Numerical analysis of partial differential equations on and of evolving surfaces

Habilitation thesis

by

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Tübingen
April 2018

Contents

Preface

The present cumulative habilitation thesis, *Habilitationsschrift*, collects the works of the author, over the past four years, in collaboration with B. Li (The Hong Kong Polytechnic University), Ch. Lubich (University of Tübingen), C.A. Power Guerra (University of Tübingen) on the analysis of numerical methods for parabolic problems on *evolving surfaces*. The results collected in this thesis were obtained in the papers:

[KP18a], included as Appendix A,

[Kov17], included as Appendix B,

[Kov18], included as Appendix C,

[KP18b], included as Appendix D,

[KP16], included as Appendix E,

[KLLP17], included as Appendix F,

[KL18], included as Appendix G.

The results and the scientific work of the above papers were obtained by equal contributions of their respective authors. The numerical experiments for [Kov17, Kov18, KL18] were carried out by the author (in Matlab), for [KP18b, KLLP17] by C.A. Power Guerra (in DUNE), while the ones for [KP18a, KP16] are their joint works.

Acknowledgements

I would like to thank my coauthors for our fruitful collaborations and our interesting mathematical discussions which lead to these (and further) works.

I would like to thank Christian Lubich for his enormous support (which also evolved) along my time in Tübingen. I am grateful to my colleagues at the Numerical Analysis Group in Tübingen for the inspiring atmosphere in our group.

I cannot be thankful enough to my family, especially to my wife for her tremendous support and encouragement.

The works presented in this thesis were supported by the DAAD and Deutsche Forschungsgemeinschaft, SFB 1173.

Tübingen, April 2018

Balázs Kovács

Introduction

Parabolic partial differential equations on evolving surfaces and their coupling to surface evolution equations model a wide range of real-life phenomena in physics and biology. The combined geometric and time-dependent nature of such problems attracted interest to their numerical analysis along the past years.

Evolving surface problems with a *given velocity* pose multiple hurdles within their numerical analysis: since the surface on which they hold is curved, the analysis of spatial semi-discretisation requires the study of errors related to geometric approximations; on the other hand the strong time-dependent nature of these problems renders this field also interesting from the aspect of time discretisations.

The starting point of the numerical analysis is the fundamental paper of Dziuk [Dzi88] analysing surface finite elements for elliptic partial differential equations on surfaces. This was later extended to various parabolic problems on stationary surfaces [DE07b]. The theory of partial differential equations on a closed evolving surface with a given velocity and the study of the evolving surface finite element method was started by Dziuk and Elliott [DE07a]. A great number of papers dealing with various evolving surface problems and their evolving surface finite element discretisation have been surveyed in [DDE05, DE13a]. Further references can be found later on in the text.

Further possible numerical approaches are level set methods, see [Set99] and [OF03], or the unstructured finite element methods, see [BBLO18], and the references therein.

The analysis of high-order time discretisations of evolving surface problems with a given velocity was started by Dziuk, Lubich and Mansour [DLM12], which deals with time discretisation using algebraically stable implicit Runge–Kutta methods, and by Lubich, Mansour and Venkataraman [LMV13], dealing with time discretisations by backward differentiation formulae. In both papers – and also for the results presented in this thesis – energy estimates for the matrix–vector formulation play a crucial role in the stability analysis.

In Chapter 1 some recent results are collected in the case of evolving surface problems with a given velocity. Various optimal-order error bounds for semi- and full discretisations, using evolving surface finite elements and implicit Runge–Kutta methods or backward differentiation formulae, of linear parabolic problems are presented. In Section 1.8 we give optimal-order error bounds for non-linear problems.

The sections of the chapter collect results from the papers [KP18a, Kov17, KP18b, Kov18] and [KP16], see Appendices A–E.

The development of numerical algorithms for *surface evolution* also goes back to a paper of Dziuk [Dzi90], which deals with a numerical algorithm for the mean curvature flow. In contrast to problems on an evolving surface with a given velocity, the numerical analysis of surface evolution equations or problems coupling surface evolution to diffusion on the surface, i.e. where the surface velocity depends on the solution of the problem on the surface, is far less explored.

Chapter 2 collects – to the best of our knowledge – the first error estimates of semi- and full discretisation of such solution-driven problems on *surfaces* of dimension two.

For evolving *curves* there are recent papers [PS17a] and [PS17b] on the finite element analysis of curve evolution (curve shortening flow and elastic flow) coupled to diffusion on the curve, while [BDS17] studies a fully discrete scheme. Surface evolutions under Navier–Stokes equations and Willmore flow have recently been considered in [BGN15a, BGN15b, BGN16].

The convergence of the evolving surface finite element method for mean curvature flow of closed surfaces is not understood, and has, as yet, remained an open problem since Dziuk's formulation of such an algorithm in his paper [Dzi90].

The sections of the chapter collect results from the papers [KLLP17] and [KL18], see Appendices F and G.

1. Numerical methods for parabolic problems on evolving surfaces

In this chapter, convergence results on full and semi-discretisations of parabolic problems on evolving surfaces are collected.

Parabolic partial differential equations (PDEs) on evolving surfaces arise in a wide variety of applications in physics and biology. We refer to the papers [DE13a, DE07a, DE07b] collecting many of these models, and also to the references therein.

The focus here is mostly on optimal-order error estimates for full discretisations using the evolving surface finite element method combined with high-order time integrators. In most cases, the error bounds are shown by combining stability bounds, obtained by energy techniques, cf. [LO95, AL15], and consistency estimates, obtained using the geometric error estimates and error bounds for a suitable Ritz map.

This chapter is organised as follows. The first four sections are of preliminary nature: Section 1.1 collects basic notions for linear evolving surface problems, Section 1.2 briefly describes the evolving surface finite element method, while Section 1.3 describes the used time discretisation methods. Finally Section 1.4 gives a brief literature overview on convergence results for time-dependent problems on evolving surfaces. In Section 1.5 we present error bounds for the arbitrary Lagrangian Eulerian evolving surface finite elements, and an algorithm for computing such maps. Section 1.6 gives semi-discrete error estimates in maximum norm. Section 1.7 collects error estimates using high-order basis functions. In Section 1.8 error bounds are presented for non-linear problems.

1.1. Parabolic problems on evolving surfaces: preliminaries and notation

Research of the theory and, especially, of the numerical analysis of parabolic partial differential equations on evolving surfaces was started by the paper of Dziuk and Elliott [DE07a], which in turn finds its roots in the fundamental paper of Dziuk [Dzi88]. We collect here the basic definitions and notations. Although most of them became quite standard in the literature, cf. [DE07a, DE13b, DE13a], it is worth to recall them below, allowing a clear and self-contained presentation of our results.

We consider an evolving closed surface $\Gamma(t) \subset \mathbb{R}^{m+1}$ ($m = 2, 3$) for $0 \leqslant t \leqslant T$, given by

$$\Gamma(t) = \{X(p,t) \mid p \in \Gamma^0\},$$

of a sufficiently regular (non-degenerate and at least C^2) function $X : \Gamma^0 \times [0,T] \to \mathbb{R}^{m+1}$, where Γ^0 is a closed smooth initial surface, and $X(\cdot, 0) = \mathrm{Id}$. Sometimes it is convenient to use the surface representation through a sufficiently regular (at least C^2) signed distance function d (see e.g. [DE07a]). The surface is then given by

$$\Gamma(t) = \{x \in \mathbb{R}^{m+1} \mid d(x,t) = 0\}.$$

The surface moves with a given smooth *velocity* $v : \cup_{t\in[0,T]}\Gamma(t) \times \{t\} \to \mathbb{R}^{m+1}$, which satisfies the ordinary differential equation (ODE), for all $p \in \Gamma^0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}X(p,t) = v(X(p,t),t), \tag{1.1}$$

with $X(p,0) = p$. Note that with a known velocity field v, any point $x = X(p,t)$ on $\Gamma(t)$ at time t and for fixed $p \in \Gamma^0$ can be obtained by integrating the ODE (1.1) from 0 to t.

The *material derivative* of a function u is given by

$$\partial^\bullet u(\cdot,t) = \frac{\mathrm{d}}{\mathrm{d}t}u(X(\cdot,t),t). \tag{1.2}$$

We denote the unit outward normal by $\nu = \nu_{\Gamma(t)}$. The *tangential gradient* for a function u is given by $\nabla_{\Gamma(t)}u = \nabla u - (\nabla u \cdot \nu)\nu$. By $\nabla_{\Gamma(t)} \cdot v$ we denote the tangential divergence of the velocity v, while the Laplace–Beltrami operator applied to u is denoted by $\Delta_{\Gamma(t)}u$, and is given by $\nabla_{\Gamma(t)} \cdot \nabla_{\Gamma(t)}u$. An important tool is Green's formula on closed surfaces, for smooth functions $u, \varphi : \Gamma(t) \to \mathbb{R}$,

$$\int_{\Gamma(t)} \nabla_{\Gamma(t)}u \cdot \nabla_{\Gamma(t)}\varphi = -\int_{\Gamma(t)} (\Delta_{\Gamma(t)}u)\varphi.$$

We use Sobolev spaces on surfaces: For a smooth surface Γ we define

$$\begin{aligned} L^2(\Gamma) &= \Big\{\eta : \Gamma \to \mathbb{R} \;\Big|\; \int_\Gamma |\eta|^2 < \infty\Big\}, \\ H^1(\Gamma) &= \Big\{\eta \in L^2(\Gamma) \;\Big|\; \nabla_\Gamma \eta \in L^2(\Gamma)^{m+1}\Big\}, \end{aligned}$$

and analogously for higher order versions $H^k(\Gamma)$ for $k \in \mathbb{N}$. See for instance [DE07a] or [DE13b] for these notions.

The simplest model problem is the heat equation on a closed evolving surface, derived in [DE07a], which reads:

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)}u &= f \qquad \text{on } \Gamma(t), \\ u(\cdot,0) &= u^0 \qquad \text{on } \Gamma(0), \end{aligned} \tag{1.3}$$

where $f(\cdot,t) : \Gamma(t) \to \mathbb{R}$ is a given inhomogeneity for all $0 \leqslant t \leqslant T$.

The variational formulation of this problem reads as: Find $u \in H^1(\Gamma(t))$ with a time-continuous material derivative $\partial^\bullet u \in L^2(\Gamma(t))$ such that, for all test functions $\varphi \in H^1(\Gamma(t))$ with $\partial^\bullet \varphi = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)}u \cdot \nabla_{\Gamma(t)}\varphi = \int_{\Gamma(t)} f\varphi, \tag{1.4}$$

with the initial value $u(\cdot,0) = u^0$.

Existence and uniqueness results for (1.4), with suitable initial values u_0, were obtained by Dziuk and Elliott [DE07a, Theorem 4.4].

1.2. The evolving surface finite element method

A starting point to surface finite elements is the fundamental paper of Dziuk [Dzi88], while the evolving surface finite element method was later developed by Dziuk and Elliott [DE07a]. Here we give a brief introduction to the evolving surface finite element method.

The surface $\Gamma(t)$ is approximated by a family of admissible triangulations denoted by $\mathcal{T}_h(t)$, with h denoting the maximum diameter. The notion of admissible triangulations, cf. [DE07a, Section 5.1], includes quasi-uniformity and shape regularity. The vertices $(x_j(t))_{j=1}^N$ of the discrete surface $\Gamma_h(t)$, given by its elements as

$$\Gamma_h(t) = \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t),$$

are sitting on the exact surface $\Gamma(t)$ for all $0 \leqslant t \leqslant T$.

The continuous, piecewise linear evolving surface finite element basis functions $\phi_j(\cdot,t)$: $\Gamma_h(t) \to \mathbb{R}$ $(j = 1,2,\dots,N)$ satisfy the property

$$\phi_j(x_k,t) = \delta_{jk} \quad \text{for all} \quad j,k = 1,2,\dots,N.$$

For every $t \in [0,T]$ the finite element space $S_h(t)$, spanned by the basis functions ϕ_j, is given by

$$S_h(t) = \operatorname{span}\big\{\phi_1(\cdot,t), \phi_2(\cdot,t), \dots, \phi_N(\cdot,t)\big\}.$$

The discrete tangential gradient of a function $u_h \in S_h(t)$ on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)} u_h = \nabla u_h - (\nabla u_h \cdot \nu_h)\nu_h,$$

understood in a piecewise sense, with $\nu_h = \nu_{\Gamma_h(t)}$ denoting the outward unit normal to $\Gamma_h(t)$.

The velocity of the discrete surface $\Gamma_h(t)$, denoted by V_h, is given by the interpolation of v using the basis functions: $V_h = \sum_{j=1}^N v(x_j(t),t)\phi_j(\cdot,t)$. Then the discrete material derivative is given by

$$\partial_h^\bullet \varphi_h = \partial_t \varphi_h + V_h \cdot \nabla\varphi_h \qquad (\varphi_h \in S_h(t)).$$

The key *transport property* derived in [DE07a, Proposition 5.4], is

$$\partial_h^\bullet \phi_k = 0 \quad \text{for} \quad k = 1,2,\dots,N. \tag{1.5}$$

Therefore, the discrete material derivative of a temporally smooth surface finite element function $u_h(\cdot,t) = \sum_{j=1}^N u_j(t)\phi_j(\cdot,t) \in S_h(t)$ is simply given by

$$\partial_h^\bullet u_h(\cdot,t) = \sum_{j=1}^N \dot u_j(t)\phi_j(\cdot,t) \in S_h(t).$$

Semi-discrete problem and matrix–vector formulation

The semi-discrete problem then reads: Find the finite element function $u_h(\cdot,t) \in S_h(t)$ with a time-continuous discrete material derivative $\partial_h^\bullet u_h(\cdot,t) \in S_h(t)$ such that, for all $\varphi_h(\cdot,t) \in S_h(t)$ with $\partial_h^\bullet \varphi_h(\cdot,t) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} u_h\varphi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)} u_h \cdot \nabla_{\Gamma_h(t)}\varphi_h = \int_{\Gamma_h(t)} f_h\varphi_h. \tag{1.6}$$

The initial value $u_h(\cdot,0)$ and the inhomogeneity f_h are taken as suitable approximations of u^0 and f, respectively.

The above semi-discrete problem translates to a matrix–vector formulation presented below. Apart form the obvious role in numerical computations, the matrix–vector formulation plays a central role in the stability analysis for many problems, see [DLM12, LMV13, KP18a, KP16] and an even more crucial role in [KLLP17, KL18], see the appendices as well.

The time-dependent mass matrix $\mathbf{M}(t)$ and stiffness matrix $\mathbf{A}(t)$ are defined by

$$\begin{aligned} \mathbf{M}(t)|_{kj} &= \int_{\Gamma_h(t)} \phi_j \phi_k, \\ \mathbf{A}(t)|_{kj} &= \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)} \phi_j \cdot \nabla_{\Gamma_h(t)} \phi_k, \end{aligned} \qquad (j,k = 1,2,\dots,N). \tag{1.7}$$

The right-hand side vector is simply given by

$$\mathbf{b}(t)|_k = \int_{\Gamma_h(t)} f_h \phi_k \qquad (j,k = 1,2,\dots,N).$$

We obtain the following ODE system for the vector of nodal values $\mathbf{u}(t) = (u_j(t))_{j=1}^N \in \mathbb{R}^N$, collecting the nodal values of $u_h(\cdot,t) = \sum_{j=1}^N u_j(t)\phi_j(\cdot,t) \in S_h(t)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(t)\mathbf{u}(t)\Big) + \mathbf{A}(t)\mathbf{u}(t) &= \mathbf{b}(t), \\ \mathbf{u}(0) &= \mathbf{u}^0. \end{aligned} \tag{1.8}$$

Concerning notation, we will apply the convention to use small boldface letters to denote vectors in $\mathbb{R}^N$ or $\mathbb{R}^{3N}$ collecting nodal values of discretised functions on the surface denoted by the same letter, and boldface capitals for matrices over the discrete spaces.

Lift

In the following we recall the *lift operator*, which was introduced in [Dzi88] and further investigated in [DE07a, DE13b]. The lift operator maps a finite element function on the discrete surface onto a function on the smooth surface.

The lift of a continuous function $\eta_h : \Gamma_h(t) \to \mathbb{R}$ is defined as

$$\eta_h^\ell(y,t) = \eta_h(x,t), \qquad x \in \Gamma_h(t),$$

where for every $x \in \Gamma_h(t)$ the point $y = y(x,t) \in \Gamma(t)$ is *uniquely* defined via the equation

$$x = y + \nu(y,t) d(x,t).$$

For vector valued functions the lift is meant componentwise. By $\eta^{-\ell}$ we mean the function whose lift is η. We also have the lifted finite element space

$$S_h^\ell(t) := \big\{ \varphi_h^\ell \mid \varphi_h \in S_h(t) \big\}.$$

1.3. Time discretisation methods

We now briefly describe the time discretisation methods used in this thesis. Instead of the linear problem (1.8) we consider a more general problem, which accommodates all subsequent problems of this chapter:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(t)\mathbf{u}(t)\Big) + \mathbf{A}(t,\mathbf{u}(t))\mathbf{u}(t) &= \mathbf{f}(t,\mathbf{u}(t)), \\ \mathbf{u}(0) &= \mathbf{u}^0. \end{aligned} \tag{1.9}$$

For example in (1.8) we have $\mathbf{A}(t,\mathbf{u}) = \mathbf{A}(t)$ and the non-linearity takes the form $\mathbf{f}(t,\mathbf{u}(t)) = \mathbf{b}(t)$.

Implicit Runge–Kutta methods

An s-stage implicit Runge–Kutta method applied to the ODE system (1.9), with constant[1] step size τ, determines the approximations $\mathbf{u}^n \approx \mathbf{u}(t_n)$ and the internal stages $\mathbf{u}^{ni}$:

$$\mathbf{M}_{ni}\mathbf{u}^{ni} = \mathbf{M}_n\mathbf{u}^n + \tau\sum_{j=1}^{s} a_{ij}\dot{\mathbf{u}}^{nj}, \qquad \text{for} \quad i = 1, 2, \dots, s, \tag{1.10a}$$

$$\mathbf{M}_{n+1}\mathbf{u}^{n+1} = \mathbf{M}_n\mathbf{u}^n + \tau\sum_{i=1}^{s} b_i\dot{\mathbf{u}}^{ni}, \tag{1.10b}$$

where the internal stages satisfy

$$\dot{\mathbf{u}}^{ni} + \mathbf{A}(t_n + c_i\tau, \mathbf{u}^{ni})\mathbf{u}^{ni} = \mathbf{f}(t_n + c_i\tau, \mathbf{u}^{ni}) \qquad \text{for} \quad i = 1, 2, \dots, s, \tag{1.10c}$$

with $\mathbf{M}_{ni} = \mathbf{M}(t_n + c_i\tau)$ and $\mathbf{M}_{n+1} = \mathbf{M}(t_{n+1})$, where $t_n = n\tau$. Note that $\dot{\mathbf{u}}^{ni}$ is not a time derivative, only a suggestive notation.

The method is determined by its coefficient matrix $\mathcal{O}\!\iota = (a_{ij})_{i,j=1}^{s}$ and its vector of weights $b = (b_i)_{i=1}^{s}$, with the nodes $c_i = \sum_{j=1}^{s} a_{ij}$. We will always consider Runge–Kutta methods that have the following important properties:

- The method has stage order $q \geqslant 1$ and classical order $p \geqslant q + 1$.
- The coefficient matrix $\mathcal{O}\!\iota$ is invertible.
- The method is *algebraically stable*, i.e. the weights b_i are positive and the following matrix is positive semi-definite:

$$\big(b_i a_{ij} - b_j a_{ji} - b_i b_j\big)_{i,j=1}^{s}. \tag{1.11}$$

- The method is *stiffly accurate*, i.e. the coefficients satisfy

$$b_j = a_{sj}, \qquad \text{and} \qquad c_s = 1, \quad \text{for} \quad j = 1, 2, \dots, s. \tag{1.12}$$

Algebraically stable Runge–Kutta methods are known to be A-stable. For the numerical solution of parabolic problems, an important class of methods – which also satisfy the above properties – are the *Radau IIA methods*. For more details we refer to [HW96, Chapter IV.].

From now on, under implicit Runge–Kutta method we always mean (unless stated otherwise) a method which satisfies the above conditions.

Backward differentiation formulae

A k-step backward differentiation formula (BDF method) applied to the ODE system (1.9), with constant step size τ, determines the approximations $\mathbf{u}^n \approx \mathbf{u}(t_n)$:

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j \mathbf{M}(t_{n-j})\mathbf{u}^{n-j} + \mathbf{A}(t_n, \mathbf{u}^n)\mathbf{u}^n = \mathbf{f}(t_n, \mathbf{u}^n), \qquad (n \geqslant k), \tag{1.13}$$

where the coefficients of the method are given by $\delta(\zeta) = \sum_{j=0}^{k} \delta_j \zeta^j = \sum_{i=1}^{k} \frac{1}{i}(1-\zeta)^i$, while the starting values $\mathbf{u}^0, \mathbf{u}^1, \dots, \mathbf{u}^{k-1}$ are assumed to be given. They can be precomputed in a way as is usual for multistep methods: using lower-order methods with smaller step sizes, or using an implicit Runge–Kutta method of the same order.

[1] This assumption is only made for simplicity. Most of our results hold for variable step sizes, cf. appendices.

The method is known to be 0-stable for $k \leqslant 6$ and have order k, furthermore, being $A(\alpha)$-stable with angles $90°, 90°, 86.03°, 73.35°, 51.84°, 17.84°$, respectively. For more details we refer to [HW96, Chapter V.].

We also consider linearly implicit BDF methods, which applied to the ODE system (1.9) determine the approximations $\mathbf{u}^n \approx \mathbf{u}(t_n)$, by solving the linear system of equations:

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j \mathbf{M}(t_{n-j})\mathbf{u}^{n-j} + \mathbf{A}(t_n, \widetilde{\mathbf{u}}^n)\mathbf{u}^n = \mathbf{f}(t_n, \widetilde{\mathbf{u}}^n), \qquad (n \geqslant k), \tag{1.14}$$

where the extrapolated vector $\widetilde{\mathbf{u}}^n$ is defined by

$$\widetilde{\mathbf{u}}^n = \sum_{j=0}^{k-1} \gamma_j \mathbf{u}^{n-1-j}, \qquad n \geqslant k.$$

The coefficients are given by the same function $\delta(\zeta)$ as for the fully implicit case, and $\gamma(\zeta) = \sum_{j=0}^{k-1} \gamma_j \zeta^j = (1-(1-\zeta)^k)/\zeta$. In general for (1.9), the linearly implicit method requires to solve a linear system with the matrix $\delta_0 \mathbf{M}(t_n) + \tau \mathbf{A}(t_n, \widetilde{\mathbf{u}}^n)$, while the fully implicit method (1.13) requires to solve a non-linear system, in each time step.

The classical order k is retained by the linearly implicit variant using the above coefficients γ_j, cf. [AL15, ALL17].

1.4. A short review on convergence results

Numerous convergence results have been obtained for discretisations of *time-dependent* evolving surface problems, here we shortly (and non comprehensively) review the earliest results.

The first H^1 norm semi-discrete error estimate was shown by [DE07a]. Dziuk and Elliott also showed an optimal L^2 norm semi-discrete error estimate in [DE13b], while a fully discrete convergence result, using the backward Euler method, was shown in [DE12]. Results have been collected (up to 2012) in the excellent review article [DE13a].

Convergence results (of classical order) of time discretisations were obtained for algebraically stable Runge–Kutta methods in [DLM12], and for backward differentiation formulae in [LMV13].

Semi- and full discretisation of wave equations have been studied in [LM15, Man15]. A unified presentation of the evolving surface finite element method and time discretisations for parabolic problems and wave equations can be found in [Man13].

The numerical analysis of first order hyperbolic problems started from [DKM13].

1.5. The arbitrary Lagrangian Eulerian evolving surface finite elements: convergence and algorithms

Dziuk and Elliott already remarked in Section 7.2 of [DE07a] that *"A drawback of our method is the possibility of degenerating grids. The prescribed velocity may lead to the effect, that the triangulation $\Gamma_h(t)$ is distorted"*, i.e. the surface evolution can yield a mesh which is not admissible, since there are triangles with very small angles. Even bad surface resolution may occur. These effects may deteriorate the approximation properties of the evolving surface finite element method. As observed in Figure 1.1: although the initial mesh (left) is quasi-uniform and the surface evolution is also not complicated (the figure shows snapshots at times $t = 0,\ 0.2,\ 0.6$, see also [ES12]), the meshes at later times (middle and right) do not preserve these good mesh properties. Small angles, quite bad surface resolution and unnecessarily fine elements occur.

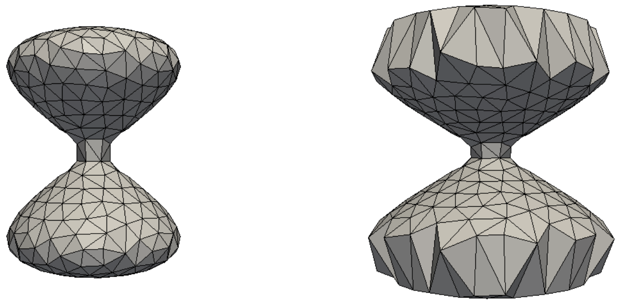

Figure 1.1: Normal evolution of a closed surface at time $t = 0, 0.2, 0.6$; see also in [ES12]

To resolve this problem, Elliott and Styles [ES12] proposed an *arbitrary Lagrangian Eulerian* (ALE) evolving surface finite element method, which in contrast to the (pure Lagrangian) evolving surface finite element method, uses an additional tangential velocity leading to a surface evolution which preserves the good properties of the initial mesh. Various numerical experiments have been presented in [ES12] where smaller numerical errors are achieved using this approach.

The idea of arbitrary Lagrangian Eulerian maps has been previously investigated for moving domains, see for example [FN99, FN04] and [BKN13b, BKN13a], and the references therein. These papers construct nice meshes, assuming that a suitable movement of the boundary is given.

Semi-discrete optimal-order convergence results for evolving surfaces have been first proved in [EV15], together with error bounds for the fully discrete schemes using first and second-order BDF methods.

In [KP18a] fully discrete convergence results using high-order time discretisation methods have been shown by extending the convergence results of [DLM12] for the Runge–Kutta discretisations, and the results of [LMV13] for the backward differentiation formulae to the ALE case. Stability and convergence of these high-order time discretisations are shown, and therefore we establish optimal-order convergence results for full discretisations of linear evolving surface parabolic PDEs when these time integrators are coupled with the ALE evolving surface finite element method as a space discretisation.

For evolving domains and surfaces Elliott and Fritz [EF17, EF16] constructed meshes with very good properties using different techniques via the DeTurck trick.

The ALE evolving surface finite element method

Let us first introduce some further notations related to the arbitrary Lagrangian Eulerian approach. We assume that the surface $\Gamma(t)$ is also given by the sufficiently smooth function $X^{\mathcal{A}} : \Gamma^0 \times [0, T] \to \mathbb{R}^{m+1}$:

$$\Gamma(t) = \{X^{\mathcal{A}}(p, t) \mid p \in \Gamma^0\}.$$

The two parametrisations X and $X^{\mathcal{A}}$ have the same image for all t, although they might differ pointwise. The parametrisation $X^{\mathcal{A}}$ is assumed to retain the good quality of the initial mesh.

The corresponding *ALE surface velocity* $w : \cup_{t\in[0,T]}\Gamma(t) \times \{t\} \to \mathbb{R}^{m+1}$ is then given, for all $p \in \Gamma^0$, by

$$\frac{\mathrm{d}}{\mathrm{d}t} X^{\mathcal{A}}(p, t) = w(X^{\mathcal{A}}(p, t), t). \tag{1.15}$$

We have that the difference $w - v$ of the ALE velocity w and the surface velocity v (from (1.1)) is a tangential vector field. The *ALE material derivative* of a function u is given by

$$\partial^{\mathcal{A}} u(\cdot,t) = \frac{\mathrm{d}}{\mathrm{d}t} u(X^{\mathcal{A}}(\cdot,t),t).$$

The arbitrary Lagrangian Eulerian weak formulation for the linear evolving surface problem (1.3) reads as: Find the unknown function $u(\cdot,t) \in H^1(\Gamma(t))$ with a time-continuous ALE material derivative $\partial^{\mathcal{A}} u(\cdot,t) \in L^2(\Gamma(t))$ such that, for all $\varphi(\cdot,t) \in H^1(\Gamma(t))$ with $\partial^{\mathcal{A}}\varphi(\cdot,t) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)}\varphi + \int_{\Gamma(t)} u\ (w-v)\cdot\nabla_{\Gamma(t)}\varphi = \int_{\Gamma(t)} f\varphi, \tag{1.16}$$

where the initial value is the same as for (1.6).

The triangulation $\Gamma_h(t)$ is obtained in a slightly different way than described in Section 1.2. The initial surface is approximated by $\Gamma_h(0)$, with nodes $(x_j^0)_{j=1}^N$, then the nodes $(x_j(t))_{j=1}^N$ are obtained by solving the ODE (1.15) for the nodes, with initial values x_j^0. The corresponding finite elements, discrete material derivatives, etc. are defined analogously as in Section 1.2, for more details we refer to [ES12, EV15, KP18a]. The analogous transport property holds in the ALE setting as well.

The ALE semi-discrete problem then reads as: Find the finite element function $u_h(\cdot,t) \in S_h(t)$ with a time-continuous discrete material derivative $\partial_h^{\mathcal{A}} u_h(\cdot,t) \in S_h(t)$ such that, for all $\varphi_h(\cdot,t) \in S_h(t)$ with $\partial_h^{\mathcal{A}}\varphi_h(\cdot,t) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} u_h\varphi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)} u_h \cdot \nabla_{\Gamma_h(t)}\varphi_h + \int_{\Gamma_h(t)} u_h\big(W_h - V_h\big)\cdot\nabla_{\Gamma_h(t)}\varphi_h = \int_{\Gamma_h(t)} f_h\phi_h, \tag{1.17}$$

where the discrete ALE and surface velocity are interpolations of their continuous counterparts, and are, respectively, given by

$$V_h(\cdot,t) = \sum_{j=1}^N v(x_j(t),t)\phi_j(\cdot,t), \qquad \text{and} \qquad W_h(\cdot,t) = \sum_{j=1}^N w(x_j(t),t)\phi_j(\cdot,t).$$

The matrix–vector formulation reads:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(t)\mathbf{u}(t)\Big) + \mathbf{A}(t)\mathbf{u}(t) + \mathbf{B}(t)\mathbf{u}(t) &= \mathbf{b}(t), \\ \mathbf{u}(0) &= \mathbf{u}^0, \end{aligned} \tag{1.18}$$

where $\mathbf{A}$, $\mathbf{M}$ and $\mathbf{b}$ are given as before, and the non-symmetric time-dependent matrix $\mathbf{B}(t)$ is given by

$$\mathbf{B}(t)|_{kj} = \int_{\Gamma_h(t)} \phi_j\big(W_h - V_h\big)\cdot\nabla_{\Gamma_h(t)}\phi_k, \qquad (j,k = 1,2,\dots,N).$$

Error estimates

The error between the lifted fully discrete numerical solution $(u_h^n)^\ell$ and the exact solution $u(\cdot,t_n)$ of the evolving surface PDE (1.3) obtained by combining ALE evolving surface finite elements and Runge–Kutta method satisfies the following optimal-order error estimates.

Theorem 1.1 (Theorem 5.7 of [KP18a], Appendix A)**.** *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method, using linear finite elements, as the space discretisation of the parabolic problem* (1.3) *with time discretisation by an s-stage implicit Runge–Kutta* *method. Let u be a sufficiently smooth solution of the problem, and assume that the initial value satisfies*

$$\|u(\cdot,0) - (u_h^0)^\ell\|_{L^2(\Gamma(0))} \leqslant C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$ such that, for $h \leqslant h_0$ and $\tau \leqslant \tau_0$, the following error estimate holds for $t_n = n\tau \leqslant T$:

$$\|u(\cdot,t_n) - (u_h^n)^\ell\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma(t_j)}\big(u(\cdot,t_j) - (u_h^j)^\ell\big)\|_{L^2(\Gamma(t_j))}^2\Big)^{\frac{1}{2}} \leqslant C\big(\tau^{q+1} + h^2\big).$$

The constant $C > 0$ is independent of h, τ and n, but depends on the final time T and on the solution u.

Assuming that we have more regularity, namely the following conditions (of Theorem 5.4 of [KP18a]) are additionally satisfied, for the nodal values of the exact solution $\tilde{\mathbf{u}}(t)$,

$$\Big|\mathbf{M}(t)^{-1}\frac{\mathrm{d}^{k_j-1}}{\mathrm{d}t^{k_j-1}}\Big(\mathbf{A}(t)\mathbf{M}(t)^{-1}\Big)\cdots\frac{\mathrm{d}^{k_1-1}}{\mathrm{d}t^{k_1-1}}\Big(\mathbf{A}(t)\mathbf{M}(t)^{-1}\Big)\frac{\mathrm{d}^{\tilde{k}-1}}{\mathrm{d}t^{\tilde{k}-1}}\Big(M(t)\tilde{\mathbf{u}}(t)\Big)\Big|_{\mathbf{M}(t)} \leqslant C',$$
$$\Big|\mathbf{M}(t)^{-1}\frac{\mathrm{d}^{k_j-1}}{\mathrm{d}t^{k_j-1}}\Big(\mathbf{A}(t)\mathbf{M}(t)^{-1}\Big)\cdots\frac{\mathrm{d}^{k_1-1}}{\mathrm{d}t^{k_1-1}}\Big(\mathbf{A}(t)\mathbf{M}(t)^{-1}\Big)\frac{\mathrm{d}^{\tilde{k}-1}}{\mathrm{d}t^{\tilde{k}-1}}\Big(\mathbf{M}(t)\tilde{\mathbf{u}}(t)\Big)\Big|_{\mathbf{A}(t)} \leqslant C',$$

with some $C' > 0$, for all $k_j \geqslant 1$ and $\tilde{k} \geqslant q+1$ with $k_1 + \cdots + k_j + \tilde{k} \leqslant p+1$, then in the error estimate we have the classical order p instead of $q+1$.

For BDF methods we have the analogous optimal-order error bounds.

Theorem 1.2 (Theorem 5.8 of [KP18a], Appendix A)**.** *Consider the arbitrary Lagrangian Eulerian evolving surface finite element method, using linear finite elements, as the space discretisation of the parabolic problem* (1.3) *with time discretisation by a k-step backward difference formula of order $k \leqslant 5$. Let u be a sufficiently smooth solution of the problem, and assume that the starting values are satisfying*

$$\max_{0\leqslant i\leqslant k-1} \|u(\cdot,t_i) - (u_h^i)^\ell\|_{L^2(\Gamma(t_i))} \leqslant C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$ such that, for $h \leqslant h_0$ and $\tau \leqslant \tau_0$, the following error estimate holds for $t_n = n\tau \leqslant T$:

$$\|u(\cdot,t_n) - (u_h^n)^\ell\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=k}^{n} \|\nabla_{\Gamma(t_j)}\big(u(\cdot,t_j) - (u_h^j)^\ell\big)\|_{L^2(\Gamma(t_j))}^2\Big)^{\frac{1}{2}} \leqslant C\big(\tau^{k} + h^2\big).$$

The constant $C > 0$ is independent of h, τ and n, but depends on the final time T and on the solution u.

Both theorems are shown using energy techniques, which are used to show stability of the numerical methods. For stiffly accurate algebraically stable implicit Runge–Kutta methods (having the Radau IIA methods in mind) we use techniques of [LO95], which were first extended to evolving surface problems in [DLM12]. Similarly, energy techniques are used to show stability for k-step BDF methods up to order five, by combining the G-stability theory of Dahlquist [Dah78] and the multiplier techniques of Nevanlinna and Odeh [NO81]. The stability analysis requires careful estimates (boundedness, perturbation errors, etc.) for the newly arising non-symmetric term, which is due the ALE formulation.

The above error bounds for BDF methods of order $k = 1$ and 2 were first shown by Elliott and Venkataraman [EV15]. The proof techniques therein are different than the ones described above.

Computing arbitrary Lagrangian Eulerian maps for evolving surfaces

As the references in the introduction of Section 1.5 show, the theory of ALE evolving surface finite elements was developed intensively, however the numerical computation of ALE maps for closed evolving surfaces has received less attention.

The ALE maps used in the experiments of [ES12, EV15, KP18a] were slightly unrealistic, obtained analytically from an *a priori knowledge* on the surface and its evolution, using deep understanding and structure of the signed distance function. No general ideas on the computation of ALE maps for evolving surfaces have been proposed in these papers.

In [Kov17] an algorithm is proposed to compute an arbitrary Lagrangian Eulerian map for closed evolving surfaces, with a focus on evolving surface finite elements, which does not use such a priori knowledge, in the following sense: the algorithm uses the distance function at each time step, but it does not use its structure or any other special properties of it.

The algorithm in [Kov17] finds the ALE map by treating the problem as a constrained system with an additional velocity, i.e. the vector field of the ODE (1.1) is extended by an additional velocity field, which aims at preserving the good properties of the mesh and in the meantime a constraint is introduced to keep the nodes of the mesh on the surface. The additional velocity law is determined based on a mechanical system, using a spring analogy.

In the various numerical experiments in [Kov17] it is illustrated that this algorithm provides an evolving surface mesh of good quality, without any a priori knowledge on the surface or its evolution. It is also demonstrated that the additional cost of the ALE computations are marginal compared to the numerical solution of the PDE.

Furthermore, we also discuss and test possible extensions of the algorithm. For example, a slight modification of the proposed ALE velocity provides surface meshes with angle conditions (i.e. acute or non-obtuse triangulations), as explored in Section 5.3 of [Kov17], which are crucial for discrete maximum principles for surfaces PDEs, see [FMSV16, FMSV17a, FMSV17b, KKK17].

1.6. Maximum norm stability and error estimates

In [KP18b] semi-discrete convergence results in the L^∞ and $W^{1,\infty}$ norms are shown for parabolic PDEs on two dimensional evolving surfaces. Error estimates in these norms are of particular interest for the numerical analysis of non-linear evolving surface problems where the velocity is not given explicitly, but depends on the solution u. Semi- and fully discrete error bounds for such problems are shown recently, for references and further details we refer to Chapter 2. Such estimates are also important for the numerical analysis of control problems on evolving surfaces, see, e.g. [HK16].

The obtained convergence bounds are optimal in terms of the powers of h (the mesh size), however they contain a non-optimal logarithmic factor. We expect that estimates with optimal logarithmic factors, or even without them for certain norms, can be obtained by extending the corresponding Euclidean theory, see [Hav84, RS82, Sch98], or [STW98].

Semi-discrete convergence estimates

The error between the semi-discrete solution $u_h(\cdot,t) \in S_h(t)$ and the solution $u(\cdot,t)$ of problem (1.3) satisfies the following error bounds in the L^∞ and $W^{1,\infty}$ norms.

Theorem 1.3 (Theorem 6.1 of [KP18b], Appendix D). *Let $\Gamma(t)$ be a smooth two dimensional evolving surface. Let u be a sufficiently smooth solution of the problem* (1.3)*, and let $u_h(t) \in S_h(t)$*

be the solution of the semi-discrete problem (1.6) *using linear basis functions. Then there exists* $h_0 > 0$ *sufficiently small such that for all* $h \leqslant h_0$ *we have the estimate*

$$\|u(\cdot,t) - (u_h(\cdot,t))^\ell\|_{L^\infty(\Gamma(t))} + h\big\|\nabla_{\Gamma(t)}\big(u(\cdot,t) - (u_h(\cdot,t))^\ell\big)\big\|_{L^\infty(\Gamma(t))} \leqslant Ch^2|\log h|^4,$$

where the constant $C > 0$ *is independent of* t *and* h*, but depends on the final time* T *and on* u*.*

The proof of this theorem relies on three main results: (i) Nitsche's weighted norm technique [Nit77] is extended to evolving surfaces, together with its basic properties, which is then used to prove L^∞ and $W^{1,\infty}$ norm error bounds for a time-dependent Ritz map. (ii) Since the Ritz map is *time-dependent* it does not commute with the material derivative. We therefore need the analogous error bounds for the material derivatives of the Ritz map. In both cases we first show the weighted norm error bounds, which in turn yield the L^∞ and $W^{1,\infty}$ norm error bounds. (iii) The weak finite element maximum principle (for Euclidean domains) of Schatz, Thomée and Wahlbin [STW80], is extended to parabolic evolving surface PDEs. This leads to the semi-discrete error bounds of Theorem 1.3. The proof of the maximum principle uses an argument using an adjoint parabolic problem and estimates for the discrete Green's function, and avoids the semigroup argument used in [STW80].

1.7. High-order evolving surface finite elements

High-order *evolving* surface finite element discretisations are of natural interest, especially in combination with time integrators of high-order, see Section 1.3. Many spatially discrete results are available for elliptic problems on *stationary surfaces*, we give a brief overview here: The high-order surface finite element method was developed by Demlow [Dem09]. Further important results for higher order surface (and bulk) finite elements were shown in [ER13]. High-order discontinuous Galerkin methods were studied in [ADM+15]. A high-order variant of *unfitted* (also called trace or cut) finite element method was analysed in [GR16].

The extensions of H^1 and L^2 norm convergence results for *evolving* surface problems discretised with high-order evolving surface finite elements are studied in [Kov18]. We study convergence of semi-discretisations, and also convergence for fully discrete schemes using an implicit Runge–Kutta, or a BDF method as a time integrator. We note here, that later the same semi-discrete results have been also obtained using a general abstract framework, but with the same techniques, see Elliott and Ranner [ER17].

It was pointed out by Grande and Reusken [GR16], that the approach of [Dem09] requires *explicit knowledge of the exact signed distance function to the surface* Γ. However, in our case the signed distance function is only used in the analysis and for computations on the initial time level. The computations only require triangulation of the initial surface given by its elements and nodes, the latter being integrated by solving the ODE (1.1) with the given velocity of the surface.

In this section we only consider the linear parabolic PDE (1.3) on evolving surfaces, however we strongly believe that our techniques and results carry over to other cases, such as to the Cahn–Hilliard equation [ER15], to wave equations [LM15, Man13], to ALE methods [EV15], [KP18a] (Section 1.5), to non-linear problems [KP16] (Section 1.8) and to evolving versions of the bulk–surface problems studied in [ER13]. This observation is strongly supported by the findings of [KLLP17, KL18], see Chapter 2.

High-order evolving surface finite elements

Here we only give a very brief introduction to the high-order evolving surface finite element method. More details are given in [Kov18], where we follow [Dem09] and [Dzi88, DE07a], while carefully treating the time-dependence.

First, the smooth *initial* surface $\Gamma(0)$ is approximated by an interpolating discrete surface of order k, denoted by $\Gamma_h^k(0)$. The discrete surface $\Gamma_h^k(t)$ is then obtained by *evolving* the high-order interpolation surface $\Gamma_h^k(0)$ in time by the a priori known surface velocity v, via the ODE (1.1). The precise details of this construction can be found in Section 3 of [Kov18]. We use continuous piecewise polynomial basis functions of degree k (meaning that on every triangle their pull-back to the reference triangle is the usual Lagrangian basis function of degree k). The basis functions spanning the high-order finite element space $S_h^k(t)$ have the exact same general properties as for the linear case, such as the transport property (1.5), lift, etc., see Section 1.1.

The semi-discrete problem on $S_h^k(t)$ and the matrix–vector formulation are formally the same as those in (1.6) and (1.8), respectively, cf. [Kov18].

Error estimates

The error between the semi-discrete solution $u_h(\cdot,t) \in S_h^k(t)$ and the solution $u(\cdot,t)$ satisfies the optimal-order convergence bound, which is a higher order extension of Theorem 4.4 in [DE13b].

Theorem 1.4 (Theorem 4.1 of [Kov18], Appendix C). *Consider the evolving surface finite element method of order k as space discretisation of the parabolic problem* (1.3). *Let the solution u be sufficiently smooth, and assume that the initial value for* (1.6) *satisfies*

$$\|u(\cdot,0) - (u_h(\cdot,0))^\ell\|_{L^2(\Gamma(0))} \leqslant C_0 h^{k+1}.$$

Then there exist $h_0 > 0$ such that for mesh size $h \leqslant h_0$, the following error estimate holds, for $t \leqslant T$:

$$\|u(\cdot,t) - (u_h(\cdot,t))^\ell\|_{L^2(\Gamma(t))} + h\Big(\int_0^t \big\|\nabla_{\Gamma(s)}\big(u(\cdot,s) - (u_h(\cdot,s))^\ell\big)\big\|_{L^2(\Gamma(s))}^2 \mathrm{d}s\Big)^{\frac12} \leqslant C h^{k+1}.$$

The constant $C > 0$ is independent of h and t, but depends on T and on the solution u.

The error between the fully discrete numerical solution u_h^n, obtained from a BDF method of order $p \leqslant 5$, and the solution $u(\cdot,t_n)$ satisfies the following optimal-order error bounds.

Theorem 1.5 (Theorem 4.4 of [Kov18], Appendix C). *Consider the evolving surface finite element method of order k as space discretisation of the parabolic problem* (1.3), *coupled to the time discretisation by a p-step backward difference formula with $p \leqslant 5$. Let u be a sufficiently smooth solution of the problem, and assume that the starting values are satisfying*

$$\max_{0\leqslant i\leqslant p-1} \|u(\cdot,t_i) - (u_h^i)^\ell\|_{L^2(\Gamma(t_i))} \leqslant C_0 h^{k+1}.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$ such that, for $h \leqslant h_0$ and $\tau \leqslant \tau_0$, the following error estimate holds for $t_n = n\tau \leqslant T$:

$$\|u(\cdot,t_n) - (u_h^n)^\ell\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=p}^{n} \big\|\nabla_{\Gamma(t_j)}\big(u(\cdot,t_j) - (u_h^j)^\ell\big)\big\|_{L^2(\Gamma(t_j))}^2\Big)^{\frac12} \leqslant C\big(\tau^p + h^{k+1}\big).$$

The constant $C > 0$ is independent of h, τ and n, but depends on T and on the solution u.

For algebraically stable implicit Runge–Kutta methods (which satisfies all the other conditions of Section 1.3) we have the following optimal-order error estimates.

Theorem 1.6 ([Kov18], Appendix C). *Consider the evolving surface finite element method of order k as space discretisation of the parabolic problem* (1.3)*, coupled to the time discretisation by an s-stage implicit Runge–Kutta method. Let u be a sufficiently smooth solution of the problem, and assume that the starting value satisfies*

$$\|u(\cdot,0) - (u_h^0)^\ell\|_{L^2(\Gamma(0))} \leqslant C_0 h^{k+1}.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$ such that, for $h \leqslant h_0$ and $\tau \leqslant \tau_0$, the following error estimate holds for $t_n = n\tau \leqslant T$:

$$\|u(\cdot,t_n) - (u_h^n)^\ell\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=1}^{n} \big\|\nabla_{\Gamma(t_j)}\big(u(\cdot,t_j) - (u_h^j)^\ell\big)\big\|^2_{L^2(\Gamma(t_j))}\Big)^{\frac{1}{2}} \leqslant C\big(\tau^{q+1} + h^{k+1}\big).$$

The constant $C > 0$ is independent of h, τ and n, but depends on T and on the solution u.

Assuming that we have more regularity, analogously as in Theorem 1.1, or see (8.3) in [DLM12], we then have the classical order p instead of $q + 1$.

In order to show optimal-order error estimates of the semi-discretisation, high-order variants of three groups of errors need to be analysed: (i) Geometric errors, resulting from the appropriate approximation of the smooth surface. Many of these results carry over from [Dem09] by careful investigation of time-dependence, while others are extended from [Man13] and [DLM12, LMV13]. (ii) High-order perturbation errors of the bilinear forms, which are shown by carefully using the core ideas of the analogous results in [DE13b]. (iii) High-order estimates for the errors of a Ritz map, and also for its material derivatives. These error bounds rely on the non-trivial combination of the mentioned geometric error bounds and on the Aubin–Nitsche duality argument.

The fully discrete error bounds are shown using the stability results from [LMV13] (for BDF methods) and [DLM12] (for Runge–Kutta methods), in combination with the semi-discrete residual bounds, which rely on the three points mentioned above.

The results of these theorems are illustrated by numerical experiments, obtained from our Matlab implementation.

1.8. Error analysis for full discretisations of non-linear parabolic problems

Many biological and physical processes are modelled by non-linear parabolic problems on evolving surfaces. Apart from general quasi-linear problems on moving surfaces, see e.g. Example 3.5 in [DE07b], more specific applications are the non-linear models: diffusion induced grain boundary motion [CFP97, FCE01, Han89, DES01, ES12]; Allen–Cahn and Cahn–Hilliard equations on evolving surfaces [CENC96, EG96, ES10, Che02]; tumour growth [CGG01, BEM11, ES12]; pattern formation models based on reaction–diffusion equations [MB14]; cell motility [ESV12]; image processing [JYS04]; Ginzburg–Landau model for superconductivity [DJ04].

A great number of non-linear problems with numerical experiments were presented in the literature, see for example the above references, in particular we refer to [DE07a, DE07b, DE13a, ES12, DES01, ESV12].

Although the literature is very rich in non-linear models and numerical experiments with them, much less is known about convergence estimates for non-linear (evolving) surface PDEs. Elliott and Ranner [ER15] give semi-discrete optimal-order error bounds for the Cahn–Hilliard equation. In [KP16] fully discrete convergence results are shown for a large class of quasi-linear and semi-linear parabolic problems on evolving surfaces. We use the evolving surface finite element method for the spatial discretisation, while in time we either use an algebraically stable implicit Runge–Kutta method, or an implicit or linearly implicit backward differentiation formula.

Abstract formulation of quasi-linear problems on evolving surfaces

We consider the following quasi-linear problem:

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \nabla_{\Gamma(t)} \cdot \Big(\mathcal{A}(u)\nabla_{\Gamma(t)} u\Big) &= f \qquad \text{on } \Gamma(t), \\ u(.,0) &= u^0 \qquad \text{on } \Gamma(0), \end{aligned} \tag{1.19}$$

where the function $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ is

$$\text{bounded and Lipschitz continuous, satisfying } \mathcal{A}(s) \geqslant \alpha > 0. \tag{1.20}$$

The results of this section can be generalized to the case of a matrix valued diffusion coefficient $\mathcal{A}(x,t,u) : T_x\Gamma(t) \to T_x\Gamma(t)$, (where $T_x\Gamma(t)$ denotes the tangent plane to $\Gamma(t)$ at x). The proofs are analogous to the ones presented in [KP16], although they require some extra care, and are more technical and lengthy as well.

This problem can be written as the general abstract parabolic problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((u,v)_t\Big) + \langle A(u)u, v\rangle_t = \langle f, v\rangle_t, \qquad \text{for all} \qquad v \in V(t),$$

with initial value $u(\cdot,0) = u^0$. This equation is cast in the following abstract framework, which is a suitable combination of [AES15, Section 2.3] and [LO95, Section 1]: Let $H(t)$ and $V(t)$ be real and separable Hilbert spaces (with norms $\|\cdot\|_{H(t)}$, $\|\cdot\|_{V(t)}$, respectively) such that $V(t)$ is densely, continuously and time-uniformly embedded into $H(t)$, and the norm of the dual space of $V(t)$ is denoted by $\|\cdot\|_{V(t)'}$. The dual space of $H(t)$ is identified with itself, and the duality $\langle\cdot,\cdot\rangle_t$ between $V(t)'$ and $V(t)$ coincides on $H(t) \times V(t)$ with the scalar product on $H(t)$ denoted by $(\cdot,\cdot)_t$, for all $t \in [0,T]$.

The operator $A(u) : V(t) \to V(t)'$ is uniformly *elliptic* with $\alpha > 0$, i.e.

$$\langle A(u)w, w\rangle_t \geqslant \alpha\|w\|^2_{V(t)}, \qquad \text{for all} \qquad w \in V(t), \tag{1.21}$$

and uniformly *bounded* with $M > 0$, i.e.

$$\big|\langle A(u)v, w\rangle_t\big| \leqslant M\|v\|_{V(t)}\|w\|_{V(t)}, \qquad \text{for all} \qquad v, w \in V(t). \tag{1.22}$$

Here uniformity is understood as uniformly in $u \in V(t)$ and in $t \in [0,T]$. We further assume that there is a subset $\mathcal{S}(t) \subset V(t)$ such that the following *Lipschitz–type* estimate holds: for every $\delta > 0$ there exists $L = L(\delta, (\mathcal{S}(t))_{0\leqslant t\leqslant T})$ such that

$$\big\|\big(A(w_1) - A(w_2)\big)u\big\|_{V(t)'} \leqslant \delta\|w_1 - w_2\|_{V(t)} + L\|w_1 - w_2\|_{H(t)}, \tag{1.23}$$

for all $u \in \mathcal{S}(t)$ and $w_1, w_2 \in V(t)$, for $0 \leqslant t \leqslant T$.

The above conditions were also used to prove error estimates using energy techniques in [LO95], or more recently in [AL15].

The weak problem corresponding to (1.19) can be formulated by choosing the setting: $V(t) = H^1(\Gamma(t))$ and $H(t) = L^2(\Gamma(t))$, and the operator, for $v, w \in V(t)$,

$$\langle A(u)v, w\rangle_t = \int_{\Gamma(t)} \mathcal{A}(u)\nabla_{\Gamma(t)} v \cdot \nabla_{\Gamma(t)} w.$$

Furthermore, we use the following subspace of $V(t)$, for $r > 0$,

$$\mathcal{S}(t) = \mathcal{S}(t,r) = \big\{ u \in H^2(\Gamma(t)) \mid \|u\|_{W^{2,\infty}(\Gamma(t))} \leqslant r \big\}.$$

It is shown in Proposition 2.1 of [KP16] that the above operator $A(u)$ for $u(\cdot,t) \in \mathcal{S}(t,r)$ satisfies (1.21), (1.22) and (1.23).

The weak formulation of the quasi-linear problem (1.19) reads as: Find $u(\cdot,t) \in H^1(\Gamma(t))$ with time-continuous $\partial^\bullet u(\cdot,t) \in L^2(\Gamma(t))$ such that, for $\varphi(\cdot,t) \in H^1(\Gamma(t))$ with $\partial^\bullet \varphi(\cdot,t) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{A}(u) \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} f\varphi, \tag{1.24}$$

with the initial value $u(\cdot,0) = u^0$.

Semi-discrete problem and matrix–vector form

The semi-discrete formulation is written in the evolving surface finite element framework from Section 1.2, and it reads as: Find $u_h(\cdot,t) \in S_h(t)$ with a time-continuous discrete material derivative $\partial_h^\bullet u_h(\cdot,t) \in S_h(t)$ such that, for all $\varphi_h(\cdot,t) \in S_h(t)$ with $\partial_h^\bullet \varphi_h(\cdot,t) = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h(t)} u_h\varphi_h + \int_{\Gamma_h(t)} \mathcal{A}(u_h) \nabla_{\Gamma_h(t)} u_h \cdot \nabla_{\Gamma_h(t)} \varphi_h = \int_{\Gamma_h(t)} f\varphi_h, \tag{1.25}$$

with the initial value $u_h(\cdot,0)$ being a sufficiently good approximation of u^0.

The corresponding ODE system for the vector of nodal values $\mathbf{u}(t) = (u_j(t))_{j=1}^N \in \mathbb{R}^N$, collecting the nodal values of $u_h(\cdot,t)$, reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(t)\mathbf{u}(t)\Big) + \mathbf{A}(\mathbf{u}(t))\mathbf{u}(t) &= \mathbf{b}(t), \\ \mathbf{u}(0) &= \mathbf{u}^0. \end{aligned} \tag{1.26}$$

The mass matrix and the right-hand side vector are both given as before, see (1.7), while the state-dependent stiffness matrix is given, for $u_h(\cdot,t) = \sum_{j=1}^N u_j(t)\phi_j(\cdot,t)$ with $\mathbf{u}(t) = (u_j(t))$, by

$$\mathbf{A}(\mathbf{u}(t))|_{kj} = \int_{\Gamma_h(t)} \mathcal{A}(u_h) \nabla_{\Gamma_h(t)} \phi_j \cdot \nabla_{\Gamma_h(t)} \phi_k, \qquad (j,k = 1,2,\dots,N). \tag{1.27}$$

This matrix–vector formulation fits into the framework of (1.9).

Error estimates

We obtain fully discrete approximations u_h^n upon applying an implicit Runge–Kutta or implicit or linearly implicit BDF method (see Section 1.3) to the non-linear ODE system (1.26), which satisfies the optimal-order error estimates.

Theorem 1.7 (Theorem 5.2 of [KP16], Appendix E)**.** *Consider the evolving surface finite element method as space discretisation of the quasi-linear parabolic problem* (1.19)*, coupled to the time discretisation by an s-stage implicit Runge–Kutta method. Let u be a sufficiently smooth solution of the problem, which satisfies $u(\cdot,t) \in \mathcal{S}(r,t)$ for $0 \leqslant t \leqslant T$, and assume that the initial value is approximated as*

$$\|u(\cdot,0) - (u_h^0)^\ell\|_{L^2(\Gamma(0))} \leqslant C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leqslant h_0$ and $\tau \leqslant \tau_0$, the following error estimate holds for $t_n = n\tau \leqslant T$:

$$\|u(\cdot,t_n) - (u_h^n)^\ell\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=1}^{n} \big\|\nabla_{\Gamma(t_j)}\big(u(\cdot,t_j) - (u_h^j)^\ell\big)\big\|^2_{L^2(\Gamma(t_j))}\Big)^{\frac{1}{2}} \leqslant C\big(\tau^{q+1} + h^2\big).$$

The constant $C > 0$ is independent of h, τ and n, but depends on α, M and L, from (1.21), (1.22) *and* (1.23)*, on T and on the solution u.*

Theorem 1.8 (Theorem 5.3 of [KP16], Appendix E)**.** *Consider the evolving surface finite element method as space discretisation of the quasi-linear parabolic problem* (1.19)*, coupled to the time discretisation by a k-step implicit or linearly implicit backward difference formula of order $k \leqslant 5$. Let u be a sufficiently smooth solution of the problem, which satisfies $u(\cdot,t) \in \mathcal{S}(r,t)$ for $0 \leqslant t \leqslant T$, and assume that the starting values are satisfying*

$$\max_{0\leqslant i\leqslant k-1} \|u(\cdot,t_i) - (u_h^i)^\ell\|_{L^2(\Gamma(t_i))} \leqslant C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$ such that, for $h \leqslant h_0$ and $\tau \leqslant \tau_0$, the following error estimate holds for $t_n = n\tau \leqslant T$:

$$\|u(\cdot,t_n) - (u_h^n)^\ell\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=k}^{n} \big\|\nabla_{\Gamma(t_j)}\big(u(\cdot,t_j) - (u_h^j)^\ell\big)\big\|^2_{L^2(\Gamma(t_j))}\Big)^{\frac{1}{2}} \leqslant C\big(\tau^{k} + h^2\big).$$

The constant $C > 0$ is independent of h, τ and n, but depends on α, M and L, from (1.21), (1.22) *and* (1.23)*, on T and on the solution u.*

These error estimates are shown using stability results for stiffly accurate algebraically stable implicit Runge–Kutta methods, and for implicit or linearly implicit p-step BDF methods up to order five, see Lemma 4.1 and 4.2 of [KP16]. These stability estimates rely on energy estimates, developed in [LO95] for Runge–Kutta methods, and in [AL15] for BDF methods using G-stability [Dah78] and the multiplier technique [NO81], and used previously in a linear evolving surface setting in [DLM12] and [LMV13], respectively.

A key tool is a *generalized Ritz map* for quasi-linear operators, together with its error estimates, shown by extending an argument of Wheeler [Whe73] from the Euclidean case to evolving surfaces, see Section 3 of [KP16]. Further important points of the analysis are the *regularity* theory of this Ritz map, and the geometric estimates due to surface approximation. Together, they yield optimal-order error bounds for the semi-discrete residual. In combination with the stability bounds this proves the above theorems.

Semilinear problems

These results can be readily extended to semilinear parabolic problems, where the function $f(\cdot,t)$ is replaced by $f(t,u)$, satisfying a local Lipschitz condition (similar to (1.23)): for every $\delta > 0$ there exists $L = L(\delta, r)$ such that

$$\|f(t,w_1) - f(t,w_2)\|_{V(t)'} \leqslant \delta\|w_1 - w_2\|_{V(t)} + L\|w_1 - w_2\|_{H(t)} \quad (0 \leqslant t \leqslant T)$$

holds for arbitrary $w_1, w_2 \in V(t)$ with $\|w_1\|_{V(t)}, \|w_2\|_{V(t)} \leqslant r$, uniformly in t. Such a condition can be satisfied by using the same $\mathcal{S}$ set as for quasi-linear problems. For more details we refer to Section VI of [KP16], Appendix E.

2. Surface evolution coupled to parabolic problems on the surface

In this chapter, convergence results on full and semi-discretisations of (two-dimensional) surface evolution coupled to a parabolic problem on the surface are collected.

Geometric partial differential equations, such as mean curvature flow (MCF) or Willmore flow, are of great interest on their own, for numerical works see [Dzi90], and [DDE05] and the references therein. Many models in biology and biophysics lead to coupled surface evolution – surface PDE problems (solution-driven problems), where the equations for surface evolution often contain terms related to the mean curvature of the surface. For such problems we refer to [DDE05, Dzi90, BEM11, ES12, CGG01], and the references therein.

Recently, many papers appeared on the numerical analysis of problems coupling curve-shortening flow (the one-dimensional, graph case of MCF) with diffusion on the curve, see [PS17a, BDS17] for semi- and fully discrete error bounds, and see [PS17b] for a coupling with elastic flow.

Approximations to the curve shortening flow and the mean curvature flow were developed in [EF17] based on the DeTurck trick. Problems coupling Navier–Stokes equations and surface evolutions under Willmore flow have recently been considered in [BGN15a, BGN15b, BGN16].

This chapter studies numerical methods and presents error estimates for a regularised or dynamic velocity law coupled to a diffusion process on the surface. Similarly to the previous chapter, the error bounds are shown by combining stability bounds (obtained via energy techniques) and consistency estimates.

This chapter is organised as follows. Section 2.1 formulates the coupled solution-driven problems, either with a regularised elliptic velocity law or with a dynamic velocity law, and recalls some basic notions. Section 2.2 describes the evolving surface finite element method used in this context. Sections 2.3 and 2.4 collects semi-discrete and fully discrete error bounds, respectively, for both the regularised and dynamic velocity laws.

2.1. Evolving surfaces driven by diffusion on the surface

Most of the notions of Section 1.1 transfer without modifications to the case where the surface velocity is not given a priori. However, in the notation we need to account for the parametrisation dependence. In order to indicate this, in this chapter we will denote the surface by

$$\Gamma(X(\cdot,t)) = \Gamma(X),$$

with the parametrisation $X : \Gamma^0 \times [0,T] \to \mathbb{R}^3$. The velocity $v : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$ still solves the ODE (1.1), the definition of the material derivative also remains the same.

The outer normal vector is denoted by $\nu_{\Gamma(X)}$, while $H_{\Gamma(X)}$ denotes the mean curvature. We denote by $\nabla_{\Gamma(X)} u$ the tangential gradient of u, by $\Delta_{\Gamma(X)} u$ the Laplace–Beltrami operator applied to u. Their definitions remain the same as in Section 1.1, but the dependence on X is made clear using the notation above. For more details on these notions we refer to [KLLP17, KL18].

We are interested in two large classes of coupled surface motion.

(i) A surface PDE is coupled to an elliptically regularized velocity law:

$$\begin{aligned} \partial^\bullet u + u \nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u &= f(u, \nabla_{\Gamma(X)} u), \\ v - \alpha \Delta_{\Gamma(X)} v + \beta H_{\Gamma(X)}(x) \nu_{\Gamma(X)}(x) &= g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)}, \end{aligned} \tag{2.1}$$

considered together with the collection of ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} X(p,t) = v(X(p,t),t) \qquad (p \in \Gamma^0). \tag{2.2}$$

Here, $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are given continuously differentiable functions, and $\alpha > 0$ and $\beta > 0$ are fixed parameters. Both functions are assumed to be locally Lipschitz continuous in their first argument and globally in the second. Initial values are specified for u and X.

The weak formulation reads: Find the functions $u(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))$ with a time-continuous $\partial^\bullet u(\cdot,t) \in L^2(\Gamma(X(\cdot,t)))$ and $v(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))^3$ such that for all test functions $\varphi(\cdot,t) \in H^1(\Gamma(X(\cdot,t)))$ with $\partial^\bullet \varphi = 0$ and $\psi(\cdot,t) \in H^1(\Gamma(X(\cdot,t)))^3$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(X)} u\varphi + \int_{\Gamma(X)} \nabla_{\Gamma(X)} u \cdot \nabla_{\Gamma(X)} \varphi &= \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u) \varphi, \\ \int_{\Gamma(X)} v \cdot \psi + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)} \psi + \beta \int_{\Gamma(X)} \nabla_{\Gamma(X)} X \cdot \nabla_{\Gamma(X)} \psi &= \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u)\, \nu_{\Gamma(X)} \cdot \psi, \end{aligned} \tag{2.3}$$

alongside the collection of ordinary differential equations (2.2) for the positions determining the surface $\Gamma(X)$. Here the term $\nabla_{\Gamma(X)} X$ is read as $\nabla_{\Gamma(X)} \mathrm{Id}_{\Gamma(X)}$, see [Dzi90].

(ii) A surface PDE coupled to a dynamic velocity law:

$$\begin{aligned} \partial^\bullet u + u \nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u &= f(u, \nabla_{\Gamma(X)} u), \\ \partial^\bullet v + v \nabla_{\Gamma(X)} \cdot v - \alpha \Delta_{\Gamma(X)} v &= g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)}, \end{aligned} \tag{2.4}$$

considered together with the collection of ordinary differential equations (2.2). Here f and g satisfy the same as above, and $\alpha > 0$. Initial values are specified for u, v and X.

The weak formulation reads: Find the functions $u(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))$ with a time-continuous $\partial^\bullet u(\cdot,t) \in L^2(\Gamma(X(\cdot,t)))$ and $v(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))^3$ with a time-continuous $\partial^\bullet v(\cdot,t) \in L^2(\Gamma(X(\cdot,t)))^3$ such that for all test functions $\varphi(\cdot,t) \in H^1(\Gamma(X(\cdot,t)))$ with $\partial^\bullet \varphi = 0$ and $\psi(\cdot,t) \in H^1(\Gamma(X(\cdot,t)))^3$ with $\partial^\bullet \psi = 0$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(X)} u\varphi + \int_{\Gamma(X)} \nabla_{\Gamma(X)} u \cdot \nabla_{\Gamma(X)} \varphi &= \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u) \varphi, \\ \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(X)} v \cdot \psi + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)} \psi &= \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)} \cdot \psi, \end{aligned} \tag{2.5}$$

alongside the collection of ordinary differential equations (2.2) for the positions determining the surface $\Gamma(X)$.

Throughout this chapter we assume that, for given initial data, the problem (2.1) or (2.4), with the ODE (2.2), has an

$$\begin{aligned} &\text{exact solution } (u, v, X) \text{ that is sufficiently smooth (say, in the Sobolev class } H^{k+1}), \\ &\text{and that the flow map } X(\cdot,t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3 \text{ is non-degenerate for } 0 \leqslant t \leqslant T, \end{aligned} \tag{2.6}$$

so that $\Gamma(t)$ is a regular surface.

2.2. Evolving surface finite elements for surface evolution

Following Section 2.3 of [KLLP17], we describe the surface finite element discretisation applied to our problems, which is based on [Dzi88, Dem09, Kov18]. By $\mathbf{x}(t) \in \mathbb{R}^{3N}$ we denote a vector collecting the evolving nodes $x_j(t)$ with $x_j(0) = x_j^0$, $j = 1, 2, \dots, N$, where the nodes $(x_j^0)_{j=1}^N$ define Γ_h^0, an admissible triangulation of the initial surface Γ^0, similarly as in Section 1.7. We use continuous piecewise polynomial basis functions of degree k, which span the finite element space

$$S_h[\mathbf{x}] = \mathrm{span}\big\{\phi_1[\mathbf{x}], \phi_2[\mathbf{x}], \dots, \phi_N[\mathbf{x}]\big\}.$$

The basis functions have the usual properties (cf. [KLLP17, Section2.3]): their pull-backs to the reference element are the usual Lagrangian basis functions, $\phi_j[\mathbf{x}(t)](x_k(t)) = \delta_{jk}$ $(j, k = 1, \dots, N)$, etc., see Section 1.2.

We set

$$X_h(p_h, t) = \sum_{j=1}^N x_j(t)\, \phi_j[\mathbf{x}(0)](p_h), \qquad p_h \in \Gamma_h^0,$$

which is the interpolation of $X(\cdot, t)$, and has the properties that $X_h(p_j, t) = x_j(t)$ for $j = 1, \dots, N$, that $X_h(p_h, 0) = p_h$ for all $p_h \in \Gamma_h^0$, and

$$\Gamma_h[\mathbf{x}(t)] = \Gamma(X_h(\cdot, t)).$$

The *discrete velocity* $v_h(x, t) \in \mathbb{R}^3$ at a point $x = X_h(p_h, t) \in \Gamma(X_h(\cdot, t))$ is given by

$$\partial_t X_h(p_h, t) = v_h(X_h(p_h, t), t).$$

A key property of the basis functions is the *transport property* [DE07a]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_j[\mathbf{x}(t)](X_h(p_h, t))\Big) = 0.$$

Therefore, the *discrete velocity* is simply

$$v_h(x, t) = \sum_{j=1}^N v_j(t)\, \phi_j[\mathbf{x}(t)](x) \quad \text{ for } x \in \Gamma_h\big(\mathbf{x}(t)\big), \qquad \text{with } \ v_j(t) = \dot{x}_j(t).$$

The *discrete material derivative* is defined analogously to the time continuous case, see (1.2).

Semi-discrete problems

The finite element semi-discretisation of the problem (2.3) reads as follows: Find the unknown nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element functions $u_h(\cdot, t) \in S_h[\mathbf{x}(t)]$ with a time-continuous $\partial_h^\bullet u_h(\cdot, t) \in S_h[\mathbf{x}(t)]$ and $v_h(\cdot, t) \in S_h[\mathbf{x}(t)]^3$ such that, for all $\varphi_h(\cdot, t) \in S_h[\mathbf{x}(t)]$ with $\partial_h^\bullet \varphi_h = 0$ and all $\psi_h(\cdot, t) \in S_h[\mathbf{x}(t)]^3$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h[\mathbf{x}]} u_h \varphi_h + \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} u_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \varphi_h = \int_{\Gamma_h[\mathbf{x}]} f(u_h, \nabla_{\Gamma_h[\mathbf{x}]} u_h)\, \varphi_h, \\ &\int_{\Gamma_h[\mathbf{x}]} v_h \cdot \psi_h + \alpha \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} v_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \psi_h \\ &\qquad\qquad + \beta \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} X_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \psi_h = \int_{\Gamma_h[\mathbf{x}]} g(u_h, \nabla_{\Gamma_h[\mathbf{x}]} u_h)\, \nu_{\Gamma_h[\mathbf{x}]} \cdot \psi_h, \end{aligned} \tag{2.7}$$

and

$$\partial_t X_h(p_h,t) = v_h(X_h(p_h,t),t), \qquad p_h \in \Gamma_h^0. \tag{2.8}$$

The initial values for u_h and the nodal vector $\mathbf{x}$ are taken as the exact initial data at the nodes x_j^0 of the triangulation of the given initial surface Γ^0:

$$x_j(0) = x_j^0, \quad u_j(0) = u(x_j^0, 0), \qquad (j = 1, \dots, N).$$

The finite element semi-discretisation of the problem (2.5) reads as follows: Find the unknown nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element functions $u_h(\cdot,t) \in S_h[\mathbf{x}(t)]$ with a time-continuous $\partial_h^\bullet u_h(\cdot,t) \in S_h[\mathbf{x}(t)]$ and $v_h(\cdot,t) \in S_h[\mathbf{x}(t)]^3$ with a time-continuous $\partial_h^\bullet v_h(\cdot,t) \in S_h[\mathbf{x}(t)]^3$ such that, for all $\varphi_h(\cdot,t) \in S_h[\mathbf{x}(t)]$ with $\partial_h^\bullet \varphi_h = 0$ and all $\psi_h(\cdot,t) \in S_h[\mathbf{x}(t)]^3$ with $\partial_h^\bullet \psi_h = 0$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h[\mathbf{x}]} u_h \varphi_h + \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} u_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \varphi_h &= \int_{\Gamma_h[\mathbf{x}]} f(u_h, \nabla_{\Gamma_h[\mathbf{x}]} u_h)\, \varphi_h, \\
\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h[\mathbf{x}]} v_h \cdot \varphi_h + \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} v_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \varphi_h &= \int_{\Gamma_h[\mathbf{x}]} g(u_h, \nabla_{\Gamma_h[\mathbf{x}]} u_h)\, \nu_{\Gamma_h[\mathbf{x}]} \cdot \psi_h,
\end{aligned} \tag{2.9}$$

with the ODE (2.8). The initial values are taken as the exact initial data at the nodes x_j^0 of the triangulation of the given initial surface Γ^0:

$$x_j(0) = x_j^0, \quad u_j(0) = u(x_j^0, 0), \quad \text{and} \quad v_j(0) = v(x_j^0, 0), \qquad (j = 1, \dots, N).$$

Matrix–vector formulation

The column vectors $\mathbf{u} \in \mathbb{R}^N$ and $\mathbf{v} \in \mathbb{R}^{3N}$ collecting the nodal values of the functions u_h and v_h, respectively, and the surface nodal vector $\mathbf{x} \in \mathbb{R}^{3N}$ (omitting the argument t), satisfy a system of differential algebraic equations (DAE).

We define the mass matrix $\mathbf{M}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ and stiffness matrix $\mathbf{A}(\mathbf{x}) \in \mathbb{R}^{N \times N}$ on the surface determined by the nodal vector $\mathbf{x}$:

$$\begin{aligned}
\mathbf{M}(\mathbf{x})|_{jk} &= \int_{\Gamma_h[\mathbf{x}]} \phi_j[\mathbf{x}] \phi_k[\mathbf{x}], \\
\mathbf{A}(\mathbf{x})|_{jk} &= \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} \phi_j[\mathbf{x}] \cdot \nabla_{\Gamma_h[\mathbf{x}]} \phi_k[\mathbf{x}],
\end{aligned} \qquad (j,k = 1, \dots, N).$$

We further let (with the identity matrix $I_3 \in \mathbb{R}^{3\times 3}$)

$$\mathbf{M}^{[3]}(\mathbf{x}) = I_3 \otimes \mathbf{M}(\mathbf{x}) \quad \text{and} \quad \mathbf{A}^{[3]}(\mathbf{x}) = I_3 \otimes \mathbf{A}(\mathbf{x}),$$

and then define

$$\mathbf{K}(\mathbf{x}) = \mathbf{M}^{[3]}(\mathbf{x}) + \alpha \mathbf{A}^{[3]}(\mathbf{x}). \tag{2.10}$$

When no confusion can arise, we write in the following $\mathbf{M}(\mathbf{x})$ for $\mathbf{M}^{[3]}(\mathbf{x})$, $\mathbf{A}(\mathbf{x})$ for $\mathbf{A}^{[3]}(\mathbf{x})$. The right-hand side vectors $\mathbf{f}(\mathbf{x},\mathbf{u}) \in \mathbb{R}^N$ and $\mathbf{g}(\mathbf{x},\mathbf{u}) \in \mathbb{R}^{3N}$ are given by

$$\begin{aligned}
\mathbf{f}(\mathbf{x},\mathbf{u})|_j &= \int_{\Gamma_h[\mathbf{x}]} f(u_h, \nabla_{\Gamma_h[\mathbf{x}]} u_h)\, \phi_j[\mathbf{x}], \\
\mathbf{g}(\mathbf{x},\mathbf{u})|_{3(j-1)+\ell} &= \int_{\Gamma_h[\mathbf{x}]} g(u_h, \nabla_{\Gamma_h[\mathbf{x}]} u_h)\, \big(\nu_{\Gamma_h[\mathbf{x}]}\big)_\ell\, \phi_j[\mathbf{x}],
\end{aligned} \qquad (j = 1, \dots, N,\ \ell = 1,2,3).$$

From (2.7)–(2.8) we then obtain the following coupled DAE system for the nodal values $\mathbf{u}, \mathbf{v}$ and $\mathbf{x}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x})\mathbf{u}\Big) + \mathbf{A}(\mathbf{x})\mathbf{u} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{K}(\mathbf{x})\mathbf{v} + \beta\mathbf{A}(\mathbf{x})\mathbf{x} &= \mathbf{g}(\mathbf{x}, \mathbf{u}), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{2.11}$$

From (2.9)–(2.8) we then obtain the following coupled DAE system for the nodal values $\mathbf{u}, \mathbf{v}$ and $\mathbf{x}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x})\mathbf{u}\Big) + \mathbf{A}(\mathbf{x})\mathbf{u} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x})\mathbf{v}\Big) + \mathbf{A}(\mathbf{x})\mathbf{v} &= \mathbf{g}(\mathbf{x}, \mathbf{u}), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{2.12}$$

Lifts

An arbitrary finite element function w_h on the discrete surface $\Gamma_h[\mathbf{x}]$, with nodal values w_j, is related to the finite element function $\widehat{w}_h$ on the interpolated surface $\Gamma_h[\mathbf{x}^*]$ (here the vector $\mathbf{x}^*(t)$ collects the nodes of the interpolation surface parametrised by $\sum_{j=1}^N X(x_j^0, t)\phi_j[\mathbf{x}^0](\cdot, t)$) with the same nodal values:

$$\widehat{w}_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}^*].$$

The lift between the interpolated surface $\Gamma_h[\mathbf{x}^*]$ and the exact surface $\Gamma(X)$ is defined exactly as before, via the distance function, described in Section 1.2.

The composite lift operator L from finite element functions on $\Gamma_h[\mathbf{x}]$ to functions on $\Gamma(X)$ via $\Gamma_h[\mathbf{x}^*]$ is given by

$$w_h^L = (\widehat{w}_h)^\ell.$$

In particular for the lifted position function we introduce the notation

$$x_h^L(x, t) = X_h^L(q, t) \in \Gamma_h[\mathbf{x}(t)] \quad \text{for} \quad x = X(q, t) \in \Gamma(X(\cdot, t)).$$

2.3. Convergence of finite elements for surface evolution

The finite element semi-discretisation of a surface PDE on a solution-driven surface as specified in (2.1) satisfy the following error bounds, for finite elements of polynomial degree $k \geqslant 2$.

Theorem 2.1 (Theorem 3.1 and Proposition 10.1 of [KLLP17], Appendix F)**.** *Consider the space discretisation* (2.7)–(2.8) *of the coupled problem* (2.1)–(2.2)*, using evolving surface finite elements of polynomial degree $k \geqslant 2$. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for u. Suppose that the problem admits an exact solution (u, v, X) satisfying* (2.6)*.*

Then, there exists $h_0 > 0$ such that for all mesh widths $h \leqslant h_0$ the following error bounds hold over the exact surface $\Gamma(t) = \Gamma(X(\cdot, t))$ for $0 \leqslant t \leqslant T$:

$$\Big(\|u_h^L(\cdot, t) - u(\cdot, t)\|_{L^2(\Gamma(t))}^2 + \int_0^t \|u_h^L(\cdot, s) - u(\cdot, s)\|_{H^1(\Gamma(s))}^2 \,\mathrm{d}s \Big)^{\frac{1}{2}} \leqslant C h^k,$$

and

$$\Big(\int_0^t \|v_h^L(\cdot,s)-v(\cdot,s)\|^2_{H^1(\Gamma(s))^3}\,\mathrm{d}s\Big)^{1/2} \leqslant Ch^k,$$
$$\|x_h^L(\cdot,t)-\mathrm{id}_{\Gamma(t)}\|_{H^1(\Gamma(t))^3} \leqslant Ch^k.$$

The constant $C>0$ is independent of t and h, but depends on bounds of the H^{k+1} norms of the solution (u,v,X), on the local and global Lipschitz constants of f and g, on the regularization parameter $\alpha>0$, on $\beta>0$ and on the length T of the time interval.

Let us note the following things. The last error bound is equivalent to

$$\|X_h^L(\cdot,t)-X(\cdot,t)\|_{H^1(\Gamma^0)^3} \leqslant Ch^k.$$

Moreover, in the case of a function g in (2.1) that only depends on the solution, i.e. that $g=g(u)$, we obtain an error bound for the velocity that is pointwise in time:

$$\|v_h^L(\cdot,t)-v(\cdot,t)\|_{H^1(\Gamma(t))^3} \leqslant Ch^k.$$

Furthermore, note that for $g=0$ the above result gives optimal-order convergence estimates for a regularised mean curvature flow. Convergence results for mean curvature flow are of great interest since the inspiring paper of Dziuk [Dzi90].

By the stability bound of Proposition 10.1 of [KLLP17] and the appropriate defect bounds (analogously to [KLLP17, Section 8], Appendix F), Theorem 2.1 extends to the coupled problem with a dynamic velocity law.

Theorem 2.2 (Section 8 of [KLLP17], Appendix F)**.** *Consider the space discretisation* (2.9)–(2.8) *of the coupled problem* (2.4)–(2.2)*, using evolving surface finite elements of polynomial degree $k\geqslant 2$. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for u and v. Suppose that the problem admits an exact solution (u,v,X) satisfying* (2.6)*.*

Then, there exists $h_0>0$ such that for all mesh widths $h\leqslant h_0$ the following error bounds hold over the exact surface $\Gamma(t)=\Gamma(X(\cdot,t))$ for $0\leqslant t\leqslant T$:

$$\Big(\|u_h^L(\cdot,t)-u(\cdot,t)\|^2_{L^2(\Gamma(t))}+\int_0^t \|u_h^L(\cdot,s)-u(\cdot,s)\|^2_{H^1(\Gamma(s))}\,\mathrm{d}s\Big)^{\frac12} \leqslant Ch^k,$$

and

$$\Big(\|v_h^L(\cdot,t)-v(\cdot,t)\|^2_{L^2(\Gamma(t))^3}+\int_0^t \|v_h^L(\cdot,s)-v(\cdot,s)\|^2_{H^1(\Gamma(s))^3}\,\mathrm{d}s\Big)^{\frac12} \leqslant Ch^k,$$
$$\|x_h^L(\cdot,t)-\mathrm{id}_{\Gamma(t)}\|_{H^1(\Gamma(t))^3} \leqslant Ch^k.$$

The constant $C>0$ is independent of t and h, but depends on bounds of the H^{k+1} norms of the solution (u,v,X), on the local and global Lipschitz constants of f and g, on the parameter $\alpha>0$ and on the length T of the time interval.

Along the proof of both theorems a key issue is to ensure the smallness of the position error of the surfaces in the $W^{1,\infty}$ norm. An H^1 norm error bound in the proofs together with an inverse estimate yield an $O(h^{k-1})$ error bound in the $W^{1,\infty}$ norm, which is small only for surface finite elements of at least degree two, which is why we impose the condition $k\geqslant 2$ in the above result.

The error bounds are proved by clearly separating the issues of consistency and stability.

The main issue in the proofs is to show *stability* in the form of an h-independent bound of the error in terms of the defects. The stability analysis is done in the matrix–vector formulation. Similarly to the previous chapter, it uses energy estimates and some technical lemmas relating different surfaces, for instance transport formulae that relate the mass and stiffness matrices and the coupling terms for different nodal vectors, see [KLLP17, Section 4]. *No geometric estimates* enter in the stability proofs.

The consistency error is the estimates of the defect, which arises on inserting the interpolation of the exact solution into the discretised equation. The defect bounds involve *geometric estimates* that were obtained for the time-dependent case and for higher order finite elements $k \geqslant 2$ in [Kov18], see Section 1.7.

2.4. Linearly implicit full discretisation of surface evolution

Linearly implicit BDF methods

We apply a p-step linearly implicit BDF method for $p \leqslant 5$, as a time discretisation to the DAE system (2.11). For a step size $\tau > 0$, and with $t_n = n\tau \leqslant T$, we determine the approximations $\mathbf{u}^n$ to $\mathbf{u}(t_n)$, $\mathbf{v}^n$ to $\mathbf{v}(t_n)$ and $\mathbf{x}^n$ to $\mathbf{x}(t_n)$ by the fully discrete system of linear equations

$$\begin{aligned}
\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{M}(\widetilde{\mathbf{x}}^{n-j})\mathbf{u}^{n-j} + \mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{u}^n &= \mathbf{f}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n), \\
\mathbf{K}(\widetilde{\mathbf{x}}^n)\mathbf{v}^n + \beta\mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{x}^n &= \mathbf{g}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n), \qquad n \geqslant p, \\
\mathbf{v}^n &= \frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{x}^{n-j},
\end{aligned} \tag{2.13}$$

where the extrapolated position vector $\widetilde{\mathbf{x}}^n$ is defined by

$$\widetilde{\mathbf{x}}^n = \sum_{j=0}^{p-1}\gamma_j \mathbf{x}^{n-1-j}, \qquad n \geqslant p. \tag{2.14}$$

The starting values $\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^{p-1}$ are assumed to be given. They can be precomputed in a way as is usual for multistep methods: using lower-order methods with smaller step sizes, or using an implicit Runge–Kutta method of the same order.

The coefficients are given by $\delta(\zeta) = \sum_{j=0}^{p}\delta_j\zeta^j = \sum_{\ell=1}^{p}\frac{1}{\ell}(1-\zeta)^\ell$ and $\gamma(\zeta) = \sum_{j=0}^{p-1}\gamma_j\zeta^j = (1-(1-\zeta)^p)/\zeta$, see Section 1.3. This classical order p is retained by the linearly implicit variant using the above coefficients γ_j; cf. [AL15, ALL17].

Similarly, linearly implicit BDF discretisation of the DAE system (2.12) reads as

$$\begin{aligned}
\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{M}(\widetilde{\mathbf{x}}^{n-j})\mathbf{u}^{n-j} + \mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{u}^n &= \mathbf{f}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n), \\
\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{M}(\widetilde{\mathbf{x}}^{n-j})\mathbf{v}^{n-j} + \mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{v}^n &= \mathbf{g}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n), \qquad n \geqslant p. \\
\mathbf{v}^n &= \frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{x}^{n-j},
\end{aligned} \tag{2.15}$$

The starting values $\mathbf{x}^i, \mathbf{v}^i$ for $i = 0, \dots, p-1$ are assumed to be given.

Fully discrete convergence bounds

Stability of BDF methods for linear parabolic problems on given evolving surfaces are well understood in [LMV13], see also Chapter 1. The combination of the stability bounds in [LMV13] for the surface PDE combined with stability results obtained in [KL18], together with appropriate defect bounds, yield error bounds for full discretisations of coupled surface-evolution equations.

Theorem 2.3 (Theorem 9.1 of [KL18], Appendix G). *Consider the evolving surface finite element / BDF linearly implicit full discretisation* (2.13) *of the coupled problem* (2.1)–(2.2)*, using finite elements of polynomial degree* $k \geqslant 2$ *and BDF methods of order* $p \leqslant 5$. *We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for* u. *Suppose that the problem admits an exact solution* (u, v, X) *satisfying* (2.6) *and of class* $C^{p+1}([0,T], W^{1,\infty})$. *Suppose further that the starting values are sufficiently accurate.*

Then, there exist $h_0 > 0$, $\tau_0 > 0$ *and* $c_0 > 0$ *such that for all mesh widths* $h \leqslant h_0$ *and step sizes* $\tau \leqslant \tau_0$ *satisfying the mild stepsize restriction* $\tau^p \leqslant c_0 h$, *the following error bounds hold over the exact surface* $\Gamma(t_n) = \Gamma(X(\cdot, t_n))$ *uniformly for* $0 \leqslant t_n = n\tau \leqslant T$:

$$\|(u_h^n)^L - u(\cdot, t_n)\|_{L^2(\Gamma(t_n))} + \Big(\tau \sum_{j=p}^{n} \|(u_h^j)^L - u(\cdot, t_j)\|_{H^1(\Gamma(t_j))}^2\Big)^{1/2} \leqslant C(h^k + \tau^p),$$
$$\|(v_h^n)^L - v(\cdot, t_n)\|_{H^1(\Gamma(t_n))^3} \leqslant C(h^k + \tau^p),$$
$$\|(x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\|_{H^1(\Gamma(t_n))^3} \leqslant C(h^k + \tau^p).$$

The constant $C > 0$ *is independent of* h *and* τ *and* n *with* $n\tau \leqslant T$, *but depends on bounds of higher derivatives of the solution* (u, v, X), *and on the length* T *of the time interval.*

The error estimate for the surface PDE coupled to a dynamic velocity law is also obtained by stability for the surface PDE from [LMV13] and Proposition 8.1 in [KL18], with appropriate defect bounds shown similarly to [KL18, Section 6].

Theorem 2.4 (Theorem 8.1 of [KL18], Appendix G). *Consider the evolving surface finite element / BDF linearly implicit full discretisation* (2.15) *of the coupled problem* (2.4)–(2.2)*, using finite elements of polynomial degree* $k \geqslant 2$ *and BDF methods of order* $p \leqslant 5$. *We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for* u *and* v. *Suppose that the problem admits an exact solution* (u, v, X) *satisfying* (2.6) *and of class* $C^{p+1}([0,T], W^{1,\infty})$. *Suppose further that the starting values are sufficiently accurate.*

Then, there exist $h_0 > 0$, $\tau_0 > 0$ *and* $c_0 > 0$ *such that for all mesh widths* $h \leqslant h_0$ *and step sizes* $\tau \leqslant \tau_0$ *satisfying the mild stepsize restriction* $\tau^p \leqslant c_0 h$, *the following error bounds hold over the exact surface* $\Gamma(t_n) = \Gamma(X(\cdot, t_n))$ *uniformly for* $0 \leqslant t_n = n\tau \leqslant T$:

$$\|(u_h^n)^L - u(\cdot, t_n)\|_{L^2(\Gamma(t_n))} + \Big(\tau \sum_{j=p}^{n} \|(u_h^j)^L - u(\cdot, t_j)\|_{H^1(\Gamma(t_j))}^2\Big)^{1/2} \leqslant C(h^k + \tau^p),$$
$$\|(v_h^n)^L - v(\cdot, t_n)\|_{L^2(\Gamma(t_n))^3} + \Big(\tau \sum_{j=p}^{n} \|(v_h^j)^L - v(\cdot, t_j)\|_{H^1(\Gamma(t_j))^3}^2\Big)^{1/2} \leqslant C(h^k + \tau^p),$$
$$\|(x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\|_{H^1(\Gamma(t_n))^3} \leqslant C(h^k + \tau^p).$$

The constant $C > 0$ *is independent of* h *and* τ *and* n *with* $n\tau \leqslant T$, *but depends on bounds of higher derivatives of the solution* (u, v, X), *and on the length* T *of the time interval.*

The key step of the proofs of the fully discrete theorems is again *stability* and the $W^{1,\infty}$ norm control of the position error of the surfaces. Similarly to the results in the previous chapter, it is shown using energy techniques and the same technical lemmas relating different surfaces. In the dynamic case we again use the G-stability of Dahlquist [Dah78] and the multiplier techniques of Nevanlinna and Odeh [NO81].

In [KL18] mean curvature flow is used in a numerical experiment to study the effect of the regularising parameter $\alpha > 0$.

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Appendix A. Higher order time discretisations with ALE finite elements for parabolic problems on evolving surfaces

IMA Journal of Numerical Analysis (2018) **38**, 460–494
doi: 10.1093/imanum/drw074
Advance Access publication on April 1, 2017

Higher order time discretizations with ALE finite elements for parabolic problems on evolving surfaces

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[Received on 21 July 2014; revised on 16 August 2016]

A linear evolving surface partial differential equation is first discretized in space by an arbitrary Lagrangian Eulerian (ALE) evolving surface finite element method, and then in time either by a Runge–Kutta method, or by a backward difference formula. The ALE technique allows one to maintain the mesh regularity during the time integration, which is not possible in the original evolving surface finite element method. Stability and high order convergence of the full discretizations is shown, for algebraically stable and stiffly accurate Runge–Kutta methods, and for backward differentiation formulas of order less than 6. Numerical experiments are included, supporting the theoretical results.

Keywords: full discretizations; evolving surfaces; ESFEM; ALE; Runge–Kutta methods; BDF.

1. Introduction

There are various approaches to solve parabolic problems on evolving surfaces. A starting point of the finite element approximation of (elliptic) surface partial differential equations (PDEs) is the paper of Dziuk (1988). Later this theory was extended to general parabolic equations on stationary surfaces by Dziuk & Elliott (2007b). They introduced the *evolving surface finite element method* (ESFEM) to discretize parabolic PDEs on moving surfaces and have shown H^1-error estimates, cf. Dziuk & Elliott (2007a). They gave optimal order error estimates in the L^2-norm (see Dziuk & Elliott (2013b). There is a review by Dziuk & Elliott (2013a), which also serves as a rich source of details and references.

Dziuk and Elliott also studied fully discrete methods (see, e.g., Dziuk & Elliott (2012)). The numerical analysis of convergence of full discretizations with higher order time integrators was first studied by Dziuk *et al.* (2012). They proved optimal order convergence for the case of algebraically stable implicit Runge–Kutta (R–K) methods, and Lubich *et al.* (2013) proved optimal convergence for backward differentiation formulas (BDFs).

The ESFEM approach and convergence results were later extended to wave equations on evolving surfaces by Lubich & Mansour (2015) and Mansour (2015). A unified presentation of ESFEM for parabolic problems and wave equations is given in Mansour (2013).

These results are for the Lagrangian case.

As it was pointed out by Dziuk and Elliott, ‘A drawback of our method is the possibility of degenerating grids. The prescribed velocity may lead to the effect, that the triangulation $\Gamma_h(t)$ is distorted.'[1] To resolve this problem Elliott & Styles (2012) proposed an *arbitrary Lagrangian Eulerian* (ALE) ESFEM approach, which in contrast to the (pure Lagrangian) ESFEM method, allows the nodes of the triangulation to move with a velocity that may not be equal to the surface (or material) velocity. They presented numerous examples where smaller errors can be achieved using a *better* mesh.

ALE–FEM for moving domains were investigated by Formaggia & Nobile (1999). They also suggest some possible ways to define the new mesh if the movement of the boundary is given. Bonito *et al.* (2013a,a) proved stability and optimal order *a-priori* error estimates for discontinuous Galerkin time discrete Runge–Kutta–Radau methods of high order.

This article extends the convergence results and techniques of Dziuk *et al.* (2012) for the R–K discretizations and of Lubich *et al.* (2013) for the BDFs (both shown for the Lagrangian case), to the ALE framework.

Elliott & Styles (2012) proposed a fully discrete ALE–ESFEM algorithm to solve parabolic problems on evolving surfaces. Elliott & Venkataraman (2015) proved convergence results for this type of scheme and in addition prove convergence of fully discrete ALE–ESFEM with second-order BDFs. They also give numerous numerical experiments. The primary consideration of the present work is to prove convergence of ALE–ESFEM with higher-order time discretizations. We use different techniques to achieve this, and thus give a new proof for the convergence of the fully discrete method suggested by Elliott & Styles (2012).

We prove stability and convergence of these higher-order time discretizations classes, and also their convergence as a full discretization for evolving surface linear parabolic PDEs when coupled with the ALE–ESFEM as a space discretization. The stability results do not require a time step restriction by powers of the mesh size, i.e., no CFL-type condition is required.

First, the stability of stiffly accurate algebraically stable implicit R–K methods (having the Radau IIA methods in mind) is shown using energy estimates and the algebraic stability as a key property, using some of the basic ideas that appeared in Lubich & Ostermann (1995) for quasilinear parabolic problems.

Secondly, we show stability for the k-step BDFs up to order five. Because of the lack of A-stability of the BDF methods of order greater than two, our proof requires a different technique than Elliott & Venkataraman (2015), namely, we used G-stability results of Dahlquist (1978), and multiplier techniques of Nevanlinna & Odeh (1981). Therefore, we handle all BDF ($k = 1, 2, \dots, 5$) methods at once.

For the fully discrete convergence results, in both cases, the study of the error of a generalized Ritz map, and also for the error in its material derivatives, plays an important role.

In the presentation we focus on the main differences compared to the previous results, and put less emphasis on those parts where minor modifications of the cited proofs are sufficient. In most cases the Lagrangian proof can be repeated in the ALE case, and these are therefore omitted.

Our convergence estimates for BDF 1 and BDF 2 match the ones achieved with a different technique in Elliott & Venkataraman (2015).

This article is organized as follows. In Section 2 we formulate the considered evolving surface parabolic problem and describe the concept of ALE methods together with other basic notions. The ALE weak formulation of the problem is also given. In Section 3 we define the mesh approximating our moving surface and derive the semidiscrete version of the ALE weak form, which is equivalent to a system of ODEs. Then we recall some properties of the evolving matrices and some estimates of

[1] Quoted from Dziuk & Elliott (2007a), Section 7.2.

bilinear forms. We also prove the analogous estimate for the new term appearing in the ALE formulation. The definition of the used generalized Ritz map is also given here. In Section 4 we prove the stability of high order R–K methods applied to the ALE–ESFEM semidiscrete problem and the same results for the BDF methods. Section 5 contains the main results of this article: convergence of the fully discrete methods, ALE–ESFEM together with R–K or BDF method, having a high order convergence both in time. Finally, in Section 6 we present numerical experiments to illustrate our theoretical results.

2. The ALE approach for evolving surface PDEs

In the following we consider a smooth evolving closed hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$, $0 \leq t \leq T$, with $m \leq 3$, which moves with a given smooth velocity v. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla_\Gamma u$ denote the material derivative of u, where ∇_Γ is the tangential gradient given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \mathrm{n}\mathrm{n}$, with unit normal n. We denote by $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ the Laplace–Beltrami operator.

We consider the following linear problem derived by Dziuk & Elliott (2007a):

$$\begin{cases} \partial^\bullet u(x,t) + u(x,t)\nabla_{\Gamma(t)} \cdot v(x,t) - \Delta_{\Gamma(t)} u(x,t) = f(x,t) & \text{on } \Gamma(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad u(x,0) = u_0(x) & \text{on } \Gamma(0). \end{cases} \tag{2.1}$$

Basic and detailed references on evolving surface PDEs are Dziuk & Elliott (2007a, 2013a,b) and Mansour (2013). We are working in the same framework as these references.

For simplicity reasons we set in all sections $f = 0$, since the extension of our results to the inhomogeneous case are straightforward.

An important tool is the Green's formula (on closed surfaces), which takes the form

$$\int_\Gamma \nabla_\Gamma z \cdot \nabla_\Gamma \phi = -\int_\Gamma (\Delta_\Gamma z)\phi.$$

We use Sobolev spaces on surfaces: For a smooth surface Γ we define

$$H^1(\Gamma) = \big\{\eta \in L^2(\Gamma) \mid \nabla_\Gamma \eta \in L^2(\Gamma)^{m+1}\big\},$$

and analogously $H^k(\Gamma)$ for $k \in \mathbb{N}$ (Dziuk & Elliott, 2007a, Section 2.1). Finally, $\mathcal{G}_T$ denotes the space-time manifold, i.e., $\mathcal{G}_T := \cup_{t\in[0,T]} \Gamma(t) \times \{t\}$. We assume that $\mathcal{G}_T \subset \mathbb{R}^{m+2}$ is a smooth hypersurface (with boundary $\partial\mathcal{G}_T = \big(\Gamma(0) \times \{0\}\big) \cup \big(\Gamma(T) \times \{T\}\big)$).

The weak formulation of this problem reads as

Definition 2.1 (weak solution, Dziuk & Elliott (2007a) Definition 4.1) A function $u \in H^1(\mathcal{G}_T)$ is called a *weak solution* of (2.1), if for almost every $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u\partial^\bullet \varphi \tag{2.2}$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(.,0) = u_0$.

For suitable u_0 existence and uniqueness results for (2.2) were obtained by Dziuk & Elliott (2007a, Theorem 4.4) and in a more abstract framework in Alphonse *et al.* (2015, Theorem 3.6) (both works consider inhomogeneous problems).

2.1 *The ALE map and ALE velocity*

We assume that for each $t \in [0, T]$, $T > 0$, $\Gamma^m(t) \subset \mathbb{R}^{m+1}$ is a closed surface. We call a subset $\Gamma^m \subset \mathbb{R}^{m+1}$ a *closed surface*, if Γ is an oriented compact submanifold of codimension 1 without boundary. Moreover we assume $m = 1, 2$ or 3 and that $\Gamma \in C^\infty$, evolving smoothly (cf. Dziuk & Elliott (2013a)). We assume that there exists a smooth map $\mathrm{n}\colon \mathcal{G}_T \to \mathbb{R}^{m+1}$ such that for each t the restriction

$$\mathrm{n}(\,.\,, t)\colon \Gamma(t) \to \mathbb{R}^{m+1}$$

is the smooth normal field on $\Gamma(t)$.

Now we shortly recall the surface description by diffeomorphic parametrization also used by Dziuk & Elliott (2007a) and by Bonito *et al.* (2013a). Another important representation of the surface is based on a signed distance function. For this we refer to Dziuk & Elliott (2007a).

We assume that there exists a smooth map $\Phi\colon \Gamma(0) \times [0, T] \to \mathbb{R}^{m+1}$ which we call a *dynamical system* or *diffeomorphic parametrization*, satisfying that

$$\Phi_t\colon \Gamma(0) \to \Gamma(t), \qquad \Phi_t(y) := \Phi(y, t)$$

is a diffeomorphism for every $t \in [0, T]$. (Φ_t) is called the *flow* of Φ. We observe:

- If $F\colon U \subset \mathbb{R}^m \to \Gamma(0)$ is a smooth parametrization of $\Gamma(0)$, then $F_t := \Phi_t \circ F$ is a smooth parametrization of $\Gamma(t)$, hence the name diffeomorphic parametrization.
- If we interpret $\Gamma(0) \times [0, T] \subset \mathbb{R}^{m+2}$ as a hypersurface, then Φ gives rise to a diffeomorphism

$$\widetilde{\Phi}\colon \Gamma(0) \times [0, T] \to \mathcal{G}_T, \quad \widetilde{\Phi}(y, t) := \big(\Phi_t(y), t\big).$$

The dynamical system Φ defines a (special) vector field v and (special) time derivative $\partial^\bullet$ as follows: First, consider the differential equation (for Φ)

$$\partial_t\Phi(\,.\,, t) = v\big(\Phi(\,.\,, t), t\big), \qquad \Phi(\,.\,, 0) = \mathrm{Id}. \tag{2.3}$$

The unique vector field v is called the *velocity of the surface evolution*, or the *material velocity*. We assume that the material velocity is the same velocity as in problem (2.1). It has the normal component v^{N}. Secondly, the derivative $\partial^\bullet$ is defined as follows (see, e.g., Dziuk & Elliott (2007a), Section 2.2 or Bonito *et al.* (2013a), Section 1): for smooth $f\colon \mathcal{G}_T \to \mathbb{R}$ and $x \in \Gamma(t)$, such that $y \in \Gamma(0)$ for which $\Phi_t(y) = x$, the *material derivative* is defined as

$$\partial^\bullet f(x, t) := \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{(y,t)} f \circ \widetilde{\Phi}. \tag{2.4}$$

Suppose that f has a smooth extension $\bar{f}$ onto an open neighbourhood of $\Gamma(t)$, then by the chain rule the following identity for the material derivative holds:

$$\partial^{\bullet} f(x,t) = \frac{\partial \bar{f}}{\partial t}\bigg|_{(x,t)} + v(x,t) \cdot \nabla \bar{f}(x,t),$$

which is clearly independent of the extension by (2.4). In Section 2.3 Dziuk & Elliott (2013a) has shown how to use the oriented distance function to construct an extension $\bar{f}$.

REMARK 2.2 An evolving surface $\Gamma(t)$ generally posseses many different dynamical systems. Consider, for example, the (constant) evolving surface $\Gamma(t) = \Gamma(0) = S^m \subset \mathbb{R}^{m+1}$ with the two (different) dynamical systems $\Phi(x,t) = x$ and $\Psi(x,t) = \alpha(t)x$, where $\alpha\colon [0,T] \to O(m+1)$ is a smooth curve in the orthogonal matrices.

DEFINITION 2.3 Let $\mathcal{A} \neq \Phi$ be any other dynamical system for $\Gamma(t)$. It is called an *ALE map*. The associated velocity will be denoted by w, which we refer as the *ALE velocity* and finally $\partial^{\mathcal{A}}$ denotes the ALE material derivative.

One can show that for all $t \in [0,T]$ and $x \in \Gamma(t)$

$$v(x,t) - w(x,t) \quad \text{is a tangential vector.} \tag{2.5}$$

The formula for the differentiation of a parameter-dependent surface integral played a decisive role in the analysis of evolving surface problems. In the following lemma we will state its ALE version, together with the connection between the material derivative and the ALE material derivative.

LEMMA 2.4 Let $\Gamma(t)$ be an evolving surface and f be a function defined in $\mathcal{G}_T$ such that all the following quantities exist.

(a) (Leibniz formula Dziuk & Elliott (2007a)/ Reynolds transport identity Bonito *et al.* (2013a)) There holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} f = \int_{\Gamma(t)} \partial^{\mathcal{A}} f + f\, \nabla_{\Gamma(t)} \cdot w. \tag{2.6}$$

(b) There also holds

$$\partial^{\mathcal{A}} f = \partial^{\bullet} f + (w - v) \cdot \nabla_{\Gamma(t)} f. \tag{2.7}$$

Proof. At first we prove (b): consider an extension $\bar{f}$ of f. Use the chain rule for $\partial^{\mathcal{A}} f$ and $\partial^{\bullet} f$ and note the identity (cf. (2.5))

$$(w(.,t) - v(.,t)) \cdot \nabla \bar{f}(.,t) = (w(.,t) - v(.,t)) \cdot \nabla_{\Gamma} f(.,t).$$

To prove (a) use the original Leibniz formula from Dziuk & Elliott (2007a):

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma f = \int_\Gamma \partial^\bullet f + f\,\nabla_\Gamma \cdot v.$$

Now use (b) and Greens identity for surfaces to complete the proof. □

2.2 *Weak formulation*

Now we have everything at our hands to derive the ALE version of the weak form of the evolving surface PDE (2.1).

LEMMA 2.5 (ALE weak solution) The ALE weak solution for an evolving surface PDE is a function $u \in H^1(\mathcal{G}_T)$, if for almost every $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)}u \cdot \nabla_{\Gamma(t)}\varphi + \int_{\Gamma(t)} u(w-v)\cdot\nabla_{\Gamma(t)}\varphi = \int_{\Gamma(t)} u\partial^{\mathcal{A}}\varphi$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(\,.\,,0) = u_0$. If u solves equation (2.2) then u is an ALE weak solution.

Proof. We start by substituting the material derivative by the ALE material derivative in (2.2), using the relation (2.7), connecting the different material derivatives (cf. (2.5)), i.e., by putting

$$\partial^\bullet\varphi = \partial^{\mathcal{A}}\varphi + (v-w)\cdot\nabla_\Gamma\varphi$$

into (2.2), and rearranging the terms, we get the desired formulation. □

3. Semidiscretization: ALE–ESFEM

This section is devoted to the spatial semidiscretization of the parabolic moving surface PDE with the ALE version of the ESFEM. The ESFEM was developed by Dziuk & Elliott (2007a). In the original case the nodes were moving only with the material velocity along the surface, which could lead to degenerated meshes. One can maintain the good properties of the initial mesh by having additional tangential velocity.

The ALE–ESFEM discretization will lead to a system of ordinary differential equations (ODEs) with time-dependent matrices. We will prove the basic properties of those matrices, which will be one of our main tools to prove the stability of time discretizations and convergence of full discretizations. We will also recall the lifting operator and its properties introduced by Dziuk & Elliott (2007a), which enables us to compare functions from the discrete and continuous surface.

3.1 *Basic notations*

First, the initial surface $\Gamma(0)$ is approximated by a triangulated one denoted by $\Gamma_h(0)$, which is given as

$$\Gamma_h(0) := \bigcup_{E(0)\in\mathcal{T}_h(0)} E(0).$$

Let $a_i(0), (i = 1, 2, \dots, N)$ denote the initial nodes lying on the initial continuous surface. Now the nodes are evolved with respect to the ALE map $\mathcal{A}$, i.e., $a_i(t) := \mathcal{A}\big(a_i(0), t\big)$. Obviously they remain on the continuous surface $\Gamma(t)$ for all t. Therefore the smooth surface $\Gamma(t)$ is approximated by the triangulated one denoted by $\Gamma_h(t)$, which is given as

$$\Gamma_h(t) := \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ form an admissible triangulation $\mathcal{T}_h(t)$ with h denoting the maximum diameter. Admissible triangulations were introduced in Dziuk & Elliott (2007a, Section 5.1): every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by ch with $c > 0$, and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$.

The discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)} f := \nabla f - \nabla f \cdot \mathrm{n}_h \mathrm{n}_h = \mathrm{Pr}_h(\nabla f),$$

understood in an element-wise sense, with n_h denoting the normal to $\Gamma_h(t)$ and $\mathrm{Pr}_h := I - \mathrm{n}_h \mathrm{n}_h^T$.

For every $t \in [0, T]$ we define the finite element subspace

$$S_h(t) := \big\{\phi_h \in C(\Gamma_h(t)) \,\big|\, \phi_h|_E \text{ is linear, for all } E \in \mathcal{T}_h(t)\big\}.$$

The piecewise linear moving basis functions χ_j are defined by $\chi_j(a_i(t), t) = \delta_{ij}$ for all $i, j = 1, 2, \dots, N$, and hence

$$S_h(t) = \mathrm{span}\big\{\chi_1(\,.\,, t), \chi_2(\,.\,, t), \dots, \chi_N(\,.\,, t)\big\}.$$

We continue with the definition of the interpolated velocities on the discrete surface $\Gamma_h(t)$:

$$V_h(\,.\,, t) = \sum_{j=1}^{N} v(a_j(t), t)\chi_j(\,.\,, t), \qquad W_h(\,.\,, t) = \sum_{j=1}^{N} w(a_j(t), t)\chi_j(\,.\,, t) \tag{3.1}$$

are the discrete velocity and the discrete ALE velocity, respectively. The discrete material derivative and its ALE version is given by

$$\partial_h^{\bullet}\phi_h = \partial_t\phi_h + V_h \cdot \nabla\phi_h, \qquad \partial_h^{\mathcal{A}}\phi_h = \partial_t\phi_h + W_h \cdot \nabla\phi_h,$$

where $\partial_t\phi_h(x, t_0)$ and $\nabla\phi_h(x, t_0)$ is meant in the following sense: Denote by $\mathcal{G}_h := \bigcup_{t\in[0,T]} \Gamma_h(t) \times \{t\} \subset \mathbb{R}^{m+2}$ the discrete time space manifold, and for simplicity assume that the coefficients of $\phi_h\colon \mathcal{G}_h \to \mathbb{R}$ w.r.t. to the standard finite element basis are smooth in t. Assume that x is lying in the interior of $E(t_0) \subset \Gamma_h(t_0)$ and denote by $E(t)$ the evolution of $E(t_0)$. Finally set $\mathcal{E} := \bigcup_{t\in[0,T]} E(t) \times \{t\}$. For the restricted function $\phi_h|\mathcal{E}$ exists a smooth extension $\bar{\phi}_h$ on a $(m+2)$-dimensional neighborhood of $\mathcal{E}$. We set $\partial_t\phi_h = \partial_t\bar{\phi}_h$ and $\nabla\phi_h = \nabla\bar{\phi}_h$. A straightforward calculation shows that $\partial_h^{\bullet}\phi_h$ is well defined.

In the ALE setting the key *transport property* is the following:

$$\partial_h^{\mathcal{A}}\chi_k = 0 \qquad \text{for} \quad k = 1, 2, \dots, N. \tag{3.2}$$

It can be shown as its non-ALE version (see Dziuk & Elliott (2007a, Proposition 5.4)).

The spatially discrete ALE problem for evolving surfaces is formulated in

PROBLEM 3.1 (Semidiscretization in space) Find $U_h \in S_h(t)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} U_h\phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}U_h \cdot \nabla_{\Gamma_h(t)}\phi_h + \int_{\Gamma_h(t)} U_h(W_h - V_h)\cdot\nabla_{\Gamma_h(t)}\phi_h = \int_{\Gamma_h(t)} U_h \partial_h^{\mathcal{A}}\phi_h, \qquad (\forall \phi_h \in S_h(t)), \tag{3.3}$$

with the initial condition $U_h(\,.\,,0) = U_h^0 \in S_h(0)$ is a sufficient approximation of u_0.

3.2 *The ODE system*

The ODE form of the above problem can be derived by setting

$$U_h(\,.\,,t) = \sum_{j=1}^{N} \alpha_j(t)\chi_j(\,.\,,t) \tag{3.4}$$

and testing by $\phi_h = \chi_k$ for $k = 1, 2, \ldots, N$ in (3.3) and using the transport property for evolving surfaces (3.2).

PROPOSITION 3.2 (ODE system for evolving surfaces) The spatially semidiscrete problem is equivalent to the ODE system for the vector $\alpha(t) = (\alpha_j(t)) \in \mathbb{R}^N$, representing $U_h(\,.\,,t)$,

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\alpha(t)\big) + A(t)\alpha(t) + B(t)\alpha(t) = 0 \\ \hfill \alpha(0) = \alpha_0, \end{cases} \tag{3.5}$$

where $M(t)$ and $A(t)$ are the evolving mass and stiffness matrices defined by

$$M(t)_{kj} = \int_{\Gamma_h(t)} \chi_j\chi_k, \qquad A(t)_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h(t)}\chi_j \cdot \nabla_{\Gamma_h(t)}\chi_k,$$

and the evolving matrix $B(t)$ is given by

$$B(t)_{kj} = \int_{\Gamma_h(t)} \chi_j(W_h - V_h)\cdot\nabla_{\Gamma_h(t)}\chi_k. \tag{3.6}$$

The proof of this proposition is analogous to the corresponding one in Dziuk *et al.* (2012, Section 3).

REMARK 3.3 In the original ESFEM setting there was no direct involvement of velocities, but in the ALE formulation there is. We remark here that since the normal components of the continuous ALE and material velocity are equal, during computations one can work only with the difference of the two discrete velocities. We keep the above formulation to leave the presentation plain and simple.

3.3 *Lifting process*

In the following we define the so called *lift operator*, which was introduced by Dziuk (1988) and further investigated by Dziuk & Elliott (2007a, 2013b). The lift operator can be interpreted as a geometric projection: it projects a finite element function $\eta_h \colon \Gamma_h(t) \to \mathbb{R}$ on the discrete surface $\Gamma_h(t)$ onto a function $\eta_h^l \colon \Gamma(t) \to \mathbb{R}$ on the smooth surface $\Gamma(t)$. Therefore it is crucial for our error estimates.

We assume that there exists an open bounded set $U(t) \subset \mathbb{R}^{m+1}$ such that $\partial U(t) = \Gamma(t)$. The *oriented distance function d* is defined as

$$\mathbb{R}^{m+1} \times [0, T] \to \mathbb{R}, \quad d(x,t) := \begin{cases} \operatorname{dist}\big(x, \Gamma(t)\big), & x \in \mathbb{R}^{m+1} \setminus U(t), \\ -\operatorname{dist}\big(x, \Gamma(t)\big), & x \in U(t). \end{cases}$$

For $\mu > 0$ we define $\mathcal{N}(t)_\mu := \big\{x \in \mathbb{R}^{m+1} \mid \operatorname{dist}\big(x, \Gamma(t)\big) < \mu\big\}$. Clearly $\mathcal{N}(t)_\mu$ is an open neighbourhood of $\Gamma(t)$. Gilbarg & Trudinger (1983) in Lemma 14.16 have shown the following important regularity result about d.

Lemma 3.4 Let $U(t) \subset \mathbb{R}^{m+1}$ be bounded and $\Gamma(t) \in C^k$ for $k \geq 2$. Then there exists a positive constant μ depending on U such that $d \in C^k\big(\mathcal{N}(t)_\mu\big)$.

In Gilbarg & Trudinger (1983, Lemma 14.16) it is also mentioned that μ^{-1} bounds the principal curvatures of $\Gamma(t)$.

In the following we recall the lift operator from Dziuk (1988, equation (2)). For each $x \in \Gamma(t)_\mu$ there exists a unique $p = p(x,t) \in \Gamma(t)$ such that $|x - p| = \operatorname{dist}\big(x, \Gamma(t)\big)$, then x and p are related by the important equation:

$$x = p + \mathrm{n}(p,t)d(x,t). \tag{3.7}$$

We assume that $\Gamma_h(t) \subset \mathcal{N}(t)$. The *lift operator* $\mathcal{L}$ maps a continuous function $\eta_h \colon \Gamma_h \to \mathbb{R}$ onto a function $\mathcal{L}(\eta_h) \colon \Gamma \to \mathbb{R}$ as follows: for every $x \in \Gamma_h(t)$ exists via equation (3.7) a unique $p = p(x,t)$. We set pointwise

$$\mathcal{L}(\eta_h)(p,t) := \eta_h^l(p,t) := \eta_h(x,t).$$

$\mathcal{L}(\eta_h) \colon \Gamma \to \mathbb{R}$ is continuous. If η_h has weak derivatives then $\mathcal{L}(\eta_h)$ also has weak derivatives.

Finally, we have the lifted finite element space

$$S_h^l(t) := \big\{\varphi_h = \phi_h^l \mid \phi_h \in S_h(t)\big\}.$$

3.4 *Properties of the evolving matrices*

Clearly the evolving stiffness matrix is symmetric, positive semi-definite, and the mass matrix is symmetric, positive definite. Through the paper we will work with the norm and semi-norm introduced by Dziuk *et al.* (2012):

$$|z(t)|_{M(t)} = \|Z_h\|_{L^2(\Gamma_h(t))} \qquad \text{and} \qquad |z(t)|_{A(t)} = \|\nabla_{\Gamma_h} Z_h\|_{L^2(\Gamma_h(t))} \tag{3.8}$$

for arbitrary $z(t) \in \mathbb{R}^N$, where $Z_h(\,.\,,t) = \sum_{j=1}^N z_j(t)\chi_j(\,.\,,t)$.

A very important lemma in our analysis is the following.

LEMMA 3.5 (Dziuk *et al.* (2012) Lemma 4.1 and Lubich *et al.* (2013) Lemma 2.2) There are constants μ, κ (independent of h, but depending on $\|\nabla_\Gamma \cdot w\|_{L^\infty(\Gamma(t))}$) such that

$$z^T\big(M(s) - M(t)\big)y \leq (e^{\mu(s-t)} - 1)|z|_{M(t)}|y|_{M(t)} \tag{3.9}$$

$$z^T\big(M^{-1}(s) - M^{-1}(t)\big)y \leq (e^{\mu(s-t)} - 1)|z|_{M^{-1}(t)}|y|_{M^{-1}(t)} \tag{3.10}$$

$$z^T\big(A(s) - A(t)\big)y \leq (e^{\kappa(s-t)} - 1)|z|_{A(t)}|y|_{A(t)} \tag{3.11}$$

for all $y, z \in \mathbb{R}^N$ and $s, t \in [0, T]$.

We will use this lemma with s close to t (usually, $t = s + k\tau$ for some positive integer k independent of the time step τ). Hence, $(e^{\mu(s-t)} - 1) \leq 2\mu(s-t)$ holds. In particular for $y = z$ we have

$$|z|^2_{M(s)} \leq \big(1 + 2\mu(t-s)\big)|z|^2_{M(t)}, \tag{3.12}$$

$$|z|^2_{A(s)} \leq \big(1 + 2\kappa(t-s)\big)|z|^2_{A(t)}. \tag{3.13}$$

The following technical lemma will play a crucial role in this article, while handling the nonsymmetric term.

LEMMA 3.6 Let $y, z \in \mathbb{R}^N$ and $t \in [0, T]$ be arbitrary, then

$$\big|\langle B(t)z|y\rangle\big| \leq c_{\mathcal{A}}|z|_{M(t)}|y|_{A(t)},$$

where the constant $c_{\mathcal{A}} > 0$ depends only on the velocity difference $w - v$.

Proof. Using the definition of the matrix $B(t)$ (see (3.6)) we can write

$$\big|\langle B(t)z|y\rangle\big| = \Big|\int_{\Gamma_h} Z_h(W_h - V_h)\cdot\nabla_{\Gamma_h}Y_h\Big| \leq \|W_h - V_h\|_{L^\infty(\Gamma_h(t))}\int_{\Gamma_h}|Z_h|\,|\nabla_{\Gamma_h}Y_h|.$$

For a first-order finite element function $\varphi_h \in S_h(t)$ it holds $\|\varphi_h\|_{L^\infty(\Gamma_h)} = |\varphi_h(p)|$ for an appropriate node $p \in \Gamma_h(t)$. Hence using (3.1) we can estimate as

$$\|W_h - V_h\|_{L^\infty(\Gamma_h(t))} \leq (m+1)\|w - v\|_{L^\infty(\Gamma(t))}. \tag{3.14}$$

Now apply the Cauchy–Schwarz inequality and using the equivalence of norms over the discrete and continuous surface (cf. Dziuk & Elliott (2007a), Lemma 5.2) to obtain the stated result. □

3.5 *Interpolation estimates*

Let $I_h\colon C(\Gamma(t)) \to S_h^l(t)$ be the lifted Lagrange interpolation operator, where $C(\Gamma(t))$ denotes the space of continuous functions on $\Gamma(t)$; cf. Dziuk & Elliott (2007a) for further details on the interpolation operator. The following interpolation estimate holds.

LEMMA 3.7 For $m \le 3$ and $p \in \{2, \infty\}$ there exists a constant $c > 0$ independent of h and t such that for $u \in W^{2,p}\big(\Gamma(t)\big)$:

$$\|u - I_h u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma(u - I_h u)\|_{L^p(\Gamma(t))} \le ch^2\big(\|\nabla_\Gamma^2 u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma u\|_{L^p(\Gamma(t))}\big).$$

Proof. Since $m \le 3$ and $\Gamma(t)$ is smooth and compact, a Sobolev embedding theorem (cf. Aubin (1998, Theorem 2.20)), implies $W^{2,p}(\Gamma(t)) \subset C(\Gamma(t))$. Hence $I_h u$ is well defined.

The estimate for the case $p = 2$ is stated in Dziuk & Elliott (2007a, Lemma 5.3). On the reference element an interpolation estimate for the case $p = \infty$ was shown in Strang & Fix (1973, Theorem 3.1). Using the estimates appearing in the proof of Dziuk (1988, Lemma 3) and combining these with standard estimates of the reference element technique, we obtain the stated result. □

3.6 *Discrete geometric estimates*

We recall some notions using the lifting process from Dziuk (1988), Dziuk & Elliott (2007a) and Mansour (2013) using the notation of the last reference. By δ_h we denote the quotient between the continuous and discrete surface measures, $\mathrm{d}A$ and $\mathrm{d}A_h$, defined as $\delta_h \mathrm{d}A_h = \mathrm{d}A$. Further, we recall that

$$\mathrm{Pr} := \big(\delta_{ij} - \mathrm{n}_i \mathrm{n}_j\big)_{i,j=1}^N \quad \text{and} \quad \mathrm{Pr}_h := \big(\delta_{ij} - \mathrm{n}_{h,i} \mathrm{n}_{h,j}\big)_{i,j=1}^N$$

are the projections onto the tangent spaces of Γ and Γ_h. Finally $\mathcal{H}$ ($\mathcal{H}_{ij} = \partial_{x_j} \mathrm{n}_i$) is the (extended) Weingarten map. For these quantities we recall some results from Dziuk & Elliott (2007a, 2013b) and Mansour (2013), having the exact same proofs for the ALE case.

LEMMA 3.8 Assume that $\Gamma_h(t)$ and $\Gamma(t)$ satisfy the above, and $\Gamma(t)$ is C^ℓ in time, then we have the estimates

$$\begin{aligned} &\|d\|_{L^\infty(\Gamma_h)} \le ch^2, && \|\mathrm{n} - \mathrm{n}_h\|_{L^\infty(\Gamma_h)} \le ch, && \|1 - \delta_h\|_{L^\infty(\Gamma_h)} \le ch^2, \\ &\|(\partial_h^{\mathcal{A}})^{(\ell)} d\|_{L^\infty(\Gamma_h)} \le ch^2, && \|\,\mathrm{Pr} - \mathrm{Pr}_h \,\mathrm{Pr}\, \mathrm{Pr}_h\,\|_{L^\infty(\Gamma_h)} \le ch^2, \end{aligned}$$

where $(\partial_h^{\mathcal{A}})^{(\ell)}$ denotes the ℓth discrete ALE material derivative.

Proof. The first three inequalities have been proven in Dziuk & Elliott (2013b, Lemma 5.4). The fourth inequality for $\ell \ge 1$ is presented in Mansour (2013, Lemma 6.1). The last inequality has been shown in Dziuk & Elliott (2007a, Lemma 4.1). □

3.7 *Bilinear forms and their properties*

We use the time dependent bilinear forms defined as in Dziuk & Elliott (2013b, Section 3.3): for $z, \varphi \in H^1(\Gamma(t))$ and their discrete analogs for $Z_h, \phi_h \in S_h(t)$:

$$
\begin{aligned}
a(z,\varphi) &= \int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a_h(Z_h,\phi_h) &= \sum_{E\in\mathcal{T}_h} \int_E \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h,\\
m(z,\varphi) &= \int_{\Gamma(t)} z\varphi, & m_h(Z_h,\phi_h) &= \int_{\Gamma_h(t)} Z_h\phi_h,\\
g(w;z,\varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot w) z\varphi, & g_h(W_h;Z_h,\phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot W_h) Z_h\phi_h,\\
b(w;z,\varphi) &= \int_{\Gamma(t)} \mathcal{B}(w)\nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & b_h(W_h;Z_h,\phi_h) &= \sum_{E\in\mathcal{T}_h} \int_E \mathcal{B}_h(W_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h,
\end{aligned}
$$

where the discrete tangential gradients are understood in a piecewise sense and with the matrices

$$
\begin{aligned}
\mathcal{B}(w)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot w) - \big((\nabla_\Gamma)_i w_j + (\nabla_\Gamma)_j w_i\big), && (i,j = 1,2,\ldots,m),\\
\mathcal{B}_h(W_h)_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot W_h) - \big((\nabla_{\Gamma_h})_i (W_h)_j + (\nabla_{\Gamma_h})_j (W_h)_i\big), && (i,j = 1,2,\ldots,m).
\end{aligned}
$$

Following Dziuk & Elliott (2013b), the ALE velocity of lifted material points is defined as follows: Denote by $\mathcal{L}_0 \colon \Gamma_h(0) \to \Gamma(0)$ the Lift for the initial surface and denote by $\mathcal{L}^t \colon \Gamma_h(t) \to \Gamma(t)$ the lift at time t, cf. equation (3.7). In a straightforward way, the ALE dynamical system $\mathcal{A}$ on $\Gamma(t)$ defines a discrete ALE dynamical system $\mathcal{A}_h$ on $\Gamma_h(t)$. $\mathcal{A}_h$ can be interpreted as the interpolation of $\mathcal{A}$. It holds

$$\frac{\mathrm{d}\mathcal{A}_h}{\mathrm{d}t}(x_h^0, t) = W_h\big(\mathcal{A}_h(x_h^0,t),t\big).$$

Define

$$\mathcal{A}_h^l \colon \Gamma_0 \times [0,T] \to \mathbb{R}^{m+1}, \quad (x_0,t) \mapsto \mathcal{L}^t\Big(\mathcal{A}_h\big(\mathcal{L}_0^{-1}(x_0),t\big)\Big).$$

Obviously it holds $\mathcal{A}_h^l(\Gamma_0,t) = \Gamma(t)$. We note that $\mathcal{A}_h^l$ is just curved element wise smooth. Analogous to equation (2.3), we define the corresponding velocity $\Gamma(t) \to \mathbb{R}^{m+1}$, $x \mapsto w_h(x,t)$ via

$$w_h\big(\mathcal{A}_h^l(x_0,t),t\big) := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_h^l(x_0,t). \tag{3.15}$$

Again, as in Section 2.1, the map

$$\widetilde{\mathcal{A}_h^l} \colon \Gamma_0 \times [0,T] \to \mathcal{G}_T$$

is bijective and now, analogous to equation (2.4), we define the corresponding discrete ALE material derivative for the function on the smooth surface as

$$\partial_h^{\mathcal{A}} f(x,t) := \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{(\widetilde{\mathcal{A}_h^l})^{-1}(x,t)} f \circ \widetilde{\mathcal{A}_h^l}.$$

If $\bar{f}$ denotes an extension of f on an open neighborhood of $\mathcal{G}_T$, then we have

$$\partial_h^{\mathcal{A}} f(x,t) = \frac{\partial \bar{f}}{\partial t}\bigg|_{(x,t)} + w_h(x,t) \cdot \nabla \bar{f}(x,t).$$

Lemma 3.9 There exists an h independent constant $c > 0$ such that the following estimate holds

$$\|w - w_h\|_{L^\infty(\mathcal{G}_T)} \leq ch^2.$$

Proof. The proof has been done in Dziuk & Elliott (2013b, Lemma 5.6). For the convenience of the reader we recap the main arguments. Applying the chain rule at the right-hand side of (3.15) leads to

$$\begin{aligned} w_h(x,t) = I_h w(x,t) - d\Big((\mathcal{L}^t)^{-1}(x),t\Big)\Big(\mathcal{H}(x,t) I_h w(x,t) + \frac{\partial \mathrm{n}}{\partial t}(x,t)\Big) \\ - \mathrm{n}(x,t)\Big(\frac{\partial d}{\partial t}\big((\mathcal{L}^t)^{-1}(x),t\big) + \mathrm{n}(x,t)\cdot I_h w(x,t)\Big). \end{aligned}$$

Since $w_h(x,t)\cdot \mathrm{n}(x,t) = w(x,t)\cdot \mathrm{n}(x,t)$ and $\frac{\partial \mathrm{n}}{\partial t}(x,t)\cdot \mathrm{n}(x,t) = 0$, it follows that multiplying the equation above by $\mathrm{n}(x,t)$ yields $\frac{\partial d}{\partial t}\big((\mathcal{L}^t)^{-1}(x),t\big) = w(x,t)\cdot \mathrm{n}(x,t)$. The claim now follows by Lemma 3.8 and Lemma 3.7. □

With the definition of w_h the following semi-discrete transport lemma holds:

Lemma 3.10 (Dziuk & Elliott (2013b) Lemma 4.2, Elliott & Venkataraman (2015) Lemma 3.8) For $z_h,\ \varphi_h,\ \partial_h^{\mathcal{A}} z_h,\ \partial_h^{\mathcal{A}} \varphi_h \in S_h^l(t) \subset H^1(\Gamma)$ we have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} m(z_h,\varphi_h) &= m(\partial_h^{\mathcal{A}} z_h,\varphi_h) + m(z_h,\partial_h^{\mathcal{A}}\varphi_h) + g(w_h; z_h,\varphi_h), \\ \frac{\mathrm{d}}{\mathrm{d}t} a(z_h,\varphi_h) &= a(\partial_h^{\mathcal{A}} z_h,\varphi_h) + a(z_h,\partial_h^{\mathcal{A}}\varphi_h) + b(w_h; z_h,\varphi_h). \end{aligned}$$

Versions of this lemma with continuous non-ALE material derivatives or discrete bilinear forms are also true, see e.g., (Mansour, 2013, Lemma 6.4).

We will need the following estimates between the continuous and discrete bilinear forms.

Lemma 3.11 (Dziuk & Elliott (2013b), Elliott & Venkataraman (2015)) For arbitrary $Z_h, \phi_h \in S_h(t)$, with corresponding lifts $z_h, \varphi_h \in S_h^l(t)$ we have the bound

$$\begin{aligned} \big|m(z_h,\varphi_h) - m_h(Z_h,\phi_h)\big| &\leq ch^2 \|z_h\|_{L^2(\Gamma(t))}\|\varphi_h\|_{L^2(\Gamma(t))}, \\ \big|a(z_h,\varphi_h) - a_h(Z_h,\phi_h)\big| &\leq ch^2 \|\varphi_h\|_{L^2(\Gamma(t))}\|\nabla_\Gamma \varphi_h\|_{L^2(\Gamma(t))}, \\ \big|g(w_h; z_h,\varphi_h) - g_h(W_h; Z_h,\phi_h)\big| &\leq ch^2 \|z_h\|_{L^2(\Gamma(t))}\|\varphi_h\|_{L^2(\Gamma(t))}, \\ \big|m(z_h,(w-v)\cdot\nabla_\Gamma\varphi_h) - m_h(Z_h,(W_h - V_h)\cdot\nabla_{\Gamma_h}\phi_h)\big| &\leq ch^2 \|z_h\|_{L^2(\Gamma(t))}\|\nabla_\Gamma\varphi_h\|_{L^2(\Gamma(t))}. \end{aligned}$$

Proof. For the first three inequalities we refer to Dziuk & Elliott (2013b, Lemma 5.5). For the last inequality observe that

$$
\begin{aligned}
&\Big|m(z_h,(w-v)\cdot\nabla_\Gamma\varphi_h)-m_h(Z_h,(W_h-V_h)\cdot\nabla_{\Gamma_h}\phi_h)\Big| \\
&\quad\le \Big|m\Big(z_h,\Big((w-v)-(W_h^l-V_h^l)\Big)\cdot\nabla_\Gamma\varphi_h\Big)\Big| + \Big|m(z_h,(W_h^l-V_h^l)\cdot\nabla_\Gamma\varphi_h)-m_h(Z_h,(W_h-V_h)\cdot\nabla_{\Gamma_h}\phi_h)\Big| \\
&\quad\le ch^2\|z_h\|_{L^2(\Gamma(t))}\|\nabla_\Gamma\varphi_h\|_{L^2(\Gamma(t))} + \Big|m(z_h,(W_h^l-V_h^l)\cdot\nabla_\Gamma\varphi_h)-m_h(Z_h,(W_h-V_h)\cdot\nabla_{\Gamma_h}\phi_h)\Big|,
\end{aligned}
$$

where we have used Lemma 3.7 for the last inequality. The inequality

$$
\Big|m(z_h,(W_h^l-V_h^l)\cdot\nabla_\Gamma\varphi_h)-m_h(Z_h,(W_h-V_h)\cdot\nabla_{\Gamma_h}\phi_h)\le ch^2\|z_h\|_{L^2(\Gamma(t))}\|\nabla_\Gamma\varphi_h\|_{L^2(\Gamma(t))}
$$

follows from Elliott & Venkataraman (2015, Lemma B.3). □

3.8 *The Ritz map*

We use nearly the same Ritz map introduced in Lubich & Mansour (2015, Definition 8.1), but for the parabolic case a much simpler version suffices:

DEFINITION 3.12 For a given $z\in H^1(\Gamma(t))$ there is a unique $\widetilde{\mathcal{P}}_h z\in S_h(t)$ such that for all $\phi_h\in S_h(t)$, with the corresponding lift $\varphi_h=\phi_h^l$, we have

$$
a_h^*(\widetilde{\mathcal{P}}_h z,\phi_h)=a^*(z,\varphi_h), \tag{3.16}
$$

where $a^*:=a+m$ and $a_h^*:=a_h+m_h$, to make the forms a and a_h positive definite. Then $\mathcal{P}_h z\in S_h^l(t)$ is defined as the lift of $\widetilde{\mathcal{P}}_h z$, i.e., $\mathcal{P}_h z=(\widetilde{\mathcal{P}}_h z)^l$.

REMARK 3.13 The Ritz map in (3.16) is a simplified version of the Ritz map considered in Mansour (2013, Definition 7.1) and Lubich & Mansour (2015, Definition 8.1). The Ritz map in the first reference is actually a more general one then (3.16), since for the choice $\zeta\equiv 0$ there, we obtain our Ritz map (see the proof below).

A different Ritz projection has been used in Dziuk & Elliott (2013b) and in Elliott & Venkataraman (2015, Appendix C). In these works a Ritz projection is defined via

$$
a(\mathcal{P}_h z,\varphi_h)=a(z,\varphi_h),\quad \forall\varphi\in S_h^l(t)\quad\text{and}\quad \int_{\Gamma(t)}\mathcal{P}_h z\,\mathrm{d}A=\int_{\Gamma(t)} z\,\mathrm{d}A=0.
$$

More recently Elliott & Ranner (2015, Section 3.6) defined a different Ritz map via

$$
a_h(\widetilde{\mathcal{P}}_h z,\phi_h)=a(z,\phi^l),\quad \forall\phi\in S_h(t)\quad\text{and}\quad \int_{\Gamma_h(t)}\widetilde{\mathcal{P}}_h z\,\mathrm{d}A_h=\int_{\Gamma(t)} z\,\mathrm{d}A.
$$

LEMMA 3.14 The Ritz map satisfies the bounds, for $0 \leq t \leq T$ and $h \leq h_0$ with a sufficiently small h_0,

$$\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma (z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} \leq ch^2 \|z\|_{H^2(\Gamma(t))},$$

$$\|(\partial_h^{\mathcal{A}})^{(\ell)}(z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma \big((\partial_h^{\mathcal{A}})^{(\ell)}(z - \mathcal{P}_h z)\big)\|_{L^2(\Gamma(t))} \leq c_\ell h^2 \sum_{j=0}^{\ell} \|(\partial^{\mathcal{A}})^{(j)} z\|_{H^2(\Gamma(t))},$$

where the constants c and c_ℓ are independent of h and $t \in [0, T]$.

Proof. Mansour (2013) has defined a Ritz map as follows. For given $\zeta \in H^1(\Gamma(t))$ he defined $\widetilde{\mathcal{P}}_h \colon H^1(\Gamma(t)) \to S_h(t)$ via the equation

$$a_h^*(\widetilde{\mathcal{P}}_h z, \phi_h) = a^*(z, \phi_h^l) + m(\zeta, (v_h - v) \cdot \nabla_\Gamma \phi_h^l) \quad \forall \phi_h \in S_h(t),$$

where v_h plays no role in our setting. Nevertheless, since the proof includes the case $\zeta \equiv 0$ our claim follows from Mansour (2013, Theorem 7.2 and 7.3). □

4. Stability

4.1 *Stability of implicit R–K methods*

We consider an s-stage implicit R–K method for the time discretization of the ODE system (3.5), coming from the ALE–ESFEM space discretization of the parabolic evolving surface PDE.

In the following we extend the stability result for R–K methods of Dziuk *et al.* (2012), Lemma 7.1, to the case of ALE–ESFEM. Apart from the properties of the ALE–ESFEM, the proof is based on the energy estimation techniques of Lubich & Ostermann (1995, Theorem 1.1).

For the convenience of the reader, we recall the method: for simplicity, we assume equidistant time steps $t_n := n\tau$, with step size τ. Our results can be straightforwardly extended to the case of nonuniform time steps. The s-stage implicit R–K method, defined by the given Butcher tableau.

$$\begin{array}{c|c} (c_i) & (a_{ij}) \\ \hline & (b_i) \end{array} \qquad \text{for} \quad i, j = 1, 2, \ldots, s$$

applied to the system (3.5):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\alpha(t)\big) + A(t)\alpha(t) + B(t)\alpha(t) = 0 \\ \qquad\qquad\qquad\qquad \alpha(0) = \alpha_0 \qquad \big(U_h^0 = \sum_{j=1}^N \alpha_{0,j}\chi_j(\,.\,,0)\big) \end{cases}$$

reads as

$$M_{ni}\alpha_{ni} = M_n\alpha_n + \tau \sum_{j=1}^{s} a_{ij}\dot{\alpha}_{nj} \qquad \text{for} \quad i = 1, 2, \ldots, s \tag{4.1a}$$

$$M_{n+1}\alpha_{n+1} = M_n\alpha_n + \tau \sum_{i=1}^{s} b_i\dot{\alpha}_{ni}, \tag{4.1b}$$

where the internal stages satisfy

$$0 = \dot{\alpha}_{ni} + B_{ni}\alpha_{ni} + A_{ni}\alpha_{ni} \qquad \text{for} \quad i = 1, 2, \ldots, s \tag{4.1c}$$

with $A_{ni} := A(t_n + c_i\tau)$, $B_{ni} := B(t_n + c_i\tau)$, $M_{ni} := M(t_n + c_i\tau)$ and $M_{n+1} := M(t_{n+1})$. Here $\dot{\alpha}_{ni}$ is not a derivative but a suggestive notation.

We recall that $U_h(\,.\,, t) = \sum_{j=1}^{N} \alpha_j(t)\chi_j(\,.\,, t)$ for the semidiscrete case from Section 3.2 and for the fully discrete case we define $U_h^n = \sum_{j=1}^{N} \alpha_{n,j}\chi_j(\,.\,, t_n)$.

Assumption 4.1 We assume that

- The method has stage order $q \geq 1$ and classical order $p \geq q + 1$.
- The coefficient matrix (a_{ij}) is invertible; the inverse will be denoted by upper indices (a^{ij}).
- The method is *algebraically stable*, i.e., $b_j > 0$ for $j = 1, 2, \ldots, s$ and the following matrix is positive semi-definite:

$$\big(b_i a_{ij} - b_j a_{ji} - b_i b_j\big)_{i,j=1}^{s}. \tag{4.2}$$

- The method is *stiffly accurate*, i.e., for $j = 1, 2, \ldots, s$ it holds

$$b_j = a_{sj}, \qquad \text{and} \qquad c_s = 1. \tag{4.3}$$

Instead of (3.5), let us consider the following perturbed equation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\widetilde{\alpha}(t)\big) + A(t)\widetilde{\alpha}(t) + B(t)\widetilde{\alpha}(t) = M(t)r(t), \\ \hfill \widetilde{\alpha}(0) = \widetilde{\alpha}_0. \end{cases} \tag{4.4}$$

The substitution of the true solution $\widetilde{\alpha}(t)$ of the perturbed problem into the R–K method yields the defects Δ_{ni} and δ_{ni}, by setting $e_n = \alpha_n - \widetilde{\alpha}(t_n)$, $E_{ni} = \alpha_{ni} - \widetilde{\alpha}(t_n + c_i\tau)$ and $\dot{E}_{ni} = \dot{\alpha}_{ni} - \dot{\widetilde{\alpha}}(t_n + c_i\tau)$, again $\dot{E}_{ni}$ is not a derivative. Then by subtraction the following *error equations* hold:

$$M_{ni}E_{ni} = M_n e_n + \tau \sum_{j=1}^{s} a_{ij}\dot{E}_{nj} - \Delta_{ni} \qquad \text{for} \quad i = 1, 2, \ldots, s, \tag{4.5a}$$

$$M_{n+1}e_{n+1} = M_n e_n + \tau \sum_{i=1}^{s} b_i\dot{E}_{ni} - \delta_{n+1}, \tag{4.5b}$$

where the internal stages satisfy:

$$\dot{E}_{ni} + A_{ni}E_{ni} + B_{ni}E_{ni} = -M_{ni}r_{ni} \qquad \text{for} \quad i = 1, 2, \ldots, s, \tag{4.5c}$$

with $r_{ni} := r(t_n + c_i\tau)$.

Similar to Dziuk *et al.* (2012, Lemma 7.1) or Mansour (2013, Lemma 3.1), we present a stability estimate (such that the choice of τ is independent of h) for the above class of R–K methods. Since the method (4.1) and the error equation (4.5) both involve only matrices and vectors, we first establish this stability estimate in terms of nodal error vectors with corresponding time-dependent norms (3.8). Using (3.8) this estimate can be translated into L^2- and H^1-norms of the corresponding finite element error functions. This result will be related to the norms of U_h, through the error, later in Theorems 5.2 and 5.4.

LEMMA 4.2 For an s-stage implicit R–K method satisfying Assumption 4.1, there exists a $\tau_0 > 0$, depending only on the constants μ and κ, such that for $\tau \le \tau_0$ and $t_n = n\tau \le T$, that the error e_n is bounded by

$$|e_n|_{M_n}^2 + \tau \sum_{k=1}^{n} |e_k|_{A_k}^2 \le C\Big\{ |e_0|_{M_0}^2 + \tau \sum_{k=1}^{n-1}\sum_{i=1}^{s} \|M_{ki} r_{ki}\|_{*,t_{ki}}^2 + \tau \sum_{k=1}^{n} |{}^{\delta_k}/{}_{\tau}|_{M_k}^2 \\ + \tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \Big(|M_{ki}^{-1}\Delta_{ki}|_{M_{ki}}^2 + |M_{ki}^{-1}\Delta_{ki}|_{A_{ki}}^2 \Big) \Big\},$$

where $\|w\|_{*,t}^2 = w^T(A(t) + M(t))^{-1}w$. The constant C is independent of h, τ and n, but depends on μ, κ, T and on the norm of the difference of the velocities. The constant τ_0 depends on the ALE velocity (see Lemma 3.5).

Proof. (a) By using (4.5a)–(4.5c) and algebraic stability (4.2) the following inequality holds for the ALE setting:

$$|e_{n+1}|_{M_{n+1}}^2 \le (1 + 2\mu\tau)|e_n|_{M_n}^2 + 2\tau \sum_{i=1}^{s} b_i \,\langle \dot{E}_{ni} | M_{n+1}^{-1} | M_{ni}E_{ni} + \Delta_{ni} \rangle \\ + \tau |E_{n+1}|_{M_{n+1}}^2 + (1+3\tau)\tau \left| {}^{\delta_{n+1}}/{}_{\tau} \right|_{M_{n+1}^{-1}}^2 . \tag{4.6}$$

We want to estimate the second term on the right-hand side of (4.6). Obviously the equation

$$\langle \dot{E}_{ni} | M_{n+1}^{-1} | M_{ni}E_{ni} + \Delta_{ni} \rangle = \langle \dot{E}_{ni} | M_{ni}^{-1} | M_{ni}E_{ni} + \Delta_{ni} \rangle \\ + \langle \dot{E}_{ni} | M_{n+1}^{-1} - M_{ni}^{-1} | M_{ni}E_{ni} + \Delta_{ni} \rangle \tag{4.7}$$

holds. The second term on the right-hand side of (4.7) can be estimated by (cf. (Mansour, 2013, Lemma 3.1, (3.14))):

$$\langle \dot{E}_{ni} | M_{n+1}^{-1} - M_{ni}^{-1} | M_{ni}E_{ni} + \Delta_{ni} \rangle \le C\Big\{ |e_n|_{M_n}^2 + \sum_{j=1}^{s} |E_{nj}|_{M_{nj}}^2 + |\Delta_{nj}|_{M_{nj}^{-1}}^2 \Big\}. \tag{4.8}$$

(b) We have to modify the estimation of the first term on the right-hand side of (4.7). Using the definition of internal stages (4.5c), we have

$$\langle \dot{E}_{ni} | M_{ni}^{-1} | M_{ni}E_{ni} + \Delta_{ni} \rangle = - |E_{ni}|_{A_{ni}}^2 - \langle M_{ni} r_{ni} | E_{ni} + M_{ni}^{-1}\Delta_{ni} \rangle \\ - \langle E_{ni} | A_{ni} | M_{ni}^{-1}\Delta_{ni} \rangle - \langle B_{ni}E_{ni} | E_{ni} + M_{ni}^{-1}\Delta_{ni} \rangle \,. \tag{4.9}$$

The last term can be estimated by Lemma 3.6 as

$$\begin{aligned}|\langle B_{ni}E_{ni}|E_{ni}+M_{ni}^{-1}\varDelta_{ni}\rangle| &\leq |\langle B_{ni}E_{ni}|E_{ni}\rangle| + |\langle B_{ni}E_{ni}|M_{ni}^{-1}\varDelta_{ni}\rangle| \\ &\leq C|E_{ni}|_{M_{ni}}|E_{ni}|_{A_{ni}} + C|E_{ni}|_{M_{ni}}|M_{ni}^{-1}\varDelta_{ni}|_{A_{ni}} \\ &\leq C|E_{ni}|_{M_{ni}}^2 + \frac{1}{4}|E_{ni}|_{A_{ni}}^2 + C|E_{ni}|_{M_{ni}}^2 + C|M_{ni}^{-1}\varDelta_{ni}|_{A_{ni}}^2. \qquad (4.10)\end{aligned}$$

While the other terms can be estimated by the following inequality (shown in Lemma 3.1 in Mansour (2013)):

$$\begin{aligned}&-|E_{ni}|_{A_{ni}}^2 + |\langle M_{ni}r_{ni}|E_{ni}+M_{ni}^{-1}\varDelta_{ni}\rangle| + |\langle E_{ni}|A_{ni}|M_{ni}^{-1}\varDelta_{ni}\rangle| \\ &\leq -\frac{1}{2}|E_{ni}|_{A_{ni}}^2 + \frac{1}{4}|E_{ni}|_{M_{ni}}^2 + C\big(|M_{ni}^{-1}\varDelta_{ni}|_{M_{ni}}^2 + |M_{ni}^{-1}\varDelta_{ni}|_{A_{ni}}^2\big). \qquad (4.11)\end{aligned}$$

We continue to estimate the right-hand side of (4.9) with (4.10), (4.11) and arrive to

$$\langle \dot{E}_{ni}|M_{n+1}^{-1}|M_{ni}E_{ni}+\varDelta_{ni}\rangle \leq -\frac{1}{4}|E_{ni}|_{A_{ni}}^2 + C\big(|E_{ni}|_{M_{ni}}^2 + |M_{ni}^{-1}\varDelta_{ni}|_{M_{ni}}^2 + |M_{ni}^{-1}\varDelta_{ni}|_{A_{ni}}^2\big). \qquad (4.12)$$

(c) Now we return to the main inequality (4.6), consider equation (4.9) and plug in the inequalities (4.8) and (4.12) to get

$$\begin{aligned}|e_{n+1}|_{M_{n+1}}^2 - |e_n|_{M_n}^2 + \frac{1}{4}\tau\sum_{i=1}^s b_i|E_{ni}|_{A_{ni}}^2 \leq C\tau\Big\{&|e_n|_{M_n}^2 + \sum_{j=1}^s |E_{nj}|_{M_{nj}}^2 + \|M_{nj}r_{nj}\|_{*,nj}^2 \\ &+ \sum_{j=1}^s\big(|M_{nj}^{-1}\varDelta_{nj}|_{M_{nj}}^2 + |M_{nj}^{-1}\varDelta_{nj}|_{A_{nj}}^2\big) + \big|\delta_{n+1}/\tau\big|_{M_{n+1}^{-1}}^2\Big\}. \qquad (4.13)\end{aligned}$$

(d) Next we estimate $|E_{nj}|_{M_{nj}}^2$, in Mansour (2013, Lemma 3.1) one can find the estimate:

$$|E_{ni}|_{M_{ni}}^2 \leq C\Big(|e_n|_{M_n}^2 + \tau\sum_{j=1}^s a_{ij}\,\langle \dot{E}_{nj}|E_{ni}\rangle + |M_{ni}^{-1}\varDelta_{ni}|_{M_{ni}}^2\Big). \qquad (4.14)$$

We have to estimate $\langle \dot{E}_{nj}|E_{ni}\rangle$, with equation (4.5c) we get

$$\langle \dot{E}_{nj}|E_{ni}\rangle = -\,\langle E_{nj}|A_{nj}|E_{ni}\rangle - \langle M_{nj}r_{nj}|E_{ni}\rangle - \langle B_{nj}E_{nj}|E_{ni}\rangle. \qquad (4.15)$$

The following inequalities can be shown easily using Young's-inequality (ε will be chosen later) and Cauchy–Schwarz inequality:

$$\begin{aligned}&-\langle E_{nj}|A_{nj}|E_{ni}\rangle \leq C(\kappa)\big(|E_{nj}|_{A_{nj}}^2 + |E_{ni}|_{A_{ni}}^2\big), \\ &-\langle B_{nj}E_{nj}|E_{ni}\rangle \leq \varepsilon|E_{nj}|_{M_{nj}}^2 + \frac{1}{4\varepsilon}C(\kappa)|E_{ni}|_{A_{ni}}^2 \\ &-\langle M_{nj}r_{nj}|E_{ni}\rangle \leq C(\mu,\kappa)\Big(\frac{1}{4\varepsilon}\|M_{nj}r_{nj}\|_{*,nj}^2 + \varepsilon\big(|E_{ni}|_{M_{ni}}^2 + |E_{ni}|_{A_{ni}}^2\big)\Big).\end{aligned}$$

Using the above three inequalities to estimate (4.15), we get

$$\langle \dot{E}_{nj} | E_{ni} \rangle \le C(\mu, \kappa) \Big(\varepsilon |E_{ni}|_{M_{ni}}^2 + C(\varepsilon) |E_{ni}|_{A_{ni}}^2 + |E_{nj}|_{A_{nj}}^2 + C(\varepsilon) \| M_{nj} r_{nj} \|_{*,nj}^2 \Big). \tag{4.16}$$

Using this for a sufficiently small ε (independent of τ) we can proceed by estimating (4.14) further as

$$|E_{ni}|_{M_{ni}}^2 \le C\Big(|e_n|_{M_n}^2 + \tau \sum_{j=1}^{s} a_{ij} \big(|E_{nj}|_{A_{nj}}^2 + \| M_{nj} r_{nj} \|_{*,nj}^2 \big) + |M_{ni}^{-1} \Delta_{ni}|_{M_{ni}}^2 \Big).$$

(e) Now for a sufficiently small τ we can use the above inequality to estimate (4.13) to

$$\begin{aligned} |e_{n+1}|_{M_{n+1}}^2 - |e_n|_{M_n}^2 + \frac{1}{8} \tau \sum_{i=1}^{s} b_i |E_{ni}|_{A_{ni}}^2 &\le C\tau \Big\{ |e_n|_{M_n}^2 + \sum_{i=1}^{s} \| M_{ni} r_{ni} \|_{*,ni}^2 \\ &+ \sum_{i=1}^{s} \big(|M_{ni}^{-1} \Delta_{ni}|_{M_{ni}}^2 + |M_{ni}^{-1} \Delta_{ni}|_{A_{ni}}^2 \big) + \big| {}^{\delta_{n+1}}/_{\tau} \big|_{M_{n+1}^{-1}}^2 \Big\}. \end{aligned}$$

Summing up over n and applying a discrete Gronwall inequality yields the desired result. □

4.2 *Stability of BDFs*

We apply a BDF as a temporal discretization to the ODE system (3.5), coming from the ALE–ESFEM space discretization of the parabolic evolving surface PDE.

In the following we extend the stability result for BDF methods of Lubich *et al.* (2013, Lemma 4.1) to the case of ALE–ESFEM. Apart from the properties of the ALE–ESFEM, the proof is based on the G-stability theory of Dahlquist (1978) and the multiplier technique of Nevanlinna & Odeh (1981). We will prove that the fully discrete method is stable for the k-step BDF methods up to order five. Again the stability holds without a CFL-type condition.

We recall the k-step BDF method, applied to the ODE system (3.5), with step size $\tau > 0$ and given starting values $\alpha_{n-k}, \dots, \alpha_{n-1}$:

$$\frac{1}{\tau} \sum_{j=0}^{k} \delta_j M(t_{n-j}) \alpha_{n-j} + A(t_n) \alpha_n + B(t_n) \alpha_n = 0, \qquad (n \ge k), \tag{4.17}$$

where the coefficients of the method is given by $\delta(\zeta) = \sum_{j=1}^{k} \delta_j \zeta^j = \sum_{\ell=1}^{k} \frac{1}{\ell} (1-\zeta)^\ell$, while the initial values are $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$. Again U_h and α is related through (3.4). The method is known to be 0-stable for $k \le 6$ (but not A-stable for $k \ge 3$) and have order k; for more details we refer to Hairer & Wanner (1996, Chapter V.).

Instead of (3.5), let us consider again the perturbed problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} \big(M(t) \tilde{\alpha}(t) \big) + A(t) \tilde{\alpha}(t) + B(t) \tilde{\alpha}(t) = M(t) r(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \tilde{\alpha}(0) = \tilde{\alpha}_0. \end{cases} \tag{4.18}$$

By substituting the true solution $\tilde{\alpha}(t)$ of the perturbed problem into the BDF method (4.17), we obtain

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\tilde{\alpha}_{n-j} + A(t_n)\tilde{\alpha}_n + B(t_n)\tilde{\alpha}_n = -d_n, \qquad (n \geq k).$$

Then by introducing the error $e_n = \alpha_n - \tilde{\alpha}(t_n)$, multiplying by τ, and by subtraction we have the error equation

$$\sum_{j=0}^{k}\delta_j M_{n-j}e_{n-j} + \tau A_n e_n + \tau B_n e_n = \tau d_n, \qquad (n \geq k). \tag{4.19}$$

We recall two important preliminary results.

Lemma 4.3 (Dahlquist (1978)) Let $\delta(\zeta)$ and $\mu(\zeta)$ be polynomials of degree at most k (at least one of them of exact degree k) that have no common divisor. Let $\langle\,.\,|\,.\,\rangle$ be an inner product on $\mathbb{R}^N$ with associated norm $\|\,.\,\|$. If

$$\mathrm{Re}\frac{\delta(\zeta)}{\mu(\zeta)} > 0, \qquad \text{for} \quad |\zeta| < 1,$$

then there exists a symmetric positive definite matrix $G = (g_{ij}) \in \mathbb{R}^{k\times k}$ and real $\gamma_0, \ldots, \gamma_k$ such that for all $v_0, \ldots, v_k \in \mathbb{R}^N$

$$\Big\langle \sum_{i=0}^{k}\delta_i v_{k-i}\Big| \sum_{i=0}^{k}\mu_i v_{k-i}\Big\rangle = \sum_{i,j=1}^{k} g_{ij}\langle v_i\,|\,v_j\rangle - \sum_{i,j=1}^{k} g_{ij}\langle v_{i-1}\,|\,v_{j-1}\rangle + \Big\|\sum_{i=0}^{k}\gamma_i v_i\Big\|^2$$

holds.

Together with this result, the case $\mu(\zeta) = 1 - \eta\zeta$ will play an important role:

Lemma 4.4 (Nevanlinna & Odeh (1981)) If $k \leq 5$, then there exists $0 \leq \eta < 1$ such that for $\delta(\zeta) = \sum_{\ell=1}^{k}\frac{1}{\ell}(1-\zeta)^\ell$,

$$\mathrm{Re}\frac{\delta(\zeta)}{1-\eta\zeta} > 0, \qquad \text{for} \quad |\zeta| < 1.$$

The smallest possible values of η is found to be $\eta = 0,\ 0,\ 0.0836,\ 0.2878,\ 0.8160$ for $k = 1, 2, \ldots, 5$, respectively.

We now state and prove the analogous stability result for the BDF methods. Again using (3.8), this estimate can be translated into L^2- and H^1-norms of the corresponding finite element error functions, see later in Theorem 5.5.

LEMMA 4.5 For a k-step BDF method with $k \leq 5$, there exists a $\tau_0 > 0$, depending only on the constants μ and κ, such that for $\tau \leq \tau_0$ and $t_n = n\tau \leq T$, that the error e_n is bounded by

$$|e_n|_{M_n}^2 + \tau \sum_{j=k}^{n} |e_j|_{A_j}^2 \leq C\tau \sum_{j=k}^{n} \|d_j\|_{*,t_j}^2 + C \max_{0\leq i\leq k-1} |e_i|_{M_i}^2,$$

where $\|w\|_{*,t}^2 = w^T(A(t) + M(t))^{-1}w$. The constant C is independent of h, τ and n, but depends on μ, κ, T and on the norm of the difference of the velocities. The constant τ_0 depends on the ALE velocity (see Lemma 3.5).

Proof. Our proof follows the one of Lemma 4.1 in Lubich *et al.* (2013).

(a) The starting point of the proof is the following reformulation of the error equation (4.19)

$$M_n \sum_{j=0}^{k} \delta_j e_{n-j} + \tau A_n e_n + \tau B_n e_n = \tau d_n + \sum_{j=1}^{k} \delta_j \big(M_n - M_{n-j}\big) e_{n-j}$$

and using a modified energy estimate. We multiply both sides by $e_n - \eta e_{n-1}$, for $n \geq k+1$, which gives us:

$$I_n + II_n = III_n + IV_n - V_n,$$

where

$$\begin{aligned}
I_n &= \Big\langle \sum_{j=0}^{k} \delta_j e_{n-j} \big| M_n \big| e_n - \eta e_{n-1} \Big\rangle, \\
II_n &= \tau \big\langle e_n | A_n | e_n - \eta e_{n-1} \big\rangle, \\
III_n &= \tau \langle d_n | e_n - \eta e_{n-1} \rangle, \\
IV_n &= \sum_{j=1}^{k} \langle e_{n-j} | M_n - M_{n-j} | e_n - \eta e_{n-1} \rangle, \\
V_n &= \tau \langle e_n | B_n | e_n - \eta e_{n-1} \rangle.
\end{aligned}$$

(b) The estimates of I_n, II_n, III_n and IV_n are the same as in the proof of Lubich *et al.* (2013, Lemma 4.1). For the convenience of the reader, we repeat them:

$$\begin{aligned}
I_n &\geq |E_n|_{G,n}^2 - |E_{n-1}|_{G,n}^2, \\
II_n/\tau &\geq \frac{2-\eta}{2} |e_n|_{A_n}^2 - c\eta |e_{n-1}|_{A_{n-1}}^2, \\
|III_n|/\tau &\leq c \frac{1}{1-\eta} \|d_n\|_{*,n}^2 + \frac{1-\eta}{2} \big(\varepsilon |e_n|_{A_n}^2 + |e_n|_{M_n}^2\big) \\
&\quad + (1-\eta) c \big(|e_{n-1}|_{A_{n-1}}^2 + |e_{n-1}|_{M_{n-1}}^2\big), \\
|IV_n|/\tau &\leq c \big(|E_n|_{G,n}^2 + |E_{n-1}|_{G,n}^2\big).
\end{aligned}$$

We note that during the estimation of III_n we used Young's inequality with sufficiently small (τ independent) ε.

The nonsymmetric term V_n is estimated using Lemma 3.6 and Young's inequality (with sufficiently small ε, independent of τ):

$$\begin{aligned}|V_n| &\leq C\tau|e_n|_{M_n}\big(|e_n|_{A_n} + \eta|e_{n-1}|_{A_{n-1}}\big)\\ &= C\tau|e_n|_{M_n}|e_n|_{A_n} + C\eta\tau|e_n|_{M_n}|e_{n-1}|_{A_{n-1}}\\ &\leq \tau C\frac{1}{\varepsilon}|e_n|_{M_n}^2 + \varepsilon\tau|e_n|_{A_n}^2 + \tau C\frac{1}{\varepsilon}|e_n|_{M_n}^2 + \varepsilon\eta^2\tau|e_{n-1}|_{A_{n-1}}^2.\end{aligned}$$

(c) Combining all estimates, choosing a sufficiently small ε (independently of τ), and summing up gives, for $\tau \leq \tau_0$ and for $k \geq n+1$:

$$|E_n|_{G,n}^2 + (1-\eta)\frac{\tau}{8}\sum_{j=k+1}^{n}|e_j|_{A_j}^2 \leq C\tau\sum_{j=k}^{n-1}|E_j|_{G,j}^2 + C\tau\sum_{j=k+1}^{n}\|d_j\|_{*,t_j}^2 + C\eta^2\tau|e_k|_{A_k}^2, \tag{4.20}$$

where $E_n = (e_n, \ldots, e_{n-k+1})$ and the $|E_n|_{G,n}^2 := \sum_{i,j=1}^{k} g_{ij}\langle e_{n-k+1}|M_n|e_{n-k+j}\rangle$.

(d) To achieve the stated result we have to estimate the extra term $|e_k|_{M_k}^2 + \tau|e_k|_{A_k}^2$. For that we take the inner product of the error equation for $n = k$ with e_k to obtain

$$\delta_0|e_k|_{M_k}^2 + \tau|e_k|_{A_k}^2 = \tau\langle d_k \mid e_k\rangle - \sum_{j=1}^{k}\delta_j\langle M_{k-j}e_{k-j} \mid e_k\rangle + \tau|\langle e_k \mid B_k \mid e_k\rangle|.$$

Then the use of Lemma 3.6 and Young's inequality (again with sufficiently small ε) and (3.9) yields

$$|e_k|_{M_k}^2 + \tau|e_k|_{A_k}^2 \leq C\tau\|d_k\|_{*,t_k}^2 + C\max_{0\leq i\leq k-1}|e_i|_{M_i}^2.$$

Similarly as in Lubich *et al.* (2013, Lemma 4.1), using the discrete Gronwall inequality for (4.20) and the above estimate concludes the result. □

5. Error bounds for the fully discrete solutions

We start by connecting the stability results of the previous section with the continuous solution of the parabolic problem. Then using the Ritz map of u we will show the convergence of the error which – together with the stability results – leads us to our main results. We will prove that the full discretizations, ALE–ESFEM coupled with R–K or BDF methods converges. The convergence does not require a bound on τ in terms of h.

5.1 *Bound of the semidiscrete residual*

Before turning to the fully discrete problem we show error bounds for the semidiscretization.

Since the stability analysis only uses the matrix-vector formulation (4.4), (4.18), but not the semidiscrete weak form, we follow Lubich *et al.* (2013, Section 5), using the Ritz map to define the finite element residual,

$$R_h(.,t) = \sum_{j=1}^{N} r_j(t)\chi_j(.,t) \in S_h(t),$$

by duality pointwise in time, as follows. Let

$$\int_{\Gamma_h(t)} R_h(.,t)\phi_h = L_t(\phi_h) \qquad \text{for all } \phi_h \in S_h(t), \tag{5.1}$$

where, for a fixed $t \in [0,T]$, the linear functional $L_t : S_h(t) \to \mathbb{R}$ is defined as follows: for a given finite element function

$$\phi_h = \sum_{j=1}^{N} c_j \chi_j(.,t) \in S_h(t)$$

define the temporal extension $\varphi_h(s) \in S_h(s)$ as the finite element function with the same nodal values

$$\varphi_h(s) = \sum_{j=1}^{N} c_j \chi_j(.,s) \in S_h(s) \qquad (s \in [0,T]).$$

Then, $\partial_h^{\mathcal{A}} \varphi_h(s) = 0$ for all s, by the transport property (3.2) of the basis functions.

We now define

$$\begin{aligned} L_t(\phi_h) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h(t)} \widetilde{\mathcal{P}}_h u(.,t)\varphi_h(.,t) + \int_{\Gamma_h(t)} \nabla_{\Gamma_h}(\widetilde{\mathcal{P}}_h u)(.,t) \cdot \nabla_{\Gamma_h}\varphi_h(.,t) \\ + \int_{\Gamma_h(t)} (\widetilde{\mathcal{P}}_h u)(.,t)(W_h - V_h)(.,t) \cdot \nabla_{\Gamma_h}\varphi_h(.,t) \end{aligned}$$

and determine the residual $R_h(.,t)$ by (5.1).

The above construction yields the following linear ODE system with the vector $r(t) = (r_j(t)) \in \mathbb{R}^N$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(M(t)\tilde{\alpha}(t)\big) + A(t)\tilde{\alpha}(t) + B(t)\tilde{\alpha}(t) = M(t)r(t),$$

which is the perturbed ODE system (4.4) and (4.18).

We show second order error bounds for the residual R_h using the bounds on the Ritz map.

THEOREM 5.1 (Bound of the semidiscrete residual) Let u, the solution of the parabolic problem, be smooth. Then there exists a constant $C > 0$ and $h_0 > 0$, such that for all $h \le h_0$ and $t \in [0,T]$, the finite element residual R_h of the Ritz map is bounded by

$$\|R_h(.,t)\|_{H_h^{-1}(\Gamma_h(t))} \le Ch^2,$$

where the constant C is independent of h and t, but depends on T and on the solution u. The H_h^{-1}-norm of R_h is defined as

$$\|R_h(.,t)\|_{H_h^{-1}(\Gamma_h(t))} := \sup_{0\neq\phi_h\in S_h(t)} \frac{\langle R_h(.,t),\phi_h\rangle_{L^2(\Gamma_h(t))}}{\|\phi_h\|_{H^1(\Gamma_h(t))}}.$$

Proof. (a) We start by applying the discrete ALE transport property to the residual equation (5.1) and using the definition of L_t, for $\widetilde{\mathcal{P}}_h u \in S_h(t)$:

$$m_h(R_h,\phi_h) = m_h(\partial_h^{\mathcal{A}}\widetilde{\mathcal{P}}_h u,\varphi_h) + a_h(\widetilde{\mathcal{P}}_h u,\varphi_h) + g_h(W_h;\widetilde{\mathcal{P}}_h u,\varphi_h) + m_h(\widetilde{\mathcal{P}}_h u,(W_h - V_h)\cdot\nabla_{\Gamma_h}\varphi_h).$$

(b) We continue by the transport property with discrete ALE material derivatives from Lemma 3.10, but for the ALE weak form (from Lemma 2.5)

$$0 = m(\partial_h^{\mathcal{A}} u,\varphi_h^l) + a(u,\varphi_h^l) + g(w_h;u,\varphi_h^l) + m(u,(w-v)\cdot\nabla_\Gamma\varphi_h^l).$$

(c) Subtraction of the two equations, and using the definition of the Ritz map (3.16), we obtain the following expression for the residual:

$$\begin{aligned} m_h(R_h,\phi_h) &= m_h(\partial_h^{\mathcal{A}}\widetilde{\mathcal{P}}_h u,\varphi_h) - m(\partial_h^{\mathcal{A}} u,\varphi_h^l) \\ &\quad + g_h(W_h;\widetilde{\mathcal{P}}_h u,\varphi_h) - g(w_h;u,\varphi_h^l) \\ &\quad - \big(m_h(\widetilde{\mathcal{P}}_h u,\varphi_h) - m(u,\varphi_h^l)\big) \\ &\quad + m_h(\widetilde{\mathcal{P}}_h u,(W_h - V_h)\cdot\nabla_{\Gamma_h}\varphi_h) - m(u,(w-v)\cdot\nabla_\Gamma\varphi_h^l). \end{aligned}$$

(d) We estimate these pairs separately, we show the basic idea by using the nonsymmetric term: We aim to use Lemma 3.8 and the error estimate for the Ritz map, Lemma 3.14; namely, we estimate as

$$\begin{aligned} & m_h(\widetilde{\mathcal{P}}_h u,(W_h - V_h)\cdot\nabla_{\Gamma_h}\varphi_h) - m(\mathcal{P}_h u,(w-v)\cdot\nabla_\Gamma\varphi_h^l) \\ &\quad + m(\mathcal{P}_h u - u,(w-v)\cdot\nabla_\Gamma\varphi_h^l) \leq Ch^2\|\varphi_h^l\|_{H^1(\Gamma(t))}. \end{aligned}$$

The other pairs can be estimated in the same way: by Lemma 3.11 and the errors in the Ritz map (in fact they can be bounded by $Ch^2\|\varphi_h\|_{L^2(\Gamma(t))}$). □

5.2 *Error bounds*

The direct application of the stability lemma for R–K methods and BDF methods (Lemma 4.2 and Lemma 4.5, respectively) gives error estimates between the projection $\widetilde{\mathcal{P}}_h u(.,t_n)$ and the fully discrete solution U_h^n (ALE–ESFEM combined with a temporal discretization), i.e.,

$$U_h^n := \sum_{j=1}^N \alpha_j^n \chi_j(.,t_n) \in S_h(t),$$

where the vectors α^n are generated, either by an s-stage implicit R–K method (4.1) or by a BDF method of order k (4.17).

5.2.1 *Implicit R–K methods*. Now we can prove the analogous error estimation result from (Dziuk *et al.*, 2012, Theorem 8.1) and Mansour (2013, Theorem 5.1).

THEOREM 5.2 Consider the ALE–ESFEM as space discretization of the parabolic problem (2.1) with time discretization by an s-stage implicit R–K method satisfying Assumption 4.1. Assume that the solution u and the surface $\Gamma(t)$ is smooth. Then there exists $\tau_0 > 0$, independent of h, but depending on the ALE velocity (see Lemma 3.5), such that for $\tau \leq \tau_0$, for the error $E_h^n = U_h^n - \widetilde{\mathcal{P}}_h u(., t_n)$ the following estimate holds for $t_n = n\tau \leq T$:

$$\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\Big)^{\frac{1}{2}}$$
$$\leq C\tilde{\beta}_{h,q}\tau^{q+1} + C\Big(\tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \|R_h(., t_k + c_i\tau)\|^2_{H_h^{-1}(\Gamma_h(t_k+c_i\tau))}\Big)^{\frac{1}{2}} + C\|E_h^0\|_{L^2(\Gamma_h(t_0))},$$

where the constant C is independent of h and τ, but depends on T, and we have

$$\tilde{\beta}_{h,q}^2 = \int_0^T \sum_{\ell=1}^{q+2} \|(\partial_h^{\mathcal{A}})^{(\ell)}(\widetilde{\mathcal{P}}_h u)(., t)\|_{L^2(\Gamma_h(t))} + \sum_{\ell=1}^{q+1} \|\nabla_{\Gamma_h(t)}(\partial_h^{\mathcal{A}})^{(\ell)}(\widetilde{\mathcal{P}}_h u)(., t)\|_{L^2(\Gamma_h(t))}\, \mathrm{d}t.$$

REMARK 5.3 Later on in the proofs we will use the existence of high order material derivatives of $\widetilde{\mathcal{P}}_h u$. This follows as a combination of the assumed regularity of the evolution of $\Gamma(t)$ and the assumed regularity of the exact solution u.

The version with the classical order p from (Dziuk *et al.*, 2012, Theorem 8.2), or Mansour (2013, Theorem 5.2) also holds in the ALE case, if the stronger regularity conditions are satisfied:

$$\left| M(t)^{-1} \frac{\mathrm{d}^{k_j-1}}{\mathrm{d}t^{k_j-1}}\Big(A(t)M(t)^{-1}\Big) \cdots \frac{\mathrm{d}^{k_1-1}}{\mathrm{d}t^{k_1-1}}\Big(A(t)M(t)^{-1}\Big) \frac{\mathrm{d}^{\tilde{k}-1}}{\mathrm{d}t^{\tilde{k}-1}}\Big(M(t)\tilde{\alpha}(t)\Big)\right|_{M(t)} \leq \gamma,$$
$$\left| M(t)^{-1} \frac{\mathrm{d}^{k_j-1}}{\mathrm{d}t^{k_j-1}}\Big(A(t)M(t)^{-1}\Big) \cdots \frac{\mathrm{d}^{k_1-1}}{\mathrm{d}t^{k_1-1}}\Big(A(t)M(t)^{-1}\Big) \frac{\mathrm{d}^{\tilde{k}-1}}{\mathrm{d}t^{\tilde{k}-1}}\Big(M(t)\tilde{\alpha}(t)\Big)\right|_{A(t)} \leq \gamma,$$

for all $k_j \geq 1$ and $\tilde{k} \geq q+1$ with $k_1 + \cdots + k_j + \tilde{k} \leq p+1$.

THEOREM 5.4 Consider the ALE–ESFEM as space discretization of the parabolic problem (2.1), with time discretization by an s-stage implicit R–K method satisfying Assumption 4.1 with $p > q+1$. Assuming the above regularity conditions. There exists $\tau_0 > 0$ independent of h, but depending on the ALE velocity (see Lemma 3.5), such that for $\tau \leq \tau_0$, for the error $E_h^n = U_h^n - \widetilde{\mathcal{P}}_h u(., t_n)$, the following estimate holds for $t_n = n\tau \leq T$:

$$\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\Big)^{\frac{1}{2}}$$
$$\leq C_0\tau^p + C\Big(\tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \|R_h(., t_k + c_i\tau)\|^2_{H_h^{-1}(\Gamma_h(t_k+c_i\tau))}\Big)^{\frac{1}{2}} + C\|E_h^0\|_{L^2(\Gamma_h(t_0))},$$

where the constant C_0 is independent of h and τ, but depends on T and γ.

Proof of Theorem 5.2 and 5.4. The proofs of the above two theorems are a combination of our previous results, especially the stability lemma (Lemma 4.2) and the relation $\|M_n r_n\|_{*,t_n} = \|R_h(.,t_n)\|_{H_h^{-1}(\Gamma_h(t_n))}$ (cf. Mansour (2013, (5.5))). Otherwise they are the same as the proof in Dziuk *et al.* (2012, Section 8) or (see Mansour (2013, Theorem 5.1–5.2)). The h and τ independency holds since the used stability lemma is also independent of them. □

5.2.2 *Backward differentiation formulas.* We prove the analogous result of Lubich *et al.* (2013, Theorem 5.1) and Mansour (2013, Theorem 5.3).

THEOREM 5.5 Consider the ALE–ESFEM as space discretization of the parabolic problem (2.1) with time discretization by a k-step backward difference formula of order $k \leq 5$. Assume that the solution u and the surface $\Gamma(t)$ is smooth. Then there exists $\tau_0 > 0$, independent of h, but depending on the ALE velocity (see Lemma 3.5), such that for $\tau \leq \tau_0$, for the error $E_h^n = U_h^n - \widetilde{\mathcal{P}}_h u(.,t_n)$ the following estimate holds for $t_n = n\tau \leq T$:

$$
\begin{aligned}
&\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\Big)^{\frac{1}{2}} \\
&\quad \leq C\tilde{\beta}_{h,k}\tau^k + \Big(\tau \sum_{j=1}^{n} \|R_h(.,t_j)\|^2_{H_h^{-1}(\Gamma_h(t_j))}\Big)^{\frac{1}{2}} + C \max_{0\leq i\leq k-1} \|E_h^i\|_{L^2(\Gamma_h(t_i))},
\end{aligned}
$$

where the constant C is independent of h and τ, but depends on T, and we have

$$
\tilde{\beta}_{h,k}^2 = \int_0^T \sum_{\ell=1}^{k+1} \|(\partial_h^{\mathcal{A}})^{(\ell)}(\widetilde{\mathcal{P}}_h u)(.,t)\|_{L^2(\Gamma_h(t))}\,\mathrm{d}t.
$$

Proof. The proof of this theorem relays on the corresponding h and τ independent stability result, i.e., Lemma 4.5. Otherwise we follow the proof of Lubich *et al.* (2013, Theorem 5.1), or Mansour (2013, Theorem 5.3). □

REMARK 5.6 The quantities $\widetilde{\beta}_{h,q}^2$ and $\widetilde{\beta}_{h,k}^2$ from Theorem 5.2 and Theorem 5.5 require existence of higher order discrete ALE material derivatives of the Ritz projection of u and, further, that they are bounded w.r. to the L^2 resp. H^1 norm. The existence of higher order material derivatives can be seen as follows: Rewrite equation (3.16) as a matrix vector equation for the coefficients of $\widetilde{\mathcal{P}}_h u$. Discrete ALE material derivatives corresponds to usual time derivatives for the coefficients of $\widetilde{\mathcal{P}}_h u$. Hence if we assume that the ALE dynamical system is smooth and that the exact solution is smooth, then it follows that higher order discrete ALE material derivatives of u exists. The boundedness of them follows from Lemma 3.14.

5.3 *Error of the full ALE discretizations*

We compare the lifted fully discrete numerical solution $u_h^n := (U_h^n)^l$ with the exact solution $u(.,t_n)$ of the evolving surface PDE (2.1), where $U_h^n = \sum_{j=1}^N \alpha_j^n \chi_j(.,t_n)$, where the vectors α^n are generated by the R–K (4.1) or the BDF method (4.17).

Now we state and prove the main results of this article.

THEOREM 5.7 (ALE–ESFEM and R–K) Consider the ALE–ESFEM as space discretization of the parabolic problem (2.1) with time discretization by an s-stage implicit R–K method satisfying Assumption 4.1. Let u be a smooth solution of the problem, as in Theorem 5.2 and 5.4, and assume that the initial value is approximated as

$$\|u_h^0 - (\mathcal{P}_h u)(.,0)\|_{L^2(\Gamma(0))} \leq C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leq h_0$ and $\tau \leq \tau_0$, the following error estimate holds for $t_n = n\tau \leq T$:

$$\|u_h^n - u(.,t_n)\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma(t_j)} u_h^j - \nabla_{\Gamma(t_j)} u(.,t_j)\|^2_{L^2(\Gamma(t_j))}\Big)^{\frac{1}{2}} \leq C\big(\tau^{q+1} + h^2\big).$$

The constant C is independent of h, τ and n, but depends on T and on the solution u.

Assuming that we have more regularity: conditions of Theorem 5.4 are additionally satisfied, then we have p instead of $q+1$.

THEOREM 5.8 (ALE–ESFEM and BDF) Consider the ALE–ESFEM as space discretization of the parabolic problem (2.1) with time discretization by a k-step backward difference formula of order $k \leq 5$. Let u be a smooth solution of the problem (as in Theorem 5.5) and assume that the starting values are satisfying

$$\max_{0\leq i\leq k-1} \|u_h^i - (\mathcal{P}_h u)(.,t_i)\|_{L^2(\Gamma(0))} \leq C_0 h^2.$$

Then there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leq h_0$ and $\tau \leq \tau_0$, the following error estimate holds for $t_n = n\tau \leq T$:

$$\|u_h^n - u(.,t_n)\|_{L^2(\Gamma(t_n))} + h\Big(\tau \sum_{j=1}^{n} \|\nabla_{\Gamma(t_j)} u_h^j - \nabla_{\Gamma(t_j)} u(.,t_j)\|^2_{L^2(\Gamma(t_j))}\Big)^{\frac{1}{2}} \leq C\big(\tau^{k} + h^2\big).$$

The constant C is independent of h, τ and n, but depends on T and on the smooth solution u.

Proof of Theorem 5.7–5.8. The global error is decomposed into two parts:

$$u_h^n - u(.,t_n) = \Big(u_h^n - (\mathcal{P}_h u)(.,t_n)\Big) + \Big((\mathcal{P}_h u)(.,t_n) - u(.,t_n)\Big),$$

and the terms are estimated by previous results.

The first term is estimated by a combination of the theorems and lemmas from the previous sections, in particular the convergence results for R–K or BDF methods: Theorem 5.2, 5.4 or 5.5, respectively, together with the residual bound Theorem 5.1, and the errors for the Ritz map and for its material derivatives (Lemma 3.14).

The second part is estimated again by the error estimates for the Ritz map (Lemma 3.14).

The h and τ independency holds, since all our previous results are shown to be independent of these quantities and therefore this property is preserved. The constant τ_0 depends on the ALE velocity (see Lemma 3.5). □

6. Numerical experiments

We present numerical experiments for an evolving surface parabolic problem discretized by the original and the ALE evolving surface finite elements coupled with various time discretizations. The fully discrete methods were implemented in Matlab and DUNE Dedner *et al.* (2010), while the initial triangulations were generated using DistMesh from Persson & Strang (2004).

The ESFEM and the ALE–ESFEM case were integrated by identical codes, except the involvement of the nonsymmetric B matrix and the evolution of the surface. The ODE system giving the surface movement (see (6.1) below) was solved by the same time discretization method as the PDE problem itself (with the same step size), while in one experiment the ALE map is given (see (6.2)).

To illustrate our theoretical results, we choose two problems which were intensively investigated in the literature before; see Dziuk *et al.* (2012); Lubich *et al.* (2013); Elliott & Venkataraman (2015) and Barreira *et al.* (2011). Specially for ALE experiments, see Elliott & Styles (2012) and Elliott & Venkataraman (2015). For all experiments the material velocity equals the normal velocity.

Observed order of convergence: With the aid of the first experiment, we will present experimental order of convergences (EOCs). We choose a problem which was presented before in, e.g., Dziuk *et al.* (2012).

Namely, the surface is given by

$$\Gamma(t) = \left\{ x \in \mathbb{R}^3 \;\middle|\; a(t)^{-1}x_1^2 + x_2^2 + x_3^2 - 1 = 0 \right\},$$

where $a(t) = 1 + 0.25\sin(2\pi t)$. The problem is considered over the time interval [0, 1]. The right-hand side f was computed as to have $u(x,t) = e^{-6t}x_1x_2$ as the true solution of the problem (2.1).

The normal velocity is given by the above distance function (cf. Dziuk & Elliott (2007a, Section 2.)). The ALE velocity is chosen to be

$$w_1(x,t) = \frac{0.25\pi\cos(2\pi t)}{1 + 0.25\sin(2\pi t)}x_1, \qquad w_2(x,t) = 0, \qquad w_3(x,t) = 0.$$

Discretization in space is always done with ALE–ESFEM. Discretization in time is done with BDF 1 and BDF 3. Let $(\mathcal{T}_k(t))_{k=1,2,\dots,n}$ and $(\tau_k)_{k=1,2,\dots,n}$ be a series of triangulations and time steps. We choose $2h_k \approx h_{k-1}$ and for BDF 1 $4\tau_k = \tau_{k-1}$ with initial step $\tau_1 = 0.1$. For BDF 3 we choose $\sqrt[3]{4}\tau_k = \tau_{k-1}$ with initial step $\tau_1 = 0.01$. By e_k we denote the error corresponding to the mesh $\mathcal{T}_k(t)$ and stepsize τ_k. Then the EOCs are given as

$$EOC_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \qquad (k = 2, 3, \dots, n).$$

In Tables 1 and 2, we report on the EOCs, for the ALE–ESFEM with backward Euler method (BDF 1) and BDF 3, respectively, corresponding to the norm and seminorm

$$L^\infty(L^2): \qquad \max_{1\le n\le N} \|u_h^n - u(.,t_n)\|_{L^2(\Gamma(t_n))},$$

$$L^2(H^1): \qquad \left(\tau\sum_{n=1}^{N} \|\nabla_{\Gamma(t_n)}\big(u_h^n - u(.,t_n)\big)\|_{L^2(\Gamma(t_n))}\right)^{1/2}.$$

The results for BDF 1 have already independently been reported in Elliott & Venkataraman (2015). The non-ALE data for the same example can be found in Dziuk *et al.* (2012) and Mansour (2013).

TABLE 1 *Errors and EOCs for BDF 1 in the $L^\infty(L^2)$ and $L^2(H^1)$ norms for the ALE case*

Level	dof	$L^\infty(L^2)$	EOCs	$L^2(H^1)$	EOCs
1	126	0.02455766	—	0.05203599	—
2	516	0.00753037	1.7053	0.01689990	1.6224
3	2070	0.00201268	1.9036	0.00583376	1.5345
4	8208	0.00051164	1.9759	0.00282697	1.0451
5	32682	0.00012858	1.9923	0.00141542	0.9980

TABLE 2 *Errors and EOCs for BDF 3 in the $L^\infty(L^2)$ and $L^2(H^1)$ norms for the ALE case*

Level	dof	$L^\infty(L^2)$	EOCs	$L^2(H^1)$	EOCs
1	126	0.00917003	—	0.02266929	—
2	516	0.00246862	1.8932	0.00977487	1.2136
3	2070	0.00061587	2.0030	0.00442116	1.1447
4	8208	0.00015516	1.9889	0.00210023	1.0739
5	32682	0.00003929	1.9815	0.00098204	1.0967

Comparison of ALE and non-ALE methods: We consider the evolving surface parabolic PDE (2.1) over the closed surface $\Gamma(t)$ given by the zero level set of the distance function

$$\mathrm{d}(x,t) := x_1^2 + x_2^2 + K(t)^2 G\Big(\frac{x_3^2}{L(t)^2}\Big) - K(t)^2, \quad \text{i.e.,} \quad \Gamma(t) := \{x \in \mathbb{R}^3 \mid \mathrm{d}(x,t) = 0\}.$$

Here the functions G, L and K are given as

$$\begin{aligned} G(s) &= 200s\Big(s - \frac{199}{200}\Big), \\ L(t) &= 1 + 0.2\sin(4\pi\ t), \\ K(t) &= 0.1 + 0.05\sin(2\pi\ t). \end{aligned}$$

The velocity v is the normal velocity of the surface defined by the differential equation (formulated for the nodes):

$$\frac{\mathrm{d}}{\mathrm{d}t} a_j = V_j \mathrm{n}_j, \quad \text{where} \quad V_j = \frac{-\partial_t d(a_j,t)}{|\nabla d(a_j,t)|}, \quad \mathrm{n}_j = \frac{\nabla d(a_j,t)}{|\nabla d(a_j,t)|}. \tag{6.1}$$

The right-hand side f is chosen as to have the function $u(x,t) = e^{-6t}x_1x_2$ to be the true solution of (2.1).

Finally, we give the applied ALE movement (from Elliott & Styles (2012) and Elliott & Venkataraman (2015)):

$$(a_i(t))_1 = (a_0(t))_1 \frac{K(t)}{K(0)}, \quad (a_i(t))_2 = (a_0(t))_2 \frac{K(t)}{K(0)}, \quad (a_i(t))_3 = (a_0(t))_3 \frac{L(t)}{L(0)}, \tag{6.2}$$

hence $d(a_i(t), t) = 0$ for every $t \in [0, T]$, for $i = 1, 2, \ldots, N$.

The discrete surfaces evolved with normal and ALE velocities are shown in Fig. 1, for time $t = 0,\ 0.2,\ 0.4,\ 0.6$.

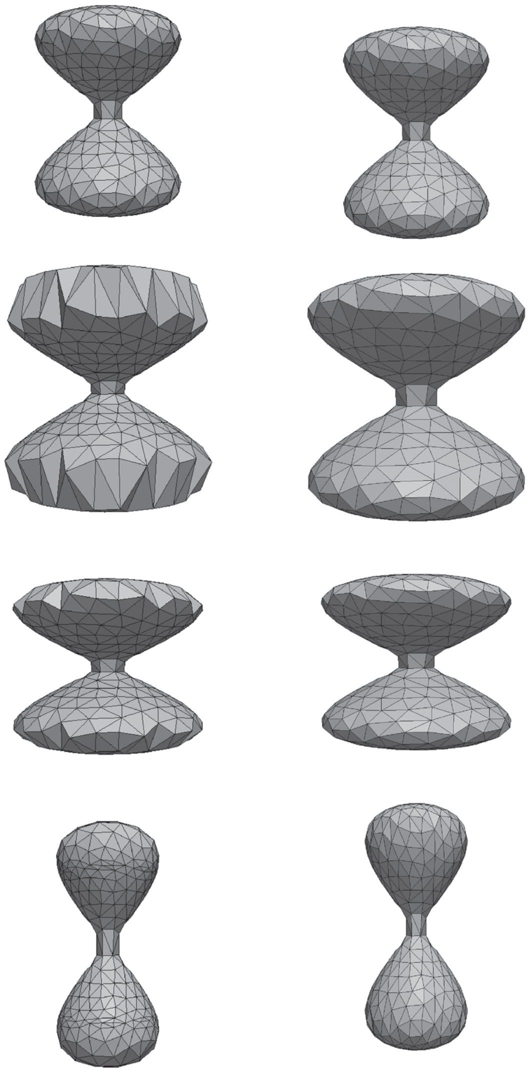

FIG. 1. Meshes with 376 nodes. Left: normal movement, with Radau IIA method ($s = 3$). Right: ALE movement.

In the following we compare the ALE and non-ALE methods with three spatial refinements and integrate the evolving surface PDE with various time discretizations, with a time step τ, until $T = 0.6$. We set $e_h(\,.\,,t) := u_h(\,.\,,T) - u(\,.\,,T)$ $(T = n\tau)$, and compute the following norm and seminorm of it

$$|e_h|_M := \|e_h(\,.\,,T)\|_{L^2(\Gamma(T))}, \qquad |e_h|_A := \|\nabla_\Gamma e_h(\,.\,,T)\|_{L^2(\Gamma(T))}.$$

The following plots show the above error norms at time $T = 0.6$ (left M-norm, right A-seminorm) plotted against the time step size τ (on logarithmic scale) and different error curves are representing different spatial discretizations.

In the experiments we used three different time discretizations. The convergence in time can be seen (note the reference line). For sufficiently small time steps τ the spatial error is dominating, in agreement with the theoretical results. The figures show that the errors in the ALE–ESFEM are significantly smaller than for the non-ALE case.

Figures 2 and 3 show the errors obtained by the backward Euler method coupled with the two different spatial discretizations.

The following plots (Figs 4 and 5) show the same norms, but they are made by the five order Radau IIA method ($s = 3$) as a time integrator.

The last two figures (Figs 6 and 7) show the results obtained by the three step BDF method.

In the case of BDF methods with non-ALE–ESFEM, for bigger values of τ, the surface itself (but not the PDE) is evolved with smaller time steps due to difficulties within the time integration of the surface.

Acknowledgements

The authors would like to thank Prof. Christian Lubich for the invaluable discussions on the topic, and for his encouragement and help during the preparation of this paper. We are grateful to the anonymous referees for their careful reading of the manuscript and helpful suggestions, which helped us to improve the quality of the paper.

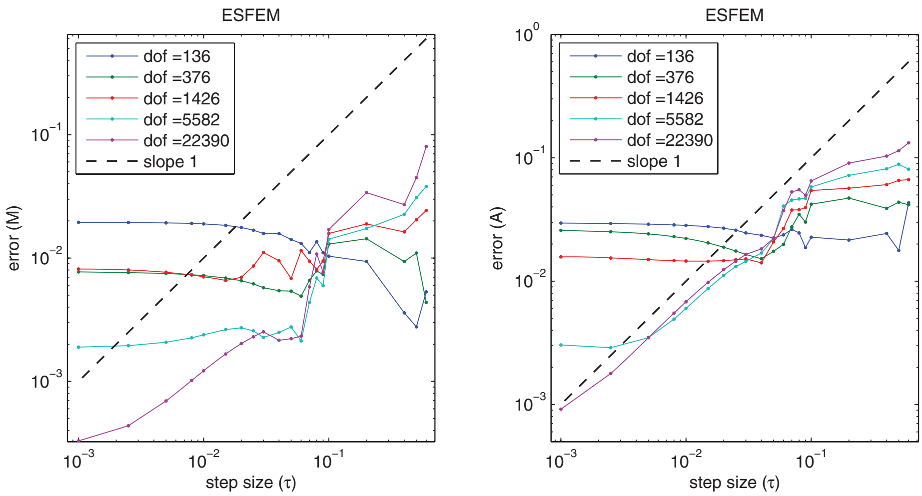

FIG. 2. Errors of the ESFEM and the implicit Euler method.

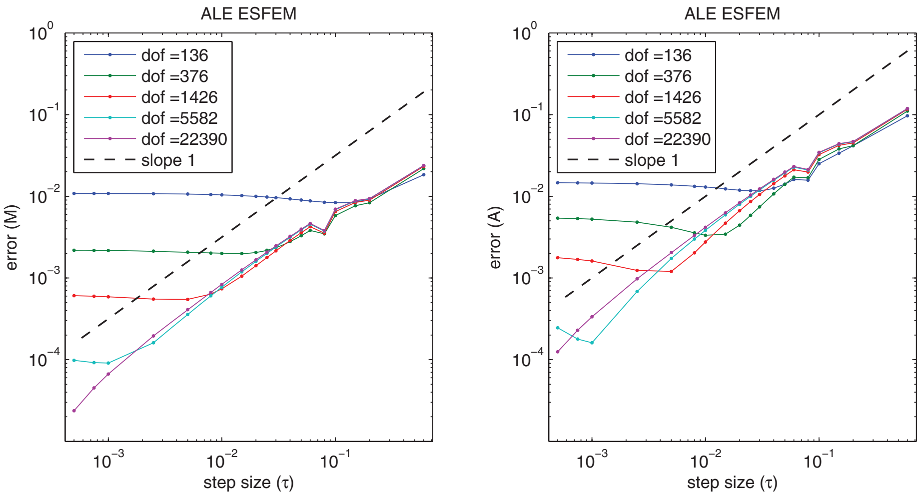

FIG. 3. Errors of the ALE–ESFEM and the implicit Euler method.

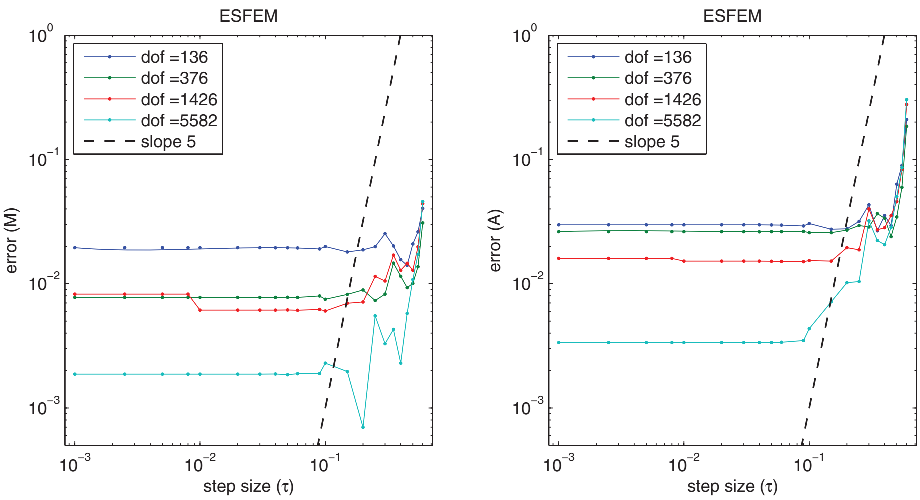

FIG. 4. Errors of the ESFEM and the three stage Radau IIA method.

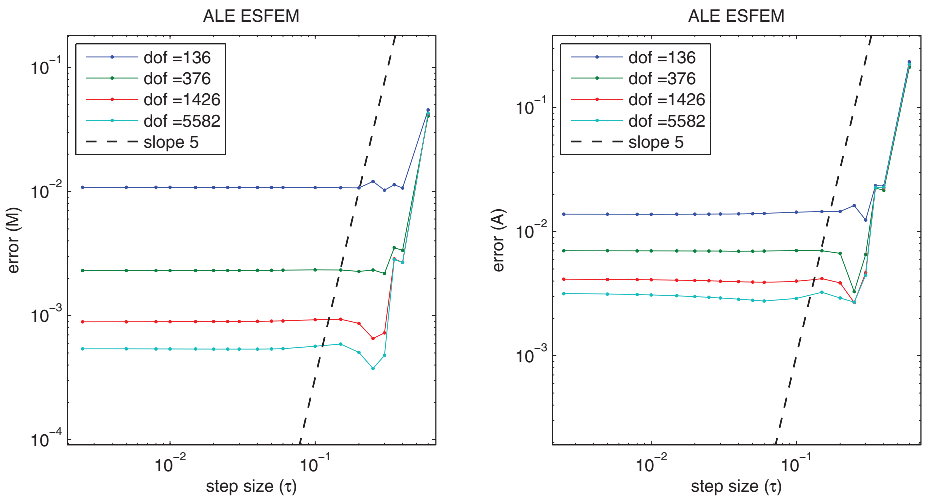

FIG. 5. Errors of the ALE–ESFEM and the three stage Radau IIA method.

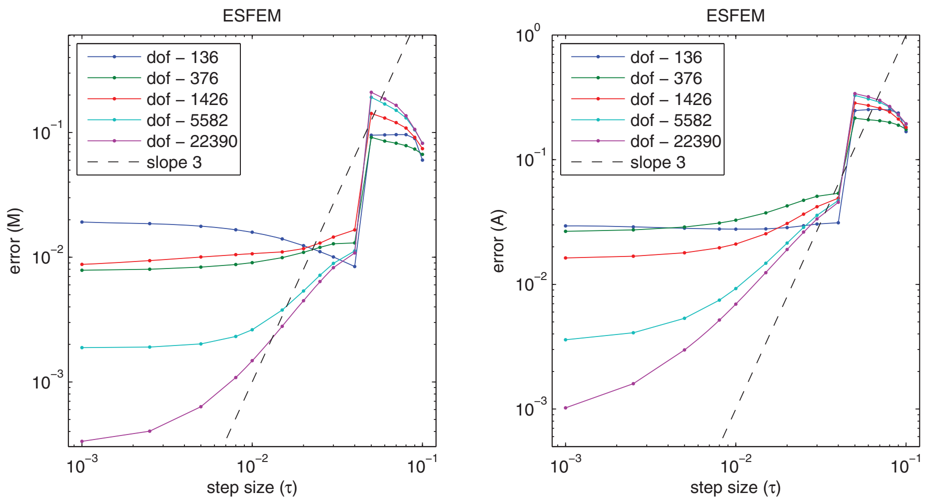

FIG. 6. Errors of the ESFEM and the BDF3 method.

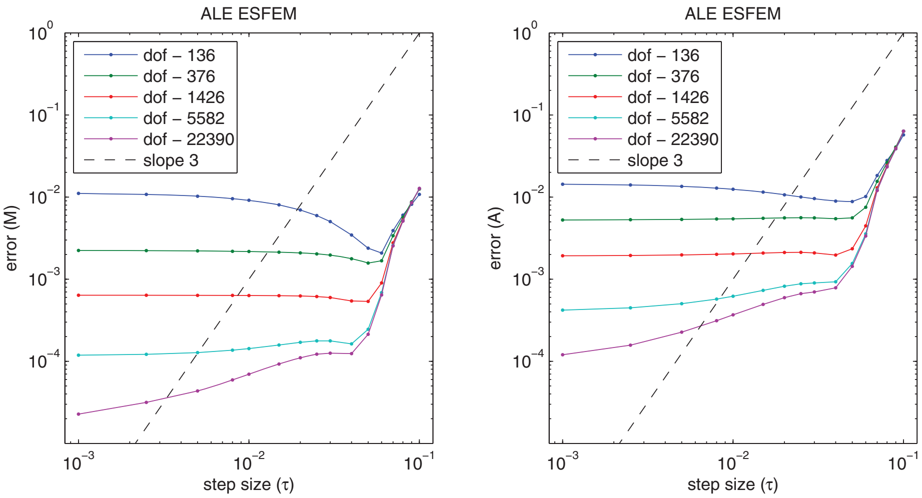

FIG. 7. Errors of the ALE–ESFEM and the BDF3 method.

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Appendix B. Computing arbitrary Lagrangian Eulerian maps for evolving surfaces

Computing arbitrary Lagrangian Eulerian maps for evolving surfaces

Balázs Kovács*

Abstract

The good mesh quality of a discretized closed *evolving* surface is often compromised during time evolution. In recent years this phenomenon has been theoretically addressed in a few ways, one of them uses arbitrary Lagrangian Eulerian (ALE) maps. However, the numerical computation of such maps still remained an unsolved problem in the literature. An approach, using differential algebraic problems, is proposed here to numerically compute an arbitrary Lagrangian Eulerian map, which preserves the mesh properties over time. The ALE velocity is obtained by finding an equilibrium of a simple spring system, based on the connectivity of the nodes in the mesh. We also consider the algorithmic question of constructing acute surface meshes. We present various numerical experiments illustrating the good properties of the obtained meshes and the low computational cost of the proposed approach.
Keywords: arbitrary Lagrangian Eulerian map, evolving surfaces, evolving surface PDE, evolving surface finite elements
AMS: 65M50, 34A09, 35R01

1 Introduction

Partial differential equations on evolving surfaces with a *given velocity* v have been discretized using a huge variety of methods. Probably one of the most popular is the evolving surface finite element method developed by Dziuk and Elliott in [DE07].

As it was pointed out by Dziuk and Elliott early on, in Section 7.2 of [DE07]: *"A drawback of our method is the possibility of degenerating grids. The prescribed velocity may lead to the effect, that the triangulation $\Gamma_h(t)$ is distorted"*. The same issue occurs for problems in moving domains.

In recent years this problem has been theoretically addressed in a few ways:

- Spatial discretizations using *arbitrary Lagrangian Eulerian* (ALE) maps for moving domains have been studied many times in the literature, see for instance [FN99, FN04] and [BKN13b, BKN13a], and the references therein.

- The ALE version of the evolving surface finite element method has been proposed by Elliott and Styles in [ES12], where a better triangulation is

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obtained using an ALE map, i.e., allowing the nodes of the mesh to evolve with a velocity having an additional tangential component compared to the (pure normal) surface velocity.

- Elliott and Fritz [EF16a, EF16b] constructed meshes with very good properties using the quite technical *DeTurck trick.*

We propose here to compute an arbitrary Lagrangian Eulerian map for closed evolving surfaces, with a focus on evolving surface finite elements, by integrating a differential algebraic equation (DAE) system for the nodes. We use a not necessarily tangential ALE velocity to achieve good mesh quality, while enforcing the points to stay on the surface.

To our knowledge there is no such ALE algorithm for evolving surfaces available in the literature, in contrast with the many papers on the theory of numerical methods involving ALE maps for both closed evolving surfaces and moving domains.

Many experiments with evolving surface finite elements have been presented in the above references, especially see [ES12, EV15, KPG17], where smaller discretization errors have been obtained by solving evolving surface problems on ALE meshes. The ALE maps used in these experiments were unrealistic, obtained analytically from an *a priori knowledge* on the surface and its evolution, using deep understanding and structure of the signed distance function (which defines the surface). No general ideas on the computation of ALE maps for evolving surfaces have been proposed in these papers. Numerical analysis of the ALE evolving surface finite element method has been studied in [EV15] and [KPG17].

Standard mesh generation algorithms (see e.g. [FG00, TWM85], and the references therein) could be used in each timestep to generate a mesh of good quality. This would require to compute a map – between the old and the new mesh – after each remeshing process. From a theoretical point of view this seems a minor issue, however from the implementation side this is undesirable, since in most cases this is a nontrivial and costly task, and hence, should be avoided. The same is true for non-ALE methods, since there it is usual that nodes disappear and/or new nodes are added to the mesh (see, e.g., [EF16a, Figure 14]).

In the present paper we propose a general algorithm to compute a suitable ALE map, *without any a priori knowledge*, for meshes of closed evolving surfaces.

The approach is based on the following idea: Usually, the surface evolution is given by an ODE system with a surface velocity. We use an additional tangential velocity for a possibly degenerated mesh to improve grid quality. In general such tangential velocities are not straightforward to construct, therefore we use a not necessarily tangential ALE velocity and introduce a constraint to keep the nodes on the surface. Altogether this is finally formulated as a DAE system for the nodes. The ALE velocity is based on a spring system, where the nodes are connected along the edges by springs. Then the algorithm approximates the equilibrium point of this spring system. The numerical solution of the DAE system gives the new mesh. We use implicit Runge–Kutta methods (in particular Radau IIA methods), and a more efficient splitting method, combined with explicit Runge–Kutta methods, to integrate the system in time.

The computation of the arbitrary Lagrangian Eulerian mesh here is free of any *a priori knowledge* in the following sense: the algorithm uses the distance function at each time, but it does not use its structure or any other special properties of it, unlike the ALE maps from the literature.

This approach for closed evolving surfaces can be used as a tool in the computation of ALE meshes for *moving domains*: In [FN99, Section 2.4] arbitrary Lagrangian Eulerian maps for moving domains are obtained by solving a parabolic problem, or the corresponding stationary problem, while in [FLM95] an elastodynamic equation system is used for the same purpose. However, for these approaches the evolution of the boundary still needs to be *known a priori*. The problem of numerically finding such a boundary evolution has not been solved in these papers. In fact, to determine such a boundary evolution is equivalent to finding an ALE map for a lower dimensional closed surface, which is the same problem as we consider in this paper. Hence, the algorithm proposed here can also serve as a tool to compute boundary evolutions, which can be used together with the well understood classical ALE methods for moving domains, for instance the ones proposed in [FLM95, FN99].

We give some further details on possible extensions of the proposed algorithm: to handle other mesh properties (e.g., acuteness), an adaptive version, and a local version as well.

We present various numerical experiments illustrating the validity of the differential algebraic model, and also the performance of the proposed algorithm compared to the ALE maps given in the literature. We also report on errors and computational times in the case of evolving surface PDEs.

2 Evolving surfaces, ALE maps and PDEs

As our main motivation lies in the numerical solution of parabolic PDEs on evolving surfaces we shortly recap the setting of [DE07]. We will also use this setting as an illustrative background to the proposed algorithm.

Let $\Gamma(t) \subset \mathbb{R}^{m+1}$, $0 \leq t \leq T$, be a smooth evolving closed hypersurface. Further, let the evolution of the surface be given by the smooth velocity v, usually assumed to be normal. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of u, the tangential gradient is denoted by ∇_Γ and given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu\nu$, with unit outward normal ν. We denote by $\Delta_\Gamma = \nabla_\Gamma \cdot \nabla_\Gamma$ the Laplace–Beltrami operator.

We consider the following linear evolving surface PDE:

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u &= f \qquad \text{on } \Gamma(t), \\ u(\cdot, 0) &= u_0 \qquad \text{on } \Gamma(0). \end{aligned} \tag{1}$$

Basic and detailed references on evolving surface PDEs and on the evolving surface finite element method (ESFEM) are [DE07, DE13a, DE13b] and [Man13]. The ALE version of the evolving surface finite element method has been proposed in the paper [ES12], which also contains a detailed description and many experiments. While numerical analysis of full discretizations can be found in [EV15] and [KPG17], the former is more concerned about spatial discretizations and BDF methods of order 1 and 2, while the latter is more focused on high-order BDF and Runge–Kutta time discretizations.

It is important to note that we aim at the numerical solution of the PDE (1) using the *evolving surface finite element method* developed by Dziuk and Elliott [DE07], the surface is discretized using a triangular mesh. The description of representation of evolving surfaces and that of discrete surfaces can be found in the following subsections.

2.1 Surface representations

Evolving surfaces are usually described in two ways, which have different advantages, hence we will use both of them for various purposes.

Distance function representation. Based on a signed distance function the evolving m-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^{m+1}$ is given by

$$\Gamma(t) = \{x \in \mathbb{R}^{m+1} \mid d(x,t) = 0\},$$

with a function $d : \mathbb{R}^{m+1} \times [0,T] \to \mathbb{R}$ (whose regularity depends on the smoothness of the surface), cf. [Dzi88, DE07].

Diffeomorphic parametrization. The surface can also be described by a diffeomorphic parametrization, cf. [KLLP17] and [DE07].

We consider the evolving m-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^{m+1}$ as the image

$$\Gamma(t) = \{X(p,t) \mid p \in \Gamma(0)\}$$

of a sufficiently regular vector-valued function $X : \Gamma(0) \times [0,T] \to \mathbb{R}^{m+1}$, where $\Gamma(0) = \Gamma^0$ is the smooth closed initial surface, and $X(p,0) = p$. It is convenient to think of $X(p,t)$ as the position at time t of a moving particle with label p, and of $\Gamma(t)$ as a collection of such particles. The parametrisation also satisfies the ODE system, for a point $p \in \Gamma(0)$,

$$\partial_t X(p,t) = v(X(p,t),t), \tag{2}$$

where $v(\cdot,t) \in \mathbb{R}^{m+1}$ is the velocity of the surface (using the distance function given by $v = V\nu$ with $\nu = \nabla d/|\nabla d|$ and $V = -\partial_t d/|\nabla d|$). Note that for a known velocity field v, the position $X(p,t)$ at time t of the particle with label p is usually obtained by solving the ordinary differential equation (2) from 0 to t for a fixed p.

We assume that the surface does not develop topological changes due to the evolution. This assumption seems to be restrictive, yet reasonable. Since, the evolving surface finite element setting is in the focus, which already cannot handle such topological changes, cf. [DE07].

2.2 Surface approximation

The smooth initial surface $\Gamma(0)$ is approximated by a triangulated surface $\Gamma_h(0)$, i.e., an admissible family of triangulations $\mathcal{T}_h(0)$ of maximal element diameter h; see [DE07] for the notion of an admissible triangulation, which includes quasi-uniformity. Let $x_j(0)$, $(j = 1, 2, \ldots, N)$ denote the nodes of $\Gamma_h(0)$ lying on the initial smooth surface $\Gamma(0)$. The nodes will be evolved in time with the given normal velocity v, by solving the ODE

$$\frac{\mathrm{d}}{\mathrm{dt}} x_j(t) = v(x_j(t),t) \qquad (j = 1, 2, \ldots, N), \tag{3}$$

which is simply (2) for the nodes (x_j). Obviously, the nodes remain on the surface $\Gamma(t)$ for all times, i.e., $d(x_j(t), t) = 0$ for $j = 1, 2, \ldots, N$ and for all $t \in [0, T]$.

Therefore, the smooth surface $\Gamma(t)$ is also approximated by a discrete surface $\Gamma_h(t)$, whose elements also form a triangulation $\mathcal{T}_h(t)$. We have

$$\Gamma_h(t) = \bigcup_{E(t) \in \mathcal{T}_h(t)} E(t).$$

The assumption on quasi-uniformity over time, i.e., there is a fixed $c > 0$ (independent of t) such that for any triangle $E(t) \in \mathcal{T}_h(t)$ the radius of the inscribed circle $\sigma_{E(t)}$ satisfies

$$\frac{h_{E(t)}}{\sigma_{E(t)}} \leq c,$$

for all $t \in [0, T]$, is generally not always satisfied during time evolution.

As an example to degenerating surface evolution, from [ES12], we evolved a surface using the ODE (3). As observed in Figure 1: however the initial mesh (left) is quasi-uniform and the surface evolution is also not complicated, the meshes at later times (middle and right) do not preserve the good mesh qualities. Both quite bad surface resolution and unnecessarily fine elements occur.

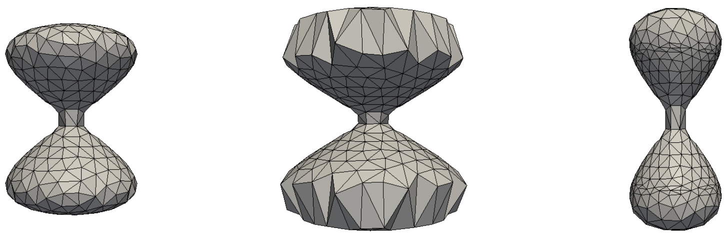

Figure 1: Normal evolution of a closed surface at time $t = 0, 0.2, 0.6$; see also in [ES12]

3 Computing ALE maps as general constrained problems

We now propose an approach which will be used to determine a suitable ALE map for evolving surfaces, to maintain mesh quality during the surface evolution. In fact, we directly compute the new positions of the nodes.

We consider another parametrization of $\Gamma(t)$, with good mesh properties, and which is different from X in (2), called an *arbitrary Lagrangian Eulerian map*. The corresponding ODE system is

$$\partial_t X(p, t) = v(X(p, t), t) + w(X(p, t), t), \tag{4}$$

with the pure tangential velocity w.

For evolving surface problems the surface velocity v is usually assumed to be known. However, such pure tangential ALE velocities are not given, and also not easy to obtain, in general. Therefore, we allow velocities (still denoted by) w which improve mesh quality, but have small non-tangential components, hence may lead points away from the surface. To compensate this, a constraint is introduced in order to keep the smooth surface $\Gamma(t)$ unaltered. Therefore, the set of ODEs (4) is modified into the following differential algebraic equation (DAE) system (of index 2) with Lagrange multiplier λ, for $p \in \Gamma(0)$,

$$\begin{aligned} \partial_t X(p,t) &= v(X(p,t),t) + w(X; X(p,t),t) - D(X(p,t))^T \lambda(p,t), \\ d(X(p,t),t) &= 0, \end{aligned} \tag{5}$$

where $D(X) = \partial\, d(\cdot,t)/\partial X$. The first argument X in w indicates that the additional velocity may (and usually does) depend on the whole surface. As it causes no confusion we will drop this argument later on.

The analogous differential algebraic equation system for the nodes of the surface approximation mesh $\Gamma_h(t)$, which are collected into the vector $\mathbf{x}(t) = (x_j(t))_{j=1}^N$, and with Lagrange multiplier $\boldsymbol{\lambda}(t) \in \mathbb{R}^N$, reads as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \mathbf{x}(t) &= v(\mathbf{x}(t),t) + w(\mathbf{x}(t),t) - D(\mathbf{x}(t))^T \boldsymbol{\lambda}(t), \\ d(\mathbf{x}(t),t) &= 0, \end{aligned} \tag{6}$$

with initial value $\mathbf{x}(0) = (x_j(0))_{j=1}^N$. The constraint $d(\mathbf{x}(t),t) = 0$ is meant pointwise, i.e., as $d(x_j(t),t) = 0$ for $j = 1,2,\ldots,N$, while the matrix $D(\mathbf{x}) = \partial\, d(\cdot,t)/\,\partial\, \mathbf{x}$. Concerning notation: we will apply the convention to use boldface letters to denote vectors in $\mathbb{R}^{3N}$ or $\mathbb{R}^N$ collecting nodal values of discretized variables.

Naturally, the DAE system is independent of the choice of the ALE velocity w, if there is some (physical, biological or modeling) knowledge on the general type of the surface evolution a user can propose a suitable ALE velocity accordingly.

In the next sections we will propose a very intuitive way to define the ALE velocity w, and we will also discuss numerical methods for the solution of the DAE system.

3.1 A spring system based arbitrary ALE velocity

We use here a simple idea to determine the velocity w: let us assume that the nodes of the mesh are connected by springs following the edges of the elements, i.e., the topology of the spring system is determined by the triangulation $\Gamma_h(t)$.

This system defines a force function F, which we use to define the ALE velocity by setting

$$w(\mathbf{x},t) = kF(\mathbf{x}), \qquad \text{with a spring constant } k \text{ chosen later.} \tag{7}$$

The force function F is computed based on the connectivity (described by the elements), and by the forces over the edges based on a length function ℓ_p (the desired length of springs). The net force $F_j(\mathbf{x})$ at a node x_j is given by, for $j = 1,2,\ldots,N$,

$$F_j(\mathbf{x}) = \sum_{e \in \{(x_j,\cdot)\}} f(e), \tag{8}$$

where the set $\{(x_j,\cdot)\}$ collects all the edges $e=(x_j,(x_j)^e)$ having x_j as one of their nodes, while $(x_j)^e$ simply denotes the other node across the edge e, see Figure 2. Then $f(e)$ is the force along the edge e, given by

$$f(e) = \big(\ell_p(e) - |e|\big)\ \nu_e, \qquad \text{with unit vector} \qquad \nu_e = \frac{x_j - (x_j)^e}{\|x_j - (x_j)^e\|},$$

and current length $|e|$.

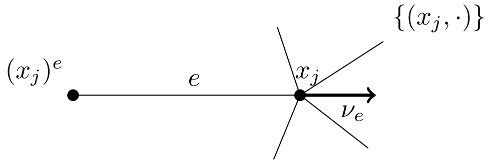

Figure 2: A typical node x_j on an edge e, the set of edges $\{(x_j,\cdot)\}$, etc. used to define $F_j(\mathbf{x})$

We propose to use the following length function, which cuts off the extremely small and large edge lengths, therefore leading to equidistribution:

$$\ell_p(e) = \begin{cases} pm_e + (1-p)M_e, & \text{if } |e| \geq pm_e + (1-p)M_e, \\ (1-p)m_e + pM_e, & \text{if } |e| \leq (1-p)m_e + pM_e, \\ |e|, & \text{otherwise,} \end{cases}$$
$$\text{where} \quad (m_e = \min_e |e|,\ M_e = \max_e |e|,\ p \in (0, 1/2)). \tag{9}$$

Other length functions can also be used, for instance the C^k version of ℓ_p (i.e. smoothed cutoff), or more complicated functions, such as the analogue of ℓ_p with more steps. Force functions based on inverse edge length are also possible to use.

The same force function (8) has been used by Strang and Persson in DistMesh [PS04]. However, constrained systems are not appearing there, and our length function has significant differences compared to the one used in DistMesh. They have not used their approach to compute ALE maps. In particular compared to DistMesh, we do not add or delete nodes (which is essential in DistMesh for the meshing), furthermore we compute the ideal spring length in a different way (DistMesh having a rule which fits better with the possibility to delete nodes and also with the startup of their process).

As Strang and Persson encourage their users to *"[...] modify the code according to their needs"*, we indeed adapt DistMesh to suit our purposes: the force function is computed using their modified code.

3.2 The DAE system

By plugging in the velocity rule (7) into the DAE system (6) we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}\mathbf{x}(t) = v(\mathbf{x}(t),t) + kF(\mathbf{x}(t)) - D(\mathbf{x}(t))^T \boldsymbol{\lambda}(t), \\ &d(\mathbf{x}(t),t) = 0. \end{aligned} \tag{10}$$

with the spring constant k, which has to be chosen by the user based on the problem at hand. The numerical approximation of this system immediately gives the position of the nodes under the ALE map.

In the following we will propose some numerical methods for the time integration.

3.2.1 Runge–Kutta solution of the DAE system

For a given stepsize $\tau > 0$, an s-stage implicit Runge–Kutta method, see [HW96, Section VII.4], applied to the DAE system (10) determines solution approximations $\mathbf{x}^{n+1}$ and approximations to the Lagrange multiplier $\boldsymbol{\lambda}^{n+1}$, as well as the internal stages $\mathbf{X}_{ni}, \boldsymbol{\Lambda}_{ni}$ by the equations

$$\mathbf{X}_{ni} = \mathbf{x}^n + \tau \sum_{j=1}^{s} a_{ij} \mathcal{F}(\mathbf{X}_{nj}, \boldsymbol{\Lambda}_{nj}), \qquad i = 1, 2, \dots, s, \tag{11}$$

$$\boldsymbol{\Lambda}_{ni} = \boldsymbol{\lambda}^n + \tau \sum_{j=1}^{s} a_{ij} \boldsymbol{\ell}_{nj}, \qquad i = 1, 2, \dots, s, \tag{12}$$

$$\mathbf{x}^{n+1} = \mathbf{x}^n + \tau \sum_{i=1}^{s} b_i \mathcal{F}(\mathbf{X}_{nj}, \boldsymbol{\Lambda}_{nj}), \quad \text{and} \quad \boldsymbol{\lambda}^{n+1} = \boldsymbol{\lambda}^n + \tau \sum_{i=1}^{s} b_i \boldsymbol{\ell}_{nj}, \tag{13}$$

where, for $i = 1, 2, \dots, s$,

$$\mathcal{F}(\mathbf{X}_{ni}, \boldsymbol{\Lambda}_{ni}) = v(\mathbf{X}_{ni}, t_{ni}) + kF(\mathbf{X}_{ni}) - D(\mathbf{X}_{ni})^T \boldsymbol{\Lambda}_{ni}, \tag{14}$$

$$\text{and} \qquad d(\mathbf{X}_{ni}, t_{ni}) = 0. \tag{15}$$

First the system (11), (14), (15) is solved, using a simplified Newton iteration, then one can compute $\boldsymbol{\ell}_{nj}$ from (12), and finally obtain $\mathbf{x}^{n+1}$, $\boldsymbol{\lambda}^{n+1}$ from (13).

The method is determined by its Butcher-tableau, i.e., the coefficient matrix $\mathcal{A} = (a_{ij})_{i,j=1}^{s}$ and its vector of weights $b = (b_i)_{i=1}^{s}$, and $c_i = \sum_{j=1}^{s} a_{ij}$ ($i = 1, 2, \dots, s$).

In the following we will use the Radau IIA methods, which are s-stage Runge–Kutta methods of classical order $2s - 1$. More details on index 2 DAEs in general, as well as on their Runge–Kutta approximations can be found in [HLR06], or [HW96, Chapter VII.].

3.2.2 Splitting methods for the constrained system

We propose to solve the DAE (10) using a splitting method, in order to decrease computational time. This is supported by making the following observations: The implicit Runge–Kutta solution of the DAE system is usually expensive. The normal movement is a pointwise nonstiff ODE system, without constraint (if integrated exactly), hence usually can be solved very cheaply. Also there are some examples where the normal velocity is not explicitly given, the surface evolution is defined by a direct mapping of a reference surface, i.e., we cannot use the system (10), see for example the surface [DKM13, Test Problem 2], [LMV13, equation (6.1)].

The most straightforward way is to split the differential algebraic problem (10) in a way that one of the subproblems is the original ODE system (3). The main benefit is that this subproblem automatically satisfies the constraint up to a small error.

The Lie splitted DAE system, over the time interval $[t_n, t_{n+1}]$, reads as

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\, \mathbf{x}^v(t) = v(\mathbf{x}^v(t), t), \\ \\ \mathbf{x}^v(t_n) = \mathbf{x}(t_n), \end{cases} \qquad \begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\, \mathbf{x}^w(t) = kF(\mathbf{x}^w(t)) - D(\mathbf{x}^w(t))^T \boldsymbol{\lambda}(t), \\ d(\mathbf{x}^w(t), t_{n+1}) = 0, \\ \mathbf{x}^w(t_n) = \mathbf{x}^v(t_{n+1}), \end{cases} \tag{16}$$

and finally, setting $\mathbf{x}(t_{n+1}) = \mathbf{x}^w(t_{n+1})$, and where F is a function is defined in (8), while the constraint is still meant coordinatewise and $D(\mathbf{x}) = \partial\, d(\cdot, t_{n+1}) / \partial\, \mathbf{x}$. In fact, this order of subproblems allows us to drop the constraint from the v-system, as the second system will eliminate the small errors mentioned above.

In order to decrease computational time, we solve the v-system using the explicit Euler method. We do not require a high-order method here, as the mesh quality depends only on the second step. For the time integration of the w-system we use the classical 4-stage Runge–Kutta method and then projecting back to the surface. Due to the stiffness of the problem, it is approximated through several substeps over interval $[t_n, t_{n+1}]$. We choose this method, since being explicit, it is very cheap, having only four right-hand side evaluations per step, but still have high-order.

We make some general remarks on the methods described above.

Remark 3.1. *If a parabolic PDE is numerically solved on the discrete surface $\Gamma_h(t_n)$ then the matrices are assembled on the improved mesh, using the ALE scheme, cf. [EV15, (4.3) and (4.4)] or [KPG17, (3.5) and (3.6)]. The tangential vector appearing in the formulas for ALE ESFEM can be computed from the obtained nodes and from the normal velocity.*

The algorithm clearly uses the surface velocity v in the exact same way as for the purely Lagrangian ESFEM algorithms: only to evolve the surface. However, the distance function is used as the constraint (being an inexpensive pointwise operation), which is avoided by the standard ESFEM method.

It is usual to use the same time discretization method for the surface evolution as for the time integration of the parabolic PDE on the surface. This time discretization scheme could be also used to solve the DAE or the splitted system.

Remark 3.2. *Higher order splitting methods, or splittings where the order of subproblems is reversed would not improve the mesh quality. Since, the mesh quality only depends on the sufficiently good numerical solution of w-system, while the v-system takes care of the surface evolution. For example, a reversed Lie splitting would first construct a good mesh, then evolve it over time in the normal direction, which can still lead to mesh distortions.*

4 Possible extensions

We now briefly describe some further extensions to the above approach. They are rather straightforward to implement, except the one using an approximative distance function (such approximative distance functions are studied, e.g., in [Cha07, HDD+] and the references therein). However, thoroughly comparing and reporting on these extensions would expand the paper enormously, therefore we will here restrict our numerical experiments to the case described above.

4.1 Construction of acute or nonobtuse surface meshes

In a couple of recent works discrete maximum principles (DMPs) and invariant regions have been studied for surface PDEs discretized by surface finite elements, see [FMSV16, FMSV17] and [KKK17]. It is well known that *acute* or *nonobtuse* meshes are required for DMPs even for flat domains, and also for triangulated surface meshes. In general the meshes generated by usual algorithms (for instance DistMesh [PS04], the grid generators of DUNE [BBD$^+$16], etc.) do not necessarily satisfy these angle conditions.

The force function proposed here can be modified in such a way that the resulting mesh is acute or nonobtuse during evolution. In practice, the function F should be obtained not only based on the length function (9), but also on an *angle function*, which does not allow angles approaching a prescribed tolerance α_{TOL} given by the user. This can be viewed as three cords added to a triangle at each angle, which does not allow the angles to expand above the specified angle α_{TOL}.

4.2 Approximating the distance function

The usage of the distance function could also be completely avoided in the splitting scheme. By using a modified distance function $\widetilde{d}(\cdot, t_{n+1})$, which is an approximation of $d(\cdot, t_{n+1})$ based on the nodes obtained from the v-system. Such an approximation can be obtained by a suitable interpolation process over the nodes $\mathbf{x}^v(t_{n+1})$. For instance by a spline interpolation over the nodes, this problem appears to be well studied in the computer science literature, see for example [HDD$^+$, Cha07] and the references therein.

This approach can be useful also in cases where the surface evolution is coupled to the problem on the surface, such as [Dzi90], or [KLLP17].

We note here that in the case of such a modified algorithm the nodes $\mathbf{x}(t_{n+1})$ no longer stay on the exact surface for all times. Therefore, the spatial convergence results of [EV15, KPG17] do not hold. However, in order to show convergence results the approach of [KLLP17] – using three surfaces: the exact surface, its interpolation and the discrete surface – can be used in this case.

4.3 Adaptivity

A cheap adaptive method can be obtained by using some mesh quality test during the time integration of the *autonomous* w-system of the splitting approach. In this case the following algorithm could be realized – assuming a fast implementation of the mesh quality tester. First test the mesh quality, if it is good accept it; else perform a timestep solving the w-system in (16), and then repeat. In such a case the numerical solution of the DAE system is avoided for already good meshes, whereas the computational overhead due to the mesh quality test is negligable.

Later on we will give some possible mesh quality measures, but it can be anything suitable specified by the user, cf. [Knu01, Fie00].

4.4 A local version of the constrained problem

To further decrease computational cost of the Runge–Kutta or the splitting method one may use an ALE map only locally. Since in most cases the evolution of the discrete surface distorts the mesh only locally, one could integrate the constrained system only on those patches which consist of ill shaped triangles, (e.g., those with too small or large angles), and additionally a few layers of neighbouring elements. The rest of the nodes are only evolved by the surface velocity.

5 Numerical experiments

5.1 ALE map tests

Now we will present some numerical experiments validating the choice of the ALE velocity (7) based on the spring system, and also illustrating the good qualities of the DAE model (10). We also compared the Runge–Kutta and the splitting approach to the pure normal evolution of the surface and also to the ALE maps given in the literature. These examples have been used many times previously, see for instance [ES12, EV15, KPG17] and the references therein. Through these ALE maps, which we call literature ALE maps, it will be also clarified further what was meant under *a priori knowledge* in the introduction.

5.1.1 A dumbbell-shaped surface [ES12]

Let the closed surface $\Gamma(t)$ be given by the zero level set of the distance function

$$d(x,t) = x_1^2+x_2^2+K(t)^2G\Big(\frac{x_3^2}{L(t)^2}\Big)-K(t)^2, \quad \text{i.e.,} \quad \Gamma(t) = \{x \in \mathbb{R}^3 \mid d(x,t) = 0\}. \tag{17}$$

Here the functions G, L and K are given by

$$\begin{aligned} G(s) &= 200s\Big(s - \frac{199}{200}\Big), \\ L(t) &= 1 + 0.2\sin(4\pi t), \\ K(t) &= 0.1 + 0.05\sin(2\pi t). \end{aligned}$$

The normal velocity v describes the surface evolution at the nodes by the ODEs:

$$\frac{\mathrm{d}}{\mathrm{dt}}\, x_j(t) = v(x_j(t),t) \tag{18}$$

for $j = 1, 2, \dots, N$. The surface velocity v in $x_j(t)$ is given by

$$v(x_j(t),t) = V_j\nu_j, \quad \text{where} \quad V_j = \frac{-\,\partial_t\, d(x_j(t),t)}{|\nabla d(x_j(t),t)|}, \quad \nu_j = \frac{\nabla d(x_j(t),t)}{|\nabla d(x_j(t),t)|}. \tag{19}$$

Finally, the literature ALE map is given by

$$(x_i(t))_1 = (x_i(0))_1\frac{K(t)}{K(0)}, \quad (x_i(t))_2 = (x_i(0))_2\frac{K(t)}{K(0)}, \quad (x_i(t))_3 = (x_i(0))_3\frac{L(t)}{L(0)}, \tag{20}$$

for every $t \in [0, T]$ and for $i = 1, 2, \ldots, N$, as suggested in [ES12]. This map clearly uses a priori knowledge on the structure of the distance function (17).

To illustrate the good qualities of the DAE model (10) we evolve the surface (17) with all four methods. In Figure 3 we can observe the evolutions of the discrete initial surface $\Gamma_h(0)$ over $[0, 0.6]$. It can be nicely observed that the quality of meshes obtained by both DAE approaches are very similar, however the splitting approach is much faster.

Firstly, plotted on the left-hand side, the purely normal surface evolution obtained by solving the ODE system (18). We have used here the explicit Euler method, with step size $\tau = 0.001$. Although, this method is clearly not the best choice, we used it in order to illustrate the performance of the proposed ALE algorithms.

Secondly, plotted second from the left, the ALE map (20) proposed in the literature, cf. [ES12], based on the structure of the distance function (17).

Thirdly, plotted third from the left, the Runge–Kutta solution of the DAE system (10), using the Radau IIA method with $s = 3$ stages, with a stepsize $\tau = 0.001$, and $k = 500$, $p = 0.4$ in (9).

Finally, plotted on the right-hand side, the ALE map obtained by the splitting method. The evolution and the ALE map is computed exactly as described in Section 3.2.2. The v-system is solved by the explicit Euler method, with $\tau = 0.01$, while the w-system is solved using the classical Runge–Kutta method of order four with 25 substeps, and again $p = 0.4$

It can be nicely observed visually that both ALE approaches provide meshes of similar quality as the literature ALE map.

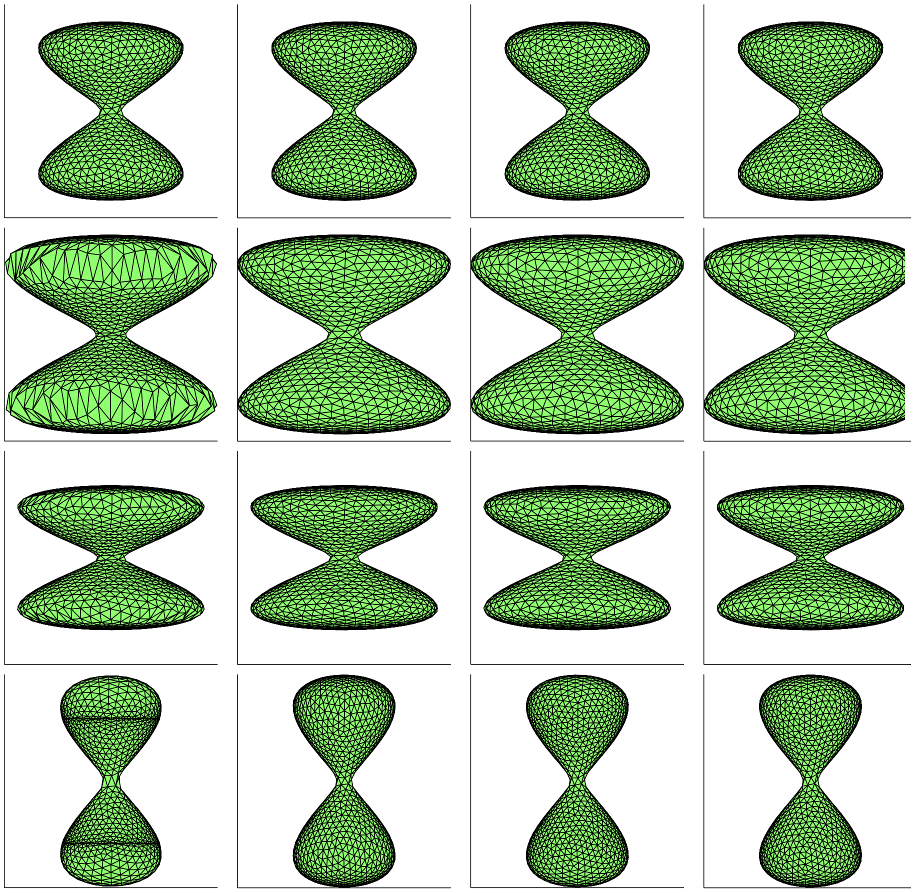

Figure 3: Surface evolution of (17) using pure normal movement (18) (first from left); literature ALE map (20) (second); Runge–Kutta ALE map (third); splitting ALE map (fourth), times $t = 0, 0.2, 0.4, 0.6$ (from top to bottom).

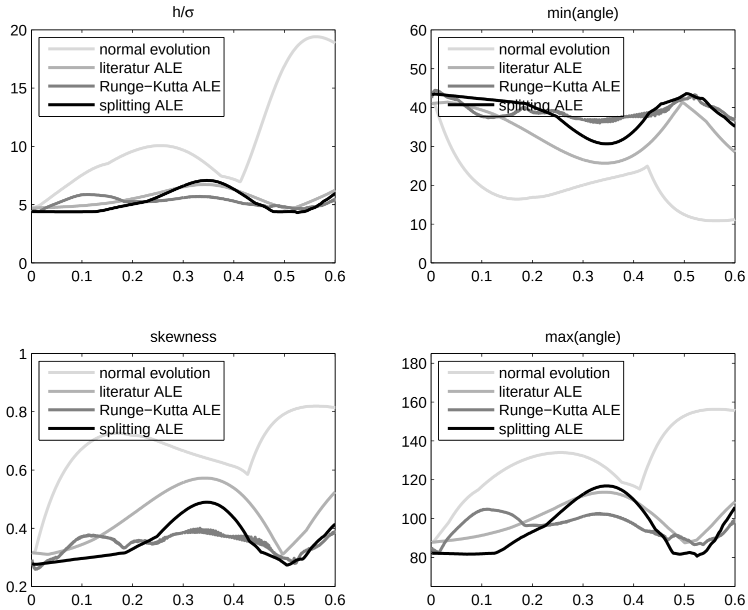

Figure 4: Mesh quality measures plotted against time

In Figure 4 we plotted the evolution of four mesh quality measures (cf. [Fie00]) for all four surface evolutions (described above) against time.

In the top left the maximal ratio of element size and the radius of the inscribed circle,

$$r(t) = \max_{E(t)\in\mathcal{T}_h(t)} \frac{h_{E(t)}}{\sigma_{E(t)}}$$

can be observed, where h_E is the maximal edge length and σ_E is the radius of the inscribed circle of a triangle $E \in \mathcal{T}_h(t)$. The same mesh quality measure is also used in [EF16a]. The plots on the right-hand side show minimum and maximum angles ($\alpha_{\min}(t)$ and $\alpha_{\max}(t)$, top and bottom, respectively) of the mesh, i.e.

$$\alpha_{\min}(t) = \min_{E\in\mathcal{T}_h(t)} \min \alpha_E, \qquad \text{and} \qquad \alpha_{\max}(t) = \max_{E\in\mathcal{T}_h(t)} \max \alpha_E,$$

where α_E contains the three angles of the triangle $E \in \mathcal{T}_h(t)$. Finally, the bottom left plot shows the maximal skewness (also called equiangular skew) $s(t) = \max_{E(t)\in\mathcal{T}_h(t)} s_{E(t)}$, where s_E is the skewness of a triangle $E \in \mathcal{T}_h(t)$, and it is defined as

$$s_E = \max\left\{\frac{\max \alpha_E - 60^\circ}{120^\circ}, \frac{60^\circ - \min \alpha_E}{60^\circ}\right\} \in [0,1].$$

The skewness (or equiangular skew) measures how irregular the triangle E is, see [ANS]. For example, for a regular triangle $s_E = 0$ and for highly irregular ones s_E tends to one. Usually, a mesh is considered good with skewness less than 0.5, while non-acceptable if its skewness exceeds 0.8.

In Figure 4 it can be clearly observed that all three ALE maps (literature and DAE solved by Runge–Kutta or splitting method) provide significantly better meshes as the purely normal evolution. Also, both ALE maps from the numerical solutions of the DAE system provide slightly better meshes then the one suggested in the literature.

5.1.2 Surface with four holes [EV15]

Let the closed surface $\Gamma(t)$ be given by the zero level set of the distance function

$$d(x,t) = \frac{x_1^2}{K(t)^2} + G(x_2^2) + K(t)^2 G\Big(\frac{x_3^2}{L(t)^2}\Big) - 1, \quad \text{i.e.,} \quad \Gamma(t) = \{x \in \mathbb{R}^3 \mid d(x,t) = 0\}. \tag{21}$$

Here the functions G, L and K are given by

$$\begin{aligned} G(s) &= 31.25s(s-0.36)(s-0.95), \\ L(t) &= 1 + 0.3\sin(4\pi t), \\ K(t) &= 0.1 + 0.01\sin(2\pi t). \end{aligned}$$

The normal velocity v describes the surface evolution at the nodes by the ODEs:

$$\frac{\mathrm{d}}{\mathrm{dt}} x_j(t) = v(x_j(t), t), \tag{22}$$

for $j = 1, 2, \ldots, N$, where the surface velocity at $x_j(t)$ is again given by the formula (19).

Finally, the ALE map from the literature is given by

$$(x_i(t))_1 = (x_i(0))_1 \frac{K(t)}{K(0)}, \quad (x_i(t))_2 = (x_i(0))_2, \quad (x_i(t))_3 = (x_i(0))_3 \frac{L(t)}{L(0)}, \tag{23}$$

for every $t \in [0,T]$ and for $i = 1, 2, \ldots, N$, as suggested in [EV15].

In Figure 5, the surface $\Gamma(0)$ (cf. (21)) is evolved over $[0,1]$ again by all four methods, exactly as for Figure 3. Again, the poor meshes of the normal movement can be nicely observed (collapsing triangles with almost zero angles), in contrast with the quasi-regular meshes obtained by the ALE maps.

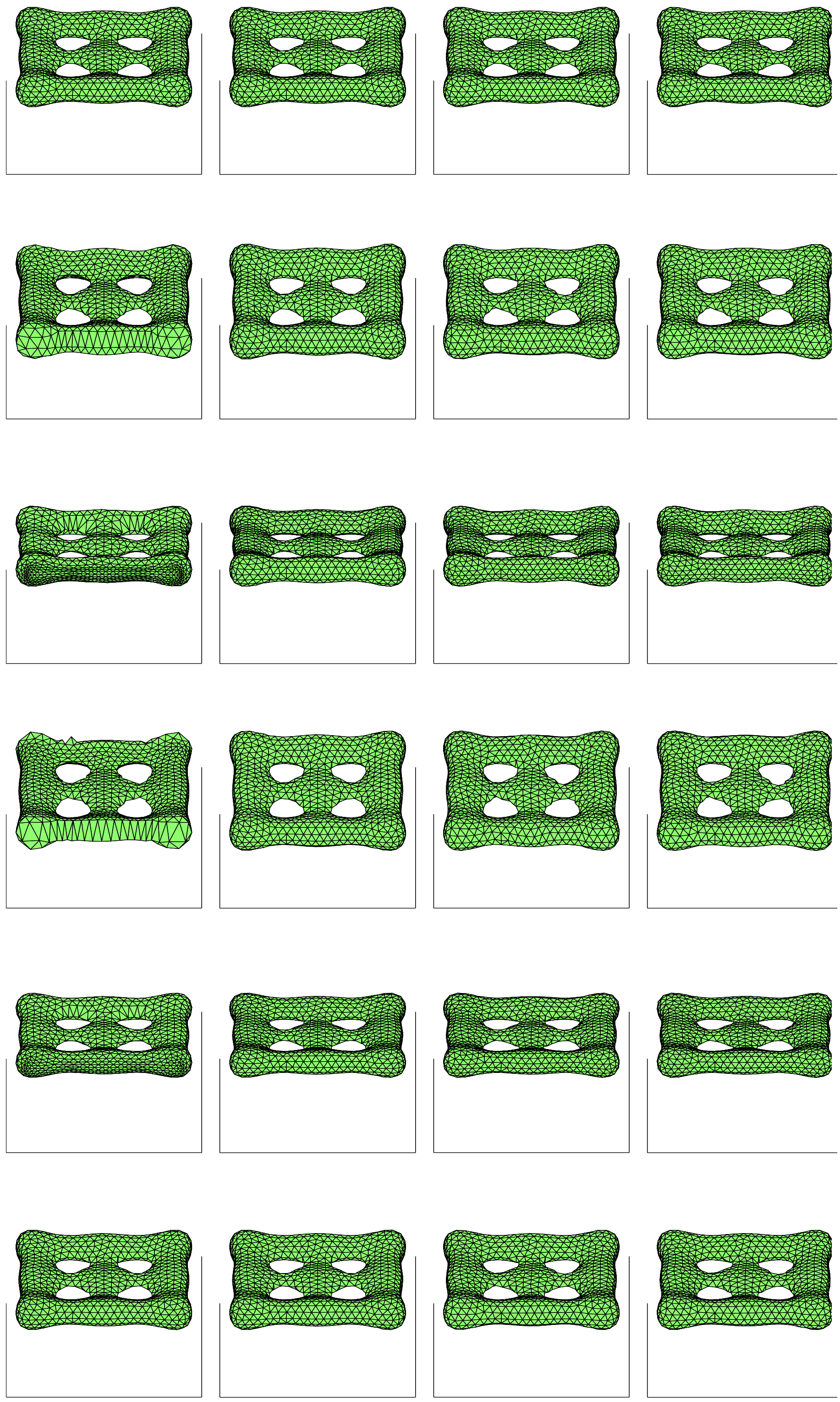

Figure 5: Surface evolution of (21) using pure normal movement (22) (first from left); literature ALE map (23) (second); Runge–Kutta ALE map (third); splitting ALE map (fourth); at times $t = 0, 0.2, 0.4, 0.6, 0.8, 1$ (from top to bottom).

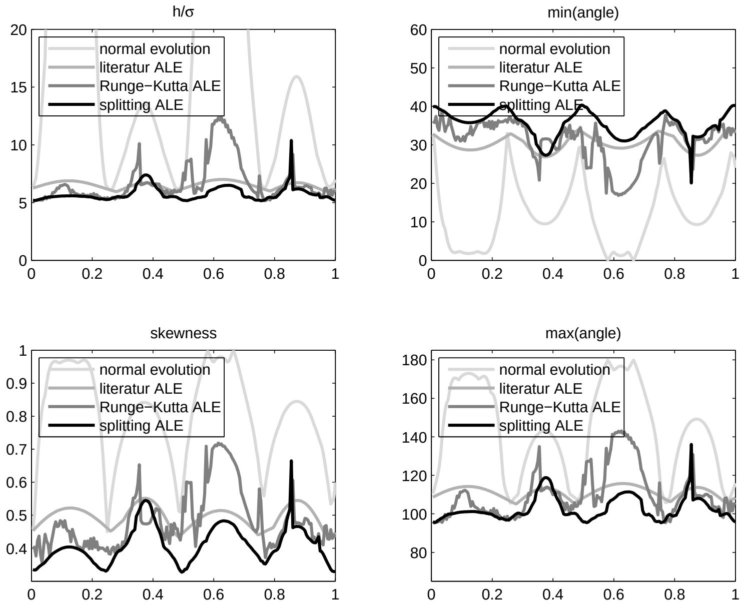

Figure 6: Mesh quality measures plotted against time

Similarly, the mesh quality measures are plotted in Figure 6. All three ALE methods provide much better meshes than the normal evolution, while they still yield meshes of similar quality.

5.2 Error behaviour of evolving surface parabolic PDEs

We tested the performance of the spring ALE algorithm by using it in the numerical solution of an evolving surface PDE. As a well studied example, we carried out the same experiments over the evolving surface of Section 5.1.1, which have been also used in [ES12, EV15, KPG17].

The evolving surface PDE (1) is discretized using ALE ESFEM, described in detail in [EV15, KPG17]. The inhomogeneity f is chosen such that the true solution is known: $u(x,t) = e^{-6t}x_1x_2$.

5.2.1 Discretization errors

The errors of the obtained lifted numerical solution is calculated in the following norms at time $T = N\tau = 1$:

$$\|u(\cdot, N\tau) - u_h^N\|_{L^2(\Gamma(N\tau))} \qquad \text{and} \qquad \|\nabla_\Gamma(u(\cdot, N\tau) - u_h^N)\|_{L^2(\Gamma(N\tau))}.$$

The logarithmic plots, in Figure 7, show the usual convergence plots: the two errors against time step size τ.

Figure 7 only serves as an illustration that the meshes from the literature ALE map and from the DAE system yield errors of the same magnitude. More spatial refinements would obfuscate the readability of the plots. For more detailed convergence tests we refer to [EV15, KPG17].

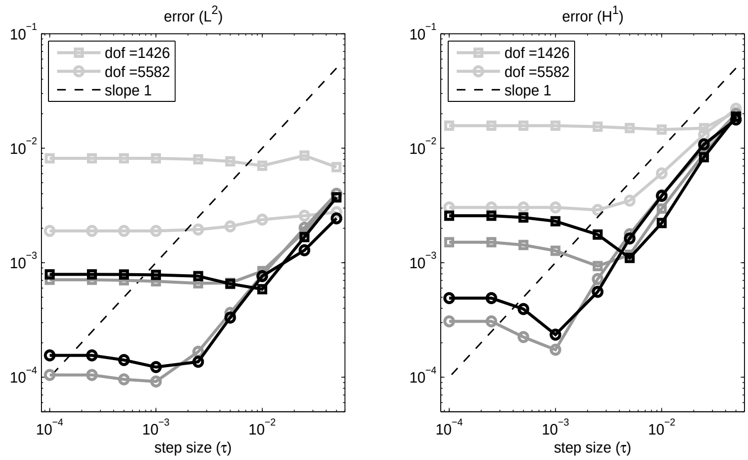

Figure 7: L^2 and H^1 norms of the errors for different spatial refinements plotted against the stepsize for the normal movement, literature and splitting ALE maps (light grey, grey and black, respectively) at time $T = 1$

Figure 7 shows the errors for all three methods concerning the evolution of the surface (normal evolution, literature ALE and splitting ALE). On the left we show the errors in the L^2 norm, while on the right in the H^1 norm. Different shades correspond to different ALE maps (light grey, grey and black, respectively), while the lines with different markers are corresponding to different mesh refinements.

As usual we can observe two regions in Figure 7: A region where the time discretization error dominates, matching to the convergence rates of the theoretical results for the backward Euler method (order 1, not the reference line). In the other region, with smaller stepsizes, the spatial discretization error is dominating (the error curves are flatