij}.$$

After a possibly further enlargement of t_0 we get a positive lower bound for the right side of the last inequality that does not depend on t , thus the desired $\delta > 0$, which completes the proof. \square

Now we are able to prove a decay of $\|A\|$.

Lemma 4.6. *For any $0 < \lambda < \gamma$ there exists $c_\lambda > 0$ such that*

$$(4.51) \quad \|A\| e^{\lambda t} \leq c_\lambda.$$

Proof. Define $\varphi = \frac{1}{2}\|A\|^2 e^{2\lambda t}$ with $0 < \lambda < \gamma$, then

$$(4.52) \quad e^{-2\lambda t}(\dot{\varphi} - \frac{1}{F^2}F^{ij}\varphi_{ij}) = -\frac{1}{F^2}F^{kl}h_{j;k}^i h_{i;l}^j + (\dot{h}_j^i - \frac{1}{F^2}F^{kl}h_{j;kl}^i)h_i^j + \lambda\|A\|^2.$$

Let $0 < T < \infty$ be large, and $x_0 = x_0(t_0)$, with $0 < t_0 \leq T$, be a point in $M(t_0)$ such that

$$(4.53) \quad \sup_{M_0} \varphi < \sup_{M(t)} \{\sup \varphi : 0 < t \leq T\} = \varphi(x_0).$$

From Lemma 4.5 we know that

$$(4.54) \quad \sup_{M(t)} \|A\| |u| \longrightarrow 0 \quad \text{as } t \longrightarrow \infty$$

so that especially in view of the homogeneity of F

$$(4.55) \quad 0 < c_1 < F^i(\check{\kappa}_i) \leq c_2 \quad \wedge \quad |F^{ij}(\check{\kappa}_i)| \leq ce^{-\gamma t}$$

(first and second derivatives of F considered as a function on Γ_+). In x_0 we have due to (4.52) and Lemma 4.3, after multiplication by F^2 and some straight-forward estimates,

$$(4.56) \quad \begin{aligned} 0 &\leq -F^{kl}h_{j;k}^i h_{i;l}^j - 2F^{-1}F^i F_j h_i^j + 2F h^{ir} h_{rj} h_i^j + F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;p} h^{pp} \\ &\quad + c|f'|^{1+\epsilon}\|A\| + c|f'|^{\frac{1}{2}}\|A\|^2 + \tilde{v}^2 f'' \|A\|^2 F^{ij} g_{ij} + \lambda F^2 \|A\|^2 \\ &\leq -\frac{1}{2}F^{kl}h_{j;k}^i h_{i;l}^j + \tilde{v}^2 f'' \|A\|^2 F^{ij} g_{ij} + cF\|A\|^2 \|A\| + c|f'|^{1+\epsilon}\|A\| \\ &\quad + c|f'|^{\frac{1}{2}}\|A\|^2 + \lambda F^2 \|A\|^2. \end{aligned}$$

For the last inequality we used that in local coordinates (such that $g_{ij} = \delta_{ij}$, h_{ij} diagonal and F^{ij} diagonal)

$$(4.57) \quad |F_i F_j| \leq c \sum_{i,k,l} |h_{kl;i}|^2 + c\|A\|^2 |f'|^{\frac{1}{2}} + c|f'|^{2+\epsilon},$$

where we used Lemma 3.10, and where $0 < \epsilon < 1$ is arbitrary but fixed, so that

$$(4.58) \quad F^{-1}F^i F_j h_i^j \leq c \frac{\|A\|}{F} \sum_{i,k,l} |h_{kl;i}|^2 + c\|A\|^2 |f'|^{-\frac{1}{2}} \|A\| + c|f'|^{1+\epsilon}\|A\|,$$

where $\frac{\|A\|}{F} \rightarrow 0$ because of Lemma 4.5.

To estimate $F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;p} h_p^p$ we used [8, inequality (2.1.73)] and (4.55).

Now we have

$$(4.59) \quad \begin{aligned} F &= |f'| F\left(\frac{\check{\kappa}_i}{|f'|}\right) = |f'| F(1, \dots, 1) + |f'| (F\left(\frac{\check{\kappa}_i}{|f'|}\right) - F(1, \dots, 1)) \\ &\leq n|f'| + |f'| c(t), \end{aligned}$$

where $0 < c(t) \rightarrow 0$, hence

$$(4.60) \quad \begin{aligned} \tilde{v}^2 f'' \|A\|^2 F^{ij} g_{ij} + \lambda F^2 \|A\|^2 &\leq c\|A\|^2 - (\gamma - \lambda)n^2 |f'|^2 \|A\|^2 \\ &\quad + \lambda c c(t) |f'|^2 \|A\|^2. \end{aligned}$$

Together with (4.56) we deduce that φ is a priori bounded from above. \square

In the next two theorems we prove the optimal decay of $\|Du\|$ and $\|A\|$ which finishes the C^2 -estimates.

Theorem 4.7. *Let $\tilde{u} = ue^{\gamma t}$, then $\|D\tilde{u}\|$ is uniformly bounded during the evolution.*

Proof. Let $\varphi = \varphi(t)$ be defined by

$$(4.61) \quad \varphi = \sup_{M(t)} \log \tilde{v} e^{2\gamma t}.$$

Then, in view of the maximum principle, we deduce from the evolution equation of \tilde{v} , cf. Lemma (3.1),

$$(4.62) \quad \begin{aligned} \dot{\varphi} &\leq ce^{-\epsilon t} + F^{-2}(f'' \|D\tilde{u}\|^2 F^{ij} g_{ij} + 2\gamma F^2 \varphi) \\ &\leq ce^{-\epsilon t} + 2F^{-2}(f'' F^{ij} g_{ij} + \gamma F^2 \varphi) \\ &\leq ce^{-\epsilon t}(1 + \varphi), \end{aligned}$$

where $\epsilon > 0$ small, i.e., φ is uniformly bounded. \square

Theorem 4.8. *The quantity $w = \frac{1}{2}\|A\|^2 e^{2\gamma t}$ is uniformly bounded during the evolution.*

Proof. Define $\varphi = \varphi(t)$ by

$$(4.63) \quad \varphi = \sup_{M(t)} w.$$

We deduce from Lemma 4.3 that for a.e. $t \geq t_0$, $t_0 > 0$ large,

$$(4.64) \quad \begin{aligned} \dot{\varphi} &= \frac{1}{F^2} F^{ij} \varphi_{ij} - \frac{1}{F^2} F^{kl} h_{j;k}^i h_{i;l}^j e^{2\gamma t} + (\dot{h}_j^i - \frac{1}{F^2} F^{kl} h_{j;kl}^i) h_i^j e^{2\gamma t} \\ &\quad + \gamma \|A\|^2 e^{2\gamma t} \\ &\leq -\frac{1}{2F^2} F^{kl} h_{j;k}^i h_{i;l}^j e^{2\gamma t} + F^{-3}(-2h^{ij} F_i F_j e^{2\gamma t} - F f''' h^{ij} \tilde{u}_i \tilde{u}_j F^{ij} g_{ij} \tilde{v}) \\ &\quad + 2F^{-2}(n f'' \tilde{v}^2 \varphi + \gamma F^2 \varphi) + ce^{-\epsilon t}(1 + \varphi) \\ &\quad + F^{-1} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h^{ij} e^{2\gamma t}, \end{aligned}$$

where $\epsilon > 0$ is small.

For the last inequality we estimated the crucial term

$$(4.65) \quad F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;q} h^{pq}$$

in the following way.

Since $F^{ij} g_{ij} \geq F(1, \dots, 1)$ and

$$(4.66) \quad F^{ij}(g_{kl}) g_{ij} = F(1, \dots, 1)$$

we deduce that the derivative vanishes in $\check{h}_{kl} = g_{kl}$

$$(4.67) \quad F^{ij,rs}(g_{kl}) g_{ij} = 0.$$

Hence

$$(4.68) \quad F^{ij,rs}(\check{h}_{kl}) g_{ij} = |u|(F^{ij,rs}(|u|\check{h}_{kl}) - F^{ij,rs}(\frac{1}{\check{\gamma}} g_{kl})) g_{ij}$$

which means by mean value theorem

$$(4.69) \quad \|F^{ij,rs}(\check{h}_{kl}) g_{ij}\| \leq c|u|^2.$$

Although the last inequality is good enough, we mention that its right side could be improved to $c|u|^{3-\epsilon}$, $\epsilon > 0$ arbitrary, cf. Lemma 4.6.

Furthermore, to estimate (4.65) we use

$$(4.70) \quad \check{h}_{ij;p} = h_{ij;p} - \tilde{v}_p f' g_{ij} - \tilde{v} f'' u_p g_{ij} + \psi_{\alpha\beta} \nu^\alpha x_p^\beta g_{ij} + \psi_\alpha x_r^\alpha h_p^r g_{ij}$$

and

$$(4.71) \quad \tilde{v}_p = \eta_{\alpha\beta} \nu^\alpha x_p^\beta - u_r h_p^r$$

So in view of Lemma 4.6 and Theorem 4.7 we have choosing coordinates such that (h_{ij}) diagonal and $g_{ij} = \delta_{ij}$

$$(4.72) \quad \begin{aligned} |F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;q} h^{pq}| &\leq |F^{ij,rs} h_{ij;p} h_{rs;q} h^{pq}| \\ &\quad + 2|F^{ij,rs} h_{rs;p} g_{ij} (-\tilde{v}_q f' - \tilde{v} f'' u_q + \psi_{\alpha\beta} \nu^\alpha x_q^\beta + \psi_\alpha x_r^\alpha h_q^r) h^{pq}| \\ &\quad + |F^{ij,rs} g_{rs} g_{ij} \sum_p (-\tilde{v}_p f' - \tilde{v} f'' u_p + \psi_{\alpha\beta} \nu^\alpha x_p^\beta + \psi_\alpha x_r^\alpha h_p^r)^2 h_p^p| \\ &\leq c|u| \|DA\|^2 \|A\| + c\|A\| |u| (\|DA\| + 1). \end{aligned}$$

The second term of the right side of inequality (4.64) can be estimated as follows

$$(4.73) \quad \begin{aligned} F^{-3} (-2h^{ij} F_i F_j e^{2\gamma t} - F f''' h^{ij} \tilde{u}_i \tilde{u}_j F^{ij} g_{ij} \tilde{v}) &\leq \\ F^{-3} (-2|f''|^2 + f' f''') h^{ij} \tilde{u}_i \tilde{u}_j \tilde{v}^2 n^2 + cF^{-3} \|DA\|^2 e^{2\gamma t} & \\ + ce^{-\epsilon t} (1 + \varphi). & \end{aligned}$$

Now, we observe that

$$(4.74) \quad (f'' + \tilde{\gamma}|f'|^2)' = f''' + 2\tilde{\gamma}f'f'' = Cf',$$

where C is a bounded function in view of (1.9). Hence

$$(4.75) \quad 2|f''|^2 - f'f''' = 2|f''|^2 + 2\tilde{\gamma}|f'|^2 f'' - C|f'|^2,$$

i.e.,

$$(4.76) \quad |2|f''|^2 - f'f'''| \leq c|f'|^2$$

because of (1.8) and we conclude that the left-hand side of (4.73) can be estimated from above by

$$(4.77) \quad ce^{-\epsilon t} (1 + \varphi) + cF^{-2} \|DA\|^2 e^{\gamma t}.$$

Next, we estimate

$$(4.78) \quad F^{-2} (nf'' \tilde{v}^2 + \gamma F^2) \varphi \leq ce^{-\epsilon t} \varphi$$

and finally

$$(4.79) \quad F^{-1} \bar{R}_{\alpha\beta\gamma\delta} \nu^\alpha x_i^\beta \nu^\gamma x_j^\delta h^{ij} e^{2\gamma t} \leq ce^{-\epsilon t} (1 + \varphi) + F^{-1} \bar{R}_{0i0j} h^{ij} e^{2\gamma t} \tilde{v}^2,$$

but

$$(4.80) \quad \bar{R}_{0i0j} \leq c|u|,$$

cf. proof of Lemma 3.5(iii).

Hence we deduce

$$(4.81) \quad \dot{\varphi} \leq ce^{-\epsilon t} (1 + \varphi)$$

for some positive ϵ and for a.e. $t \geq t_0$, i.e. φ is bounded. \square

5. APPENDIX A

Appendices A to C contain a more or less word by word adoption of the argumentation in [8, Sections 7.6-7.8] and complete the proof of Theorem 1.3.

For reasons of completeness and convenience for the reader we decided to present them here.

Appendix A proves higher order estimates.

Let us now introduce the following abbreviations

Definition 5.1. (i) For arbitrary tensors S, T denote by $S * T$ any linear combination of contractions of $S \otimes T$. The result can be a tensor or a function. Note that we do not distinguish between $S * T$ and $cS * T$, where c is a constant.

(ii) The symbol A represents the second fundamental of the hypersurfaces $M(t)$ in N , $\tilde{A} = Ae^{\gamma t}$ is the scaled version, and $D^m A$ resp. $D^m \tilde{A}$ represent the covariant derivative of order m .

(iii) For $m \in \mathbb{N}$ denote by \tilde{O}_m a tensor expression defined on $M(t)$ that satisfies the pointwise estimate

$$(5.1) \quad \|\tilde{O}_m\| \leq c_m(1 + \|\tilde{A}\|_m)^{p_m}$$

and

$$(5.2) \quad \|D\tilde{O}_m\| \leq c_m(1 + \|\tilde{A}\|_m)^{p_m}(1 + \|D^{m+1}\tilde{A}\|),$$

where $c_m, p_m > 0$ are constants and

$$(5.3) \quad \|\tilde{A}\|_m = \sum_{|\alpha| \leq m} \|D^\alpha \tilde{A}\|.$$

(iv) For arbitrary $m \in \mathbb{N}$ denote by O_m a tensor expression defined on $M(t)$ that satisfies

$$(5.4) \quad D^k O_m = \tilde{O}_{m+k} \quad \forall k \in \mathbb{N}.$$

(v) By the symbol O we denote a tensor expression such that $DO = O_0$.

Remark 5.2.

$$(5.5) \quad D^k O_m = O_{m+k} \quad \forall (k, m) \in \mathbb{N} \times \mathbb{N}.$$

Lemma 5.3. *We have*

$$(5.6) \quad D(uf') = e^{-2\gamma t} O$$

especially

$$(5.7) \quad D^m(uf') = e^{-2\gamma t} O_{m-2}, \quad m \geq 2.$$

Proof. Differentiating and adding a zero yields

$$(5.8) \quad D_i(uf') = u_i f' (1 - \tilde{\gamma} f' u) + uu_i (\tilde{\gamma} |f'|^2 + f'')$$

from which we deduce the claim in view of (1.8), (1.9) and (1.10). □

Lemma 5.4. *We have*

$$(5.9) \quad D(u\check{h}_{kl}) = e^{-2\gamma t} O_0 + e^{-2\gamma t} D\check{A}O.$$

Proof. Differentiating yields ($g_{ij} = \delta_{ij}$, $h_{ij} = \text{diagonal}$)

$$(5.10) \quad \begin{aligned} D_i(u\check{h}_{kl}) = & u_i\check{h}_{kl} + u(h_{kl;i} - \eta_{\alpha\beta}\nu^\alpha x_i^\beta f' g_{kl} \\ & + (\psi_\alpha\nu^\alpha)_i g_{kl}) + uu_i\kappa_i f' g_{kl} - u\check{v}f'' u_i g_{kl} \end{aligned}$$

and now we focus on the last term and write there

$$(5.11) \quad f'' = (f'' + \tilde{\gamma}|f'|^2) - \tilde{\gamma}|f'|^2.$$

Then all terms can be estimated obviously except for

$$(5.12) \quad u_i\check{h}_{kl} + \tilde{\gamma}|f'|^2 u\check{v}u_i g_{kl},$$

for which we use (1.28). \square

Corollary 5.5. *We have*

$$(5.13) \quad D^m(u\check{h}_{kl}) = e^{-2\gamma t} O_{m-1} + e^{-2\gamma t} D^m \tilde{A} * O.$$

Definition 5.6. We denote by $\mathcal{D}^m F$ the derivatives of order m of F with respect to \check{h}_j^i .

Lemma 5.7. *We have*

$$(5.14) \quad D^m \mathcal{D}F = e^{-2\gamma t} O_{m-1} + e^{-2\gamma t} D^m \tilde{A} * \mathcal{D}^2 F(|u|\check{h}_{kl}) * O,$$

$$(5.15) \quad |F^{ij}(\check{h}_{kl})g_{ij} - F^{ij}(g_{kl})g_{ij}| \leq ce^{-2\gamma t},$$

$$(5.16) \quad DF = \mathcal{D}F * DA + e^{-\gamma t} \mathcal{D}F * O_0 + e^{\gamma t} \mathcal{D}F * O,$$

and

$$(5.17) \quad D^m F = \mathcal{D}F * D^m A + e^{-\gamma t} O_{m-1} + e^{\gamma t} O_{m-2},$$

for $m \geq 2$.

Proof. To prove (5.14) we write

$$(5.18) \quad \mathcal{D}F(\check{h}_{kl}) = \mathcal{D}F(|u|\check{h}_{kl})$$

and infer

$$(5.19) \quad D\mathcal{D}F(\check{h}_{kl}) = \mathcal{D}^2 F(|u|\check{h}_{kl})D(|u|\check{h}_{kl}),$$

hence the desired result follows in view of (5.13) and the fact that

$$(5.20) \quad \|\mathcal{D}^m F(|u|\check{h}_{kl})\|$$

is bounded for all $m \in \mathbb{N}$.

(5.15) is proved by applying the mean value theorem.

(5.16) follows by

$$(5.21) \quad F_k = F^{ij}h_{ij;k} + F^{ij}g_{ij}(-\check{v}_k f' - \check{v}f'' u_k + \psi_{\alpha\beta}\nu^\alpha x_k^\beta + \psi_\alpha x_r^\alpha h_k^r).$$

To prove (5.17) we differentiate (5.16) and get

$$(5.22) \quad \begin{aligned} D^2 F = & D\mathcal{D}F * DA + \mathcal{D}F * D^2 A + e^{-\gamma t} D\mathcal{D}F * O_0 \\ & + e^{-\gamma t} \mathcal{D}F * O_1 + e^{\gamma t} D\mathcal{D}F * O + e^{\gamma t} \mathcal{D}F * O_0 \\ = & \mathcal{D}F * D^2 A + e^{-\gamma t} O_1 + e^{\gamma t} O_0, \end{aligned}$$

from which the claim follows easily. \square

Now we want to write the evolution equation for \tilde{h}_l^k in the form

$$(5.23) \quad \begin{aligned} \dot{\tilde{h}}_i^j - F^{-2}F^{kl}\tilde{h}_{i;kl}^j = & F^{-3}D\tilde{A} * DA * O_0 + F^{-2}D\tilde{A} * O_0 + F^{-1}O_0 \\ & + F^{-3}D\tilde{A} * DA * O. \end{aligned}$$

To check this we consider all the terms in (4.6) separately and start with

$$(5.24) \quad \boxed{(-2F^{-3}F^k F_l - F^{-2}F^{ij}g_{ij}f''' u^k u_l \tilde{v})e^{\gamma t}}.$$

We have

$$(5.25) \quad \begin{aligned} F_k = & F^{rs}h_{rs;k} - \eta_{\alpha\beta}\nu^\alpha x_k^\beta f' F^{rs}g_{rs} - \eta_\alpha x_i^\alpha h_k^i f' F^{rs}g_{rs} - \tilde{v}f'' u_k F^{rs}g_{rs} \\ & + \psi_{\alpha\beta}\nu^\alpha x_k^\beta F^{rs}g_{rs} + \psi_\alpha x_r^\alpha h_k^r F^{rs}g_{rs} \\ = & A_1 + A_2 + A_3 + A_4 + A_5 + A_6, \end{aligned}$$

hence

$$(5.26) \quad \begin{aligned} & (-2F^{-3}A_4A_4 - F^{-2}F^{ij}g_{ij}f''' u^k u_l \tilde{v})e^{\gamma t} \\ & = F^{-3}(-2|f''|^2 + f'f''')\tilde{v}^2(F^{ij}g_{ij})^2 u_l u^k e^{\gamma t} \\ & \quad - F^{-2}F^{rs}h_{rs}F^{ij}g_{ij}f''' u^k u_l \tilde{v}e^{\gamma t} \\ & \quad - F^{-2}(F^{ij}g_{ij})^2 \psi_\alpha \nu^\alpha f''' u^k u_l \tilde{v}e^{\gamma t} \\ & = F^{-1}O_0, \end{aligned}$$

where we observed that

$$(5.27) \quad \varphi = -2|f''|^2 + f'f''' = (f'' + \tilde{\gamma}|f'|^2)'f' - 2f''(f'' + \tilde{\gamma}|f'|^2).$$

In view of the assumptions on f the spatial derivatives of φ can be estimated by

$$(5.28) \quad \|D^m \varphi\| \leq c_m(1 + \|\tilde{u}\|_{m-1})^{p_{m-1}}(1 + \|D^m \tilde{u}\|)e^{2\gamma t} \quad \forall m \in \mathbb{N}^*$$

for some suitable $p_{m-1} \in \mathbb{N}$. Furthermore, we have

$$(5.29) \quad -2F^{-3}A_1A_1e^{\gamma t} = F^{-3}O_0 * D\tilde{A} * DA.$$

All remaining terms are estimated as follows

$$(5.30) \quad -2F^{-3}e^{\gamma t} \sum_{(i,j) \notin \{(1,1), (4,4)\}} A_i A_j = F^{-2}D\tilde{A} * O_0 + F^{-2}O_0$$

hence

$$(5.31) \quad \begin{aligned} & (-2F^{-3}F^k F_l - F^{-2}F^{ij}g_{ij}f''' u^k u_l \tilde{v})e^{\gamma t} = \\ & F^{-3}D\tilde{A} * DA * O_0 + F^{-2}D\tilde{A} * O_0 + F^{-1}O_0. \end{aligned}$$

Now, there are some quiet easy estimates, namely

$$(5.32) \quad \begin{aligned} & \boxed{2F^{-1}h^{kr}h_{rl}e^{\gamma t}} = F^{-2}O_0 \\ & \boxed{-F^{-2}e^{\gamma t}F^{ij}h_{aj}h_i^a h_l^k} = F^{-2}O_0 \\ & \boxed{2F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta}x_r^\alpha x_p^\beta x_i^\gamma x_l^\delta h_j^r e^{\gamma t}} = F^{-2}O_0 \\ & \boxed{F^{-2}g^{pk}F^{ij}\bar{R}_{\alpha\beta\gamma\delta;\epsilon}\nu^\alpha x_p^\beta x_i^\gamma x_l^\delta x_j^\epsilon e^{\gamma t}} = F^{-1}O_0. \end{aligned}$$

Furthermore, we have

$$(5.33) \quad \begin{aligned} \boxed{F^{-1}e^{\gamma t}\bar{R}_{\alpha\beta\gamma\delta}\nu^\alpha x_l^\beta\nu^\gamma x_r^\delta g^{rk}} &= -F^{-1}\bar{R}_{\alpha\beta\gamma\delta}\eta^\alpha\tilde{\nu}x_l^\beta\nu^\gamma x_r^\delta e^{\gamma t} \\ &- F^{-1}\bar{R}_{i\beta\gamma\delta}u^i x_l^\beta\eta^\gamma x_r^\delta g^{rk}e^{\gamma t} + F^{-1}\bar{R}_{i\beta j\delta}\tilde{\nu}^{-2}u^i x_l^\beta u^j x_r^\delta g^{rk}e^{\gamma t} \\ &= F^{-2}O, \end{aligned}$$

$$(5.34) \quad \begin{aligned} \boxed{F^{-2}e^{\gamma t}g^{pk}F^{ij,rs}\check{h}_{ij;p}\check{h}_{rs;l}} &= F^{-2}e^{\gamma t}g^{pk}\check{h}_{ij;p}D_l F^{ij} \\ &= F^{-2}D\tilde{A} * O + F^{-3}D\tilde{A} * DA * O + F^{-3}D\tilde{A} * O_0 + F^{-2}O_0 \end{aligned}$$

and

$$(5.35) \quad \boxed{F^{-2}e^{\gamma t}F^{ij}g_{ij}(\psi_\alpha x_r^\alpha h_{;l}^{rk} + f' u^r h_{l;r}^k)} = F^{-2}D\tilde{A} * O_0,$$

so that only the following term is left

$$(5.36) \quad \boxed{F^{-2}e^{\gamma t}F^{ij}g_{ij}f''\tilde{\nu}^2 h_l^k + \gamma h_l^k e^{\gamma t}} = F^{-2}e^{\gamma t}(F^{ij}g_{ij}f''\tilde{\nu}^2 h_l^k + \gamma F^2 h_l^k).$$

There holds

$$(5.37) \quad \begin{aligned} F^2 &= (F^{ij}h_{ij})^2 - 2F^{ij}h_{ij}\tilde{\nu}f'F^{rs}g_{rs} + 2F^{ij}h_{ij}\psi_\alpha\nu^\alpha F^{rs}g_{rs} \\ &+ \tilde{\nu}^2|f'|^2(F^{ij}g_{ij})^2 - 2\psi_\alpha\nu^\alpha\tilde{\nu}f'(F^{ij}g_{ij})^2 + (\psi_\alpha\nu^\alpha)^2(F^{ij}g_{ij})^2 \end{aligned}$$

and

$$(5.38) \quad \begin{aligned} f''\tilde{\nu}^2 h_l^k + \gamma h_l^k \tilde{\nu}^2 |f'|^2 F^{ij}g_{ij} &= \tilde{\nu}^2 h_l^k (f'' + \gamma |f'|^2 n) \\ &+ \tilde{\nu}^2 h_l^k \gamma |f'|^2 (F^{ij}g_{ij} - n), \end{aligned}$$

so that we infer

$$(5.39) \quad F^{-2}e^{\gamma t}F^{ij}g_{ij}f''\tilde{\nu}^2 h_l^k + \gamma h_l^k e^{\gamma t} = F^{-2}O_0.$$

Using the fact that

$$(5.40) \quad \dot{g}_{ij} = -2F^{-1}h_{ij} = F^{-2}O_0$$

(5.23) is proved.

Differentiating (5.23) covariantly with respect to a spatial variable we deduce

$$(5.41) \quad \begin{aligned} \frac{D}{dt}(D\tilde{A}) - F^{-2}F^{ij}(D\tilde{A})_{ij} &= F^{-1}O_0 + F^{-3}D^2\tilde{A} * DAO_0 \\ &+ F^{-2}O_0 * D^2\tilde{A} + F^{-4}D\tilde{A} * DA * DA * O_0 + F^{-3}O_0 * D\tilde{A} * DA \\ &+ F^{-2}D\tilde{A} * O_0 + F^{-3}D\tilde{A} * DA * DO_0 + F^{-2}D\tilde{A} * DO_0 + F^{-1}DO_0. \end{aligned}$$

And using induction we conclude for $m \in \mathbb{N}^*$

$$(5.42) \quad \begin{aligned} \frac{D}{dt}(D^{m+1}\tilde{A}) - F^{-2}F^{ij}(D^{m+1}\tilde{A})_{ij} &= F^{-1}O_m \\ &+ \Theta F^{-3}D^{m+1}\tilde{A} * D^{m+1}A * O_0 + F^{-3}D^{m+2}\tilde{A} * DA * O_0 \\ &+ F^{-2}D^{m+1}\tilde{A} * O_m + F^{-2}D^{m+2}\tilde{A} * O_0 \\ &+ F^{-2}DO_m, \end{aligned}$$

where $\Theta = 1$ if $m = 1$ and $\Theta = 0$ else.

We are now going to prove uniform bounds for $\frac{1}{2}\|D^{m+1}\tilde{A}\|$ for all $m \in \mathbb{N}$. First we observe that

$$\begin{aligned}
(5.43) \quad & \frac{D}{dt} \left(\frac{1}{2} \|D\tilde{A}\|^2 \right) - F^{-2} F^{ij} \frac{1}{2} (\|D\tilde{A}\|^2)_{ij} = -F^{-2} F^{ij} (D\tilde{A})_i (D\tilde{A})_j \\
& + F^{-1} O_0 * D\tilde{A} + F^{-3} D^2 \tilde{A} * DA * O_0 * D\tilde{A} + F^{-2} O_0 * D^2 \tilde{A} * D\tilde{A} \\
& + F^{-4} D\tilde{A} * DA * DA * O_0 * D\tilde{A} + F^{-3} O_0 * D\tilde{A} * DA * D\tilde{A} \\
& + F^{-2} D\tilde{A} * O_0 * D\tilde{A} + F^{-3} D\tilde{A} * DA * DO_0 * D\tilde{A} \\
& + F^{-2} D\tilde{A} * DO_0 * D\tilde{A} + F^{-1} DO_0 * D\tilde{A}.
\end{aligned}$$

Furthermore we have for $m \in \mathbb{N}^*$

$$\begin{aligned}
(5.44) \quad & \frac{D}{dt} \left(\frac{1}{2} \|D^{m+1}\tilde{A}\|^2 \right) - F^{-2} F^{ij} \frac{1}{2} (\|D^{m+1}\tilde{A}\|^2)_{ij} = \\
& - F^{-2} F^{ij} (D^{m+1}\tilde{A})_i (D^{m+1}\tilde{A})_j + F^{-1} O_m * D^{m+1}\tilde{A} \\
& + \Theta F^{-3} D^{m+1}\tilde{A} * D^{m+1}\tilde{A} * O_0 * D^{m+1}\tilde{A} \\
& + F^{-3} D^{m+2}\tilde{A} * DA * O_0 * D^{m+1}\tilde{A} + F^{-2} D^{m+1}\tilde{A} * O_m * D^{m+1}\tilde{A} \\
& + F^{-2} D^{m+2}\tilde{A} * O_0 * D^{m+1}\tilde{A} + F^{-2} DO_m * D^{m+1}\tilde{A}.
\end{aligned}$$

Theorem 5.8. *The quantities $\frac{1}{2}\|D^m\tilde{A}\|^2$ are uniformly bounded during the evolution for all $m \in \mathbb{N}^*$*

Proof. We prove the theorem recursively by estimating

$$(5.45) \quad \varphi = \log \frac{1}{2} \|D^{m+1}\tilde{A}\|^2 + \mu \frac{1}{2} \|D^m\tilde{A}\|^2 + \lambda e^{-\gamma t},$$

where μ is a small positive constant and $\lambda \gg 1$ large.

We shall only treat the case $m = 0$.

Fix $0 < T < \infty$, T very large, and suppose that

$$(5.46) \quad 0 < \sup_{[0, T]} \sup_{M(t)} \varphi = \varphi(t_0, \xi_0)$$

for large $0 < t_0 \leq T$.

Applying the maximum principle we deduce

$$\begin{aligned}
(5.47) \quad & 0 \leq \frac{1}{\|D\tilde{A}\|^2} \left(\frac{D}{dt} \|D\tilde{A}\|^2 - F^{-2} F^{ij} \|D\tilde{A}\|_{ij}^2 \right) + \mu \tilde{A} (\dot{\tilde{A}} - F^{-2} F^{ij} \tilde{A}_{ij}) \\
& + F^{-2} F^{ij} \tilde{A}_i \tilde{A}_j (-\mu + \mu^2 \tilde{A}^2) - \lambda \gamma e^{-\gamma t} \\
& \leq -\frac{1}{2} \frac{1}{\|D\tilde{A}\|^2} F^{-2} F^{ij} (D\tilde{A})_i (D\tilde{A})_j - \frac{\lambda}{2} \gamma e^{-\gamma t} + c F^{-4} \|D\tilde{A}\|^2 \\
& + c F^{-2} \|D\tilde{A}\| + F^{-2} F^{ij} \tilde{A}_i \tilde{A}_j (-\mu + \mu^2 \tilde{A}^2) \\
& < 0,
\end{aligned}$$

here we assumed that $\|D\tilde{A}\|$ is larger than a sufficiently large positive constant that does not depend on t_0, T .

Thus φ is a priori bounded.

The proof for $m \geq 1$ is similar. \square

6. APPENDIX B

Appendix B shows convergence of \tilde{u} and studies the behaviour of derivatives in t .

Lemma 6.1. *\tilde{u} converges in $C^m(S_0)$ for any $m \in \mathbb{N}$, if t tends to infinity, and hence $D^m \tilde{A}$ converges.*

Proof. \tilde{u} satisfies the evolution equation

$$(6.1) \quad \dot{\tilde{u}} = \frac{\tilde{v}e^{\gamma t}}{F}(1 - \gamma u f' F^{ij} g_{ij} + \frac{\gamma u}{\tilde{v}} F^{ij} h_{ij} + \frac{\gamma u}{\tilde{v}} \psi_\alpha \nu^\alpha F^{ij} g_{ij}).$$

Using (5.15) and the already known exponential decays we deduce

$$(6.2) \quad |\dot{\tilde{u}}| \leq ce^{-2\gamma t},$$

hence \tilde{u} converges uniformly. Due to Theorem 5.8 $D^m \tilde{u}$ is uniformly bounded, hence \tilde{u} converges in $C^m(S_0)$.

The convergence of $D^m \tilde{A}$ follows from Theorem 5.8 and the convergence of \tilde{h}_{ij} , which in turn can be deduced from

$$(6.3) \quad h_{ij} \tilde{v} = -u_{ij} + \tilde{h}_{ij}.$$

□

Combining the equations (5.23), (5.41) and (5.42) we immediately conclude

Lemma 6.2. *$\|\frac{D}{dt} D^m \tilde{A}\|$ and $\|\frac{D}{dt} D^m A\|$ decay by the order $e^{-\gamma t}$ for any $m \in \mathbb{N}$.*

Corollary 6.3. *$\frac{D}{dt} D^m A e^{\gamma t}$ converges, if t tends to infinity.*

Proof. Applying the product rule we obtain

$$(6.4) \quad \frac{D}{dt} D^m \tilde{A} = \frac{D}{dt} D^m A e^{\gamma t} + \gamma D^m \tilde{A},$$

hence the result, since the left-hand side converges to zero and $D^m \tilde{A}$ converges. □

Corollary 6.4. *We have*

$$(6.5) \quad \|D^m F^{-1}\| \leq c_m F^{-1} \quad \forall m \in \mathbb{N}.$$

Proof. Use (5.17). □

In the next Lemmas we prove some auxiliary estimates.

Lemma 6.5. *The following estimates are valid*

$$(6.6) \quad \|D\dot{u}\| \leq ce^{-\gamma t},$$

$$(6.7) \quad \left\| \frac{d}{dt} F^{-1} \right\| \leq c F^{-1},$$

and

$$(6.8) \quad |\dot{\tilde{v}}| \leq ce^{-2\gamma t}.$$

Proof. ”(6.6)” The estimate follows immediately from

$$(6.9) \quad \dot{u} = \frac{\tilde{v}}{F},$$

in view of Corollary 6.4.

”(6.7)” Differentiating with respect to t we obtain

$$(6.10) \quad \begin{aligned} \frac{d}{dt}F^{-1} = & -F^{-2}(F^{ij}\dot{h}_{ij} - \dot{v}f'F^{ij}g_{ij} - \tilde{v}f''\dot{u}F^{ij}g_{ij} + \frac{d}{dt}(\psi_\alpha\nu^\alpha)F^{ij}g_{ij}) \\ & + F^{ij}(-\tilde{v}f' + \psi_\alpha\nu^\alpha)\dot{g}_{ij} \end{aligned}$$

and the result follows from (6.8) and the known estimates for $|\dot{u}|$ and F .

”(6.8)” We differentiate the relation $\eta_\alpha\nu^\alpha$ to get

$$(6.11) \quad \begin{aligned} \dot{v} &= \eta_{\alpha\beta}\nu^\alpha\dot{x}^\beta + \eta_\alpha\dot{\nu}^\alpha \\ &= -\eta_{\alpha\beta}\nu^\alpha\nu^\beta F^{-1} + (F^{-1})_k u^k, \end{aligned}$$

cf. (4.2), yielding the estimate for $|\dot{v}|$, in view of Corollary 6.4. \square

Lemma 6.6.

$$(6.12) \quad \|F^{ij,kl}(\check{h}_{rs})g_{ij}\| \leq ce^{-3\gamma t},$$

$$(6.13) \quad \|F^{ij,kl}(|u|\check{h}_{rs})g_{ij}\| \leq ce^{-2\gamma t},$$

$$(6.14) \quad \left\| \frac{D}{dt}(u\check{h}_{kl}) \right\| \leq ce^{-2\gamma t},$$

$$(6.15) \quad \left\| \frac{D}{dt}F^{ij,kl}(\check{h}_{rs})g_{ij} \right\| \leq ce^{-3\gamma t},$$

$$(6.16) \quad \left\| \frac{D}{dt}F^{ij,kl}(\check{h}_{rs}) \right\| \leq ce^{-\gamma t}$$

$$(6.17) \quad \left\| \frac{D}{dt}F^{ij} \right\| \leq ce^{-2\gamma t},$$

$$(6.18) \quad \left\| \frac{D}{dt}D\check{h}_{kl} \right\| + \left\| \frac{D}{dt}DF \right\| \leq ce^{\gamma t}.$$

Proof. ”(6.12)” Use Theorem 4.8, (4.67) and (4.68).

”(6.13)” Obvious.

”(6.14)” We have

$$(6.19) \quad \begin{aligned} \frac{D}{dt}(u\check{h}_{kl}) &= \dot{u}\check{h}_{kl} + u\dot{\check{h}}_{kl} \\ &= \frac{\tilde{v}}{F}\check{h}_{kl} + u\dot{\check{h}}_{kl} - u\dot{v}f'g_{kl} - u\tilde{v}f''\frac{\tilde{v}}{F}g_{kl} - u\tilde{v}f'\dot{g}_{kl} \\ &\quad + u\frac{D}{dt}(\psi_\alpha\nu^\alpha g_{kl}), \end{aligned}$$

hence in view of (6.8)

$$(6.20) \quad \left\| \frac{D}{dt}(u\check{h}_{kl}) \right\| \leq ce^{-2\gamma t} + n\left| \frac{\tilde{v}^2}{F}(-f' - uf'') \right| \leq ce^{-2\gamma t}.$$

Here, concerning the summand

$$(6.21) \quad \frac{\tilde{v}}{F} \check{h}_{kl} - u \tilde{v} f'' \frac{\tilde{v}}{F} g_{kl},$$

we use

$$(6.22) \quad | -f' - u f'' | \leq | -f' + u \tilde{\gamma} | f' |^2 | + c | u |,$$

which follows from (1.8), and then (1.28).

"(6.15), (6.16)" We have

$$(6.23) \quad \frac{D}{dt} F^{ij,kl}(\check{h}_{rs}) = \frac{D}{dt} (|u| F^{ij,kl}(|u| \check{h}_{rs}))$$

which implies the claim together with (6.13) und (6.14).

"(6.17)" Use (6.14) and $F^{ij}(\check{h}_{rs}) = F^{ij}(|u| \check{h}_{rs})$.

"(6.18)" Obvious in view of (6.6) and (6.8). \square

Lemma 6.7. *We have*

$$(6.24) \quad |\ddot{v}| + \|D\dot{v}\| \leq c e^{-2\gamma t}$$

and $\dot{v} e^{2\gamma t}$ and $\ddot{v} e^{2\gamma t}$ converge, if t goes to infinity.

Proof. Differentiating (6.11) covariantly with respect to x we infer the estimate for $\|D\dot{v}\|$. A direct computation and easy check of each of the (many) appearing terms yield the convergence of $\dot{v} e^{2\gamma t}$ and $\ddot{v} e^{2\gamma t}$, especially the lemma is proved. \square

Finally let us estimate \check{h}_i^j and \check{h}_i^j .

Lemma 6.8. \check{h}_i^j and \check{h}_i^j decay like $e^{-\gamma t}$.

Proof. The estimate for \check{h}_i^j follows immediately by differentiating equation (4.6) covariantly with respect to t and by applying the above lemmata as well as Theorem 5.8.

Now we estimate \check{h}_i^j . We have

$$(6.25) \quad \check{h}_l^k = e^{\gamma t} \check{h}_l^k + 2\gamma e^{\gamma t} \dot{h}_l^k + \gamma^2 e^{\gamma t} h_l^k.$$

Now we insert (4.6) and the equation which results from (4.6) after covariant differentiation with respect to t into (6.25).

Then many of the appearing terms decay like $e^{-\gamma t}$ obviously. To see the decay of the remaining terms, namely

$$(6.26) \quad \begin{aligned} & \frac{D}{dt} (F^{-3} F^k F_l) e^{\gamma t} + \frac{D}{dt} (F^{-2} g^{pk} F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;l}) e^{\gamma t} \\ & - \frac{D}{dt} (F^{-2} F^{ij} g_{ij} u_l u^k \tilde{v} f''') e^{\gamma t} + \frac{D}{dt} (F^{-2} f'' \tilde{v}^2 h_l^k F^{ij} g_{ij}) e^{\gamma t} \\ & - 4\gamma F^{-3} F^k F_l e^{\gamma t} + 2\gamma F^{-2} g^{pk} F^{ij,rs} \check{h}_{ij;p} \check{h}_{rs;l} e^{\gamma t} \\ & - 2\gamma F^{-2} u_l u^k \tilde{v} f''' F^{ij} g_{ij} e^{\gamma t} \\ & + 2\gamma e^{\gamma t} F^{-2} f'' \tilde{v}^2 h_l^k F^{ij} g_{ij} + \gamma^2 e^{\gamma t} h_l^k \\ & = S_1 + \dots + S_9, \end{aligned}$$

we use the technique developed in (5.23) et seq., confer also the proof of Theorem 4.8, to rearrange terms. In this way we see the claimed decay of $S_5 + S_7$ and $\frac{1}{2} S_8 + S_9$. The summand $S_1 + S_3$ can be handled similar. The summand S_2 decays

as it should due to Lemma 6.6. S_6 is obvious. To estimate $S_4 + \frac{1}{2}S_8$ we differentiate in S_4 by product rule and use (4.6) to substitute \dot{h}_l^k . Then a little bit rearranging terms leads to the desired estimate. \square

From Corollary 6.3, Lemma 6.8 and (6.25) we infer

Corollary 6.9. *The tensor $\check{h}_i^j e^{\gamma t}$ converges, if t tends to infinity.*

The claims in Theorem 1.3 are now almost all proved with the exception of two. In order to prove the remaining claims we need:

Lemma 6.10. *The function $\varphi = e^{\tilde{\gamma}f} u^{-1}$ converges to $-\tilde{\gamma}\sqrt{m}$ in $C^\infty(S_0)$, if t tends to infinity.*

Proof. φ converges to $-\tilde{\gamma}\sqrt{m}$ in view of (1.7). Hence, we only have to show that

$$(6.27) \quad \|D^m \varphi\| \leq c_m \quad \forall m \in \mathbb{N}^*,$$

which will be achieved by induction.

We have

$$(6.28) \quad \varphi_i = \tilde{\gamma} e^{\tilde{\gamma}f} f' u_i u^{-1} - e^{\tilde{\gamma}f} u^{-2} u_i = \varphi(\tilde{\gamma} f' u - 1) u^{-1} u_i.$$

Now, we observe that

$$(6.29) \quad u^{-1} u_i = \tilde{u}^{-1} \tilde{u}_i$$

and $f' u$ have uniformly bounded C^m -norms in view of Lemma 6.1 and Lemma 5.3.

The proof of the lemma is then completed by a simple induction argument. \square

When we formulated Theorem 1.3 (ii) and (iii) we did not use the current notation where we distinguish quantities related to $\check{g}_{\alpha\beta}$ in contrast to those related to $\bar{g}_{\alpha\beta}$ by the superscript $\check{}$.

In the following two lemmas we reformulate Theorem 1.3 (iii) and (iv) using the current notation.

Lemma 6.11. *Let (\check{g}_{ij}) be the induced metric of the leaves $M(t)$ of the ICF, then the rescaled metric*

$$(6.30) \quad e^{\frac{2}{n}t} \check{g}_{ij}$$

converges in $C^\infty(S_0)$ to

$$(6.31) \quad (\tilde{\gamma}^2 m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}} \bar{\sigma}_{ij},$$

where we are slightly ambiguous by using the same symbol to denote $\tilde{u}(t, \cdot)$ and $\lim \tilde{u}(t, \cdot)$.

Proof. There holds

$$(6.32) \quad \check{g}_{ij} = e^{2f} e^{2\psi} (-u_i u_j + \sigma_{ij}(u, x)).$$

Thus, it suffices to prove that

$$(6.33) \quad e^{2f} e^{\frac{2}{n}t} \rightarrow (\tilde{\gamma}^2 m)^{\frac{1}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}}$$

in $C^\infty(S_0)$. But this is evident in view of the preceding lemma, since

$$(6.34) \quad e^{2f} e^{\frac{2}{n}t} = (-e^{\tilde{\gamma}f} u^{-1})^{\frac{2}{\tilde{\gamma}}} (-\tilde{u})^{\frac{2}{\tilde{\gamma}}}.$$

\square

Lemma 6.12. *The leaves $M(t)$ of the ICF get more umbilical, if t tends to infinity, namely*

$$(6.35) \quad \check{F}^{-1}|\check{h}_i^j - \frac{1}{n}\check{H}\check{\delta}_i^j| \leq ce^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely

$$(6.36) \quad |\check{h}_i^j - \frac{1}{n}\check{H}\check{\delta}_i^j| \leq ce^{-\frac{1}{2n}(n+\omega-4)t}.$$

Proof. Denote by $\check{h}_{ij}, \check{\nu}$, etc., the geometric quantities of the hypersurfaces $M(t)$ with respect to the original metric $(\check{g}_{\alpha\beta})$ in N , then

$$(6.37) \quad e^{\check{\psi}}\check{h}_i^j = h_i^j + \check{\psi}_\alpha \nu^\alpha \delta_i^j, \quad \check{F} = e^{-\check{\psi}}F$$

and hence,

$$(6.38) \quad \check{F}^{-1}|\check{h}_i^j - \frac{1}{n}\check{H}\check{\delta}_i^j| = F^{-1}|h_i^j - \frac{1}{n}H\delta_i^j| \leq ce^{-2\gamma t}.$$

In case $n + \omega - 4 > 0$, we even get a better estimate, namely

$$(6.39) \quad |\check{h}_i^j - \frac{1}{n}\check{H}\check{\delta}_i^j| = e^{-\psi}e^{-f}e^{-\frac{1}{n}t}|h_i^j - \frac{1}{n}H\delta_i^j|e^{\gamma t}e^{(\frac{1}{n}-\gamma)t} \leq ce^{-\frac{1}{2n}(n+\omega-4)t}$$

in view of (6.33). \square

7. APPENDIX C

Appendix C shows in which way the rescaled ICF defines a transition from big crunch to big bang and proves part (iv) of Theorem 1.3.

We shall define a new spacetime \hat{N} by reflection and time reversal such that the ICF in the old spacetime transforms to an ICF in the new one.

By switching the light cone we obtain a new spacetime \hat{N} . If we extend F , which is defined in the positive cone $\Gamma_+ \subset \mathbb{R}^n$, to $\Gamma_+ \cup (-\Gamma_+)$ by

$$(7.1) \quad F(\kappa_i) = -F(-\kappa_i)$$

for $(\kappa_i) \in -\Gamma_+$ the flow equation in N is independent of the time orientation, and we can write it as

$$(7.2) \quad \dot{x} = -\check{F}^{-1}\check{\nu} = -(-\check{F})^{-1}(-\check{\nu}) =: -\hat{F}^{-1}\hat{\nu},$$

where the normal vector $\hat{\nu} = -\check{\nu}$ is past directed in \hat{N} and the curvature $\hat{F} = -\check{F}$ negative.

Introducing a new time function $\hat{x}^0 = -x^0$ and formally new coordinates (\hat{x}^α) by setting

$$(7.3) \quad \hat{x}^0 = -x^0, \quad \hat{x}^i = x^i,$$

we define a spacetime \hat{N} having the same metric as N —only expressed in the new coordinate system—such that the flow equation has the form

$$(7.4) \quad \dot{\hat{x}} = -\hat{F}^{-1}\hat{\nu},$$

where $M(t) = \text{graph } \hat{u}(t)$, $\hat{u} = -u$, and

$$(7.5) \quad (\hat{\nu}^\alpha) = -\check{\nu}e^{-\check{\psi}}(1, \hat{u}^i)$$

in the new coordinates, since

$$(7.6) \quad \hat{\nu}^0 = -\check{\nu}^0 \frac{\partial \hat{x}^0}{\partial x^0} = \check{\nu}^0$$

and

$$(7.7) \quad \hat{\nu}^i = -\check{\nu}^i.$$

The singularity in $\hat{x}^0 = 0$ is now a past singularity, and can be referred to as a big bang singularity.

The union $N \cup \hat{N}$ is a smooth manifold, topologically a product $(-a, a) \times S_0$ —we are well aware that formally the singularity $\{0\} \times S_0$ is not part of the union; equipped with the respective metrics and time orientations it is a spacetime which has a (metric) singularity in $x^0 = 0$. The time function

$$(7.8) \quad \hat{x}^0 = \begin{cases} x^0, & \text{in } N \\ -x^0, & \text{in } \hat{N} \end{cases}$$

is smooth across the singularity and future directed.

$N \cup \hat{N}$ can be regarded as a cyclic universe with a contracting part $N = \{\hat{x}^0 < 0\}$ and an expanding part $\hat{N} = \{\hat{x}^0 > 0\}$ which are joined at the singularity $\{\hat{x}^0 = 0\}$.

We shall show that the inverse F -curvature flow, properly rescaled, defines a natural C^3 -diffeomorphism across the singularity and with respect to this diffeomorphism we speak of a transition from big crunch to big bang.

The inverse F -curvature flows in N and \hat{N} can be uniformly expressed in the form

$$(7.9) \quad \dot{\hat{x}} = -\hat{F}^{-1}\hat{\nu},$$

where (7.9) represents the original flow in N , if $\hat{x}^0 < 0$, and the flow in (7.4), if $\hat{x}^0 > 0$.

Let us now introduce a new flow parameter

$$(7.10) \quad s = \begin{cases} -\gamma^{-1}e^{-\gamma t}, & \text{for the flow in } N \\ \gamma^{-1}e^{-\gamma t}, & \text{for the flow in } \hat{N} \end{cases}$$

and define the flow $y = y(s)$ by $y(s) = \hat{x}(t)$. $y = y(s, \xi)$ is then defined in $[-\gamma^{-1}, \gamma^{-1}] \times S_0$, smooth in $\{s \neq 0\}$, and satisfies the evolution equation

$$(7.11) \quad y' := \frac{d}{ds}y = \begin{cases} -\hat{F}^{-1}\hat{\nu}e^{\gamma t}, & s < 0 \\ \hat{F}^{-1}\hat{\nu}e^{\gamma t}, & s > 0. \end{cases}$$

Theorem 7.1. *The flow $y = y(s, \xi)$ is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$ and defines a natural diffeomorphism across the singularity. The flow parameter s can be used as a new time function.*

The flow y is certainly continuous across the singularity, and also future directed, i.e., it runs into the singularity, if $s < 0$, and moves away from it, if $s > 0$. The continuous differentiability of $y = y(s, \xi)$ with respect to s and ξ up to order three will be proved in a series of lemmata.

As in the previous sections we again view the hypersurfaces as embeddings with respect to the ambient metric

$$(7.12) \quad ds^2 = -(dx^0)^2 + \sigma_{ij}(x^0, x)dx^i dx^j.$$

The flow equation for $s < 0$ can therefore be written as

$$(7.13) \quad y' = -F^{-1}\nu e^{\gamma t}.$$

To prove that y is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$ we must show that $y', y_i, y'_i, y'', y''_i, y_{ij}, y'''_i, y'_{ij}, y''_i, y_{ijk}$ (and derivatives obtained by commuting the order of differentiation) are continuous in $\{0\} \times S_0$, which means that we must show that for each of these derivatives the limits $\lim_{s \uparrow 0}, \lim_{s \downarrow 0}$ (uniformly with respect to the space variables ξ^i) exist and are the same.

Due to

$$(7.14) \quad y^0(s) = x^0(t), \quad y^i(s) = x^i(t) \quad \forall s < 0,$$

and

$$(7.15) \quad y^0(s) = -x^0(t), \quad y^i(s) = x^i(t) \quad \forall s > 0$$

we will consider the 0-component and the i -component of each of the above derivatives separately and calculate their limits as $s \uparrow 0$ and $s \downarrow 0$. Since in each case the limit $s \uparrow 0$ has the same value or the same value up to a sign as the limit $s \downarrow 0$ (provided one of them exists) it is sufficient to have a look at the limit $s \uparrow 0$ and prove its existence or that it is in addition zero respectively.

Lemma 7.2. y is of class C^1 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$.

Proof. y' is continuous across the singularity if

$$(7.16) \quad \lim_{s \uparrow 0} \frac{d}{ds} y^0, \lim_{s \uparrow 0} y_i^j \quad \text{exist,}$$

and if

$$(7.17) \quad \lim_{s \uparrow 0} \frac{d}{ds} y^i = \lim_{s \uparrow 0} y_i^0 = 0.$$

Only the limit $\lim_{s \uparrow 0} y_i^j$ is not obvious, but one easily checks that x_i^j is a 'Cauchy sequence' as $t \rightarrow \infty$ since its derivative with respect to t can be estimated by $ce^{-\gamma t}$, hence $\lim_{s \uparrow 0} y_i^j$ exists as well.

Remark 7.3. The limit relations for $\langle D^m y, \frac{\partial}{\partial x^0} \rangle$ and $\langle D^m y, \frac{\partial}{\partial x^i} \rangle$, where $D^m y$ stands for covariant derivatives of order m of y with respect to s or ξ^i are identical to those for $\langle D^m y, -\nu \rangle$ and $\langle D^m y, x_i \rangle$ because ν converges to $-\frac{\partial}{\partial x^0}$, if $s \uparrow 0$. We want to point out that we have chosen local coordinates in S_0 which are given by the limit of the embedding vector x so that we also have $x_i \rightarrow \frac{\partial}{\partial x^i}$.

□

Let us examine the second derivatives

Lemma 7.4. y is of class C^2 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$.

Proof. " y'_i ": The normal component of y'_i has to converge and the tangential components have to converge to zero as $s \uparrow 0$. For $s < 0$ we have

$$(7.18) \quad y' = -F^{-1} e^{\gamma t} \nu$$

and

$$(7.19) \quad y'_i = F^{-2} F_i e^{\gamma t} \nu - F^{-1} e^{\gamma t} \nu_i.$$

The normal component is therefore equal to

$$(7.20) \quad -F^{-2} e^{\gamma t} (F^{kl} h_{kl;i} - F^{kl} g_{kl} \tilde{\nu}_i f' - F^{kl} g_{kl} \tilde{\nu} f'' u_i + F^{kl} g_{kl} \psi_{\alpha\beta} x_i^\beta \nu^\alpha + F^{kl} g_{kl} \psi_\alpha x_r^\alpha h_i^r)$$

which converges to

$$(7.21) \quad \lim n(Fu)^{-2} f'' u^2 \tilde{u}_i.$$

The tangential components are equal to

$$(7.22) \quad -F^{-1} e^{\gamma t} h_{ik}$$

which converge to zero.

" y_{ij} ": The Gaußformula yields

$$(7.23) \quad y_{ij} = h_{ij} \nu$$

which converges to zero as it should.

" y'' ": Here, the normal component has to converge to zero, while the tangential ones have to converge.

We get for $s < 0$

$$(7.24) \quad \begin{aligned} y'' &= -\frac{D}{dt}(F^{-1}\nu)e^{2\gamma t} - F^{-1}\nu\gamma e^{2\gamma t} \\ &= -F^{-1}\dot{\nu}e^{2\gamma t} + F^{-2}\nu\dot{F}e^{2\gamma t} - F^{-1}\nu\gamma e^{2\gamma t}. \end{aligned}$$

The normal component is equal to

$$(7.25) \quad \begin{aligned} &-F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f'F^{ij}g_{ij} - \tilde{v}f''\dot{u}F^{ij}g_{ij} \\ &\quad + \psi_{\alpha\beta}\nu^\alpha\dot{x}^\beta F^{ij}g_{ij} + \psi_\alpha\nu^\alpha F^{ij}g_{ij} - \gamma F) \\ &-F^{-2}e^{2\gamma t}(-\tilde{v}f' + \psi_\alpha\nu^\alpha)F^{ij}\dot{g}_{ij}. \end{aligned}$$

$F^{-2}e^{2\gamma t}$ converges, all terms converge to zero with the possible exception of

$$(7.26) \quad -F^{ij}g_{ij}\tilde{v}f''\dot{u} - \gamma F = -F^{-1}(F^{ij}g_{ij}\tilde{v}^2f'' + \gamma F^2),$$

which however converges to zero, too.

The tangential components are equal to

$$(7.27) \quad \begin{aligned} F^{-1}D_k(F^{-1})e^{2\gamma t} &= -F^{-3}e^{2\gamma t}(F^{ij}h_{ij;k} - \tilde{v}_k f' F^{ij}g_{ij} \\ &\quad - \tilde{v}f'' u_k F^{ij}g_{ij} + \psi_{\alpha\beta}\nu^\alpha x_k^\beta F^{ij}g_{ij} + \psi_\alpha x_k^\alpha h_k^r F^{ij}g_{ij}), \end{aligned}$$

which converge to

$$(7.28) \quad \lim -\tilde{\gamma}n(Fu)^{-3}(f'u)^2\tilde{u}\tilde{u}_k.$$

□

Lemma 7.5. *y is of class C^3 in $(-\gamma^{-1}, \gamma^{-1}) \times S_0$.*

Proof. " y_{ijk} ": Now, the normal component has to converge to zero, while the tangential ones should converge. Again we look at $s < 0$ and get

$$(7.29) \quad y_{ij} = h_{ij}\nu,$$

$$(7.30) \quad y_{ijk} = h_{ijk}\nu + h_{ij}\nu_k.$$

Hence, y_{ijk} converges to zero.

" y'_{ij} ": The normal component has to converge, while the tangential ones should converge to zero.

Using the Ricci identities and Lemma 3.5 (iii) it can be easily checked that, instead of y'_{ij} , we may look at $\frac{D}{ds}(y_{ij})$.

From (7.29) we deduce

$$(7.31) \quad \frac{D}{ds}y_{ij} = \dot{h}_{ij}\nu e^{\gamma t} + h_{ij}\dot{\nu}e^{\gamma t},$$

and conclude further that the normal component converges in view of Corollary 6.3 and the tangential ones converge to zero, since $\dot{\nu}$ vanishes in the limit.

" y_i'' ": The normal component has to converge to zero and the tangential ones have to converge.

From (7.24) we infer

$$(7.32) \quad \begin{aligned} y'' &= -F^{-3}e^{2\gamma t}F^{ij}(h_{ij;^k} - \tilde{v}^k f' g_{ij} - \tilde{v} f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij})x_k \\ &\quad + F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij})\nu \\ &\quad + F^{-3}e^{2\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\ &\quad + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\ &\quad + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha \nu^\alpha (F^{ij}g_{ij})^2])\nu \\ &\quad + 2F^{-3}e^{2\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij})\nu \end{aligned}$$

and thus

$$(7.33) \quad \begin{aligned} y_l'' &= -(F^{-3}e^{2\gamma t}F^{ij}(h_{ij;^k} - \tilde{v}^k f' g_{ij} - \tilde{v} f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij})g_{ij})_l x_k \\ &\quad - F^{-3}e^{2\gamma t}F^{ij}(h_{ij;^k} - \tilde{v}^k f' g_{ij} - \tilde{v} f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij})h_{kl}\nu \\ &\quad + (F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij}))_l \nu \\ &\quad + F^{-2}e^{2\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij})\nu_l \\ &\quad + (F^{-3}e^{2\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\ &\quad + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} \\ &\quad - 2\tilde{v}f'\psi_\alpha \nu^\alpha (F^{ij}g_{ij})^2]))_l \nu + F^{-3}e^{2\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] \\ &\quad - \gamma[(F^{ij}h_{ij})^2 + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\ &\quad + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha \nu^\alpha (F^{ij}g_{ij})^2])\nu_l \\ &\quad + (2F^{-3}e^{2\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij}))_l \nu \\ &\quad + 2F^{-3}e^{2\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij})\nu_l. \end{aligned}$$

Therefore, the normal component converges to zero, while the tangential ones converge.

" y''' ": Differentiating the equation (7.32) we get

$$\begin{aligned}
y''' &= 3F^{-4}e^{3\gamma t}\dot{F}F^{ij}(h_{ij}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij})x_k \\
&\quad - 2\gamma F^{-3}e^{3\gamma t}F^{ij}(h_{ij}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij})x_k \\
&\quad - F^{-3}e^{3\gamma t}\frac{D}{dt}(F^{ij}(h_{ij}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij}))x_k \\
&\quad - F^{-3}e^{3\gamma t}F^{ij}(h_{ij}^k - \tilde{v}^k f' g_{ij} - \tilde{v}f'' u^k g_{ij} + (\psi_\alpha \nu^\alpha)^k g_{ij})\dot{x}_k \\
&\quad - 2F^{-3}e^{3\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij})\nu \\
&\quad + 2\gamma F^{-2}e^{3\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij})\nu \\
&\quad + F^{-2}e^{3\gamma t}\frac{D}{dt}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij})\nu \\
&\quad + F^{-2}e^{3\gamma t}(F^{ij}\dot{h}_{ij} - \dot{\tilde{v}}f' F^{ij}g_{ij} + \frac{D}{dt}(\psi_\alpha \nu^\alpha)F^{ij}g_{ij})\dot{\nu} \\
&\quad - 3F^{-4}e^{3\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\
&\quad + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\
&\quad + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha \nu^\alpha(F^{ij}g_{ij})^2])\nu \\
(7.34) \quad &\quad + 2\gamma F^{-3}e^{3\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\
&\quad + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\
&\quad + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha \nu^\alpha(F^{ij}g_{ij})^2])\nu \\
&\quad + F^{-3}e^{3\gamma t}\frac{D}{dt}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\
&\quad + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\
&\quad + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha \nu^\alpha(F^{ij}g_{ij})^2])\nu \\
&\quad + F^{-3}e^{3\gamma t}(-\tilde{v}^2 F^{ij}g_{ij}[f'' + \gamma F^{ij}g_{ij}|f'|^2] - \gamma[(F^{ij}h_{ij})^2 \\
&\quad + (\psi_\alpha \nu^\alpha)^2(F^{ij}g_{ij})^2 - 2\tilde{v}f' F^{ij}h_{ij}F^{ij}g_{ij} \\
&\quad + 2\psi_\alpha \nu^\alpha F^{ij}h_{ij}F^{ij}g_{ij} - 2\tilde{v}f'\psi_\alpha \nu^\alpha(F^{ij}g_{ij})^2])\dot{\nu} \\
&\quad - 6F^{-4}e^{3\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij})\nu \\
&\quad + 4\gamma F^{-3}e^{3\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij})\nu \\
&\quad + 2F^{-3}e^{3\gamma t}\frac{D}{dt}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij})\nu \\
&\quad + 2F^{-3}e^{3\gamma t}(\tilde{v}f' F^{ij}h_{ij} - \psi_\alpha \nu^\alpha F^{ij}h_{ij})\dot{\nu}.
\end{aligned}$$

We remark that

$$(7.35) \quad \dot{x}_k = F^{-2}F_k\nu - F^{-1}\nu_k$$

and

$$(7.36) \quad \dot{u}_k = F^{-1}\tilde{v}_k - F^{-2}\tilde{v}F_k$$

and that in the following especially the results of Lemma 6.5, Lemma 6.8 and Corollary 6.9 will be used.

Let us consider the normal component of y''' first, which has to converge. We will present here only how to handle the following term, the other terms are easier.

$$(7.37) \quad \begin{aligned} \frac{D}{dt}[f'' + \gamma F^{ij} g_{ij} |f'|^2] &= \frac{D}{dt}[f'' + \tilde{\gamma} |f'|^2] + \frac{D}{dt}(F^{ij} g_{ij} - n) \gamma |f'|^2 \\ &\quad + 2(F^{ij} g_{ij} - n) \gamma f' f'' \dot{u} \\ &\equiv I_1 + I_2 + I_3. \end{aligned}$$

I_1 converges due to assumption (1.11), and the convergence of I_3 is obvious. For I_2 we use

$$(7.38) \quad \frac{D}{dt} F^{ij} g_{ij} = F^{ij,rs} (|u| \check{h}_{ij}) g_{ij} \frac{D}{dt} (|u| \check{h}_{rs}) + F^{ij} \dot{g}_{ij}$$

together with (4.2), (6.13) and (6.14).

Now we consider the tangential component of y''' , i.e. we prove

$$(7.39) \quad \langle y''', x_i \rangle \rightarrow 0.$$

The crucial terms are

$$(7.40) \quad \begin{aligned} &3F^{-4} e^{3\gamma t} (F^{ij} g_{ij})^2 \tilde{v}^2 (f'')^2 \dot{u} u^k + 2\gamma F^{-3} e^{3\gamma t} \tilde{v} f'' u^k F^{ij} g_{ij} \\ &\quad + F^{-3} e^{3\gamma t} \tilde{v} f''' u^k \dot{u} F^{ij} g_{ij} + F^{-5} e^{3\gamma t} (F^{ij} g_{ij})^2 \tilde{v}^3 |f''|^2 u^k \end{aligned}$$

and can be rearranged to yield

$$(7.41) \quad F^{-5} e^{3\gamma t} n^2 \tilde{v} u^k (4f'' (f'' + \tilde{\gamma} |f'|^2) - f' (f'' + \tilde{\gamma} |f'|^2)').$$

Hence the tangential components tend to zero.

The remaining mixed derivatives of y which are obtained by commuting the order of differentiation in the derivatives we already treated are also continuous across the singularity in view of the Ricci identities and Lemma 3.5 (iii). \square

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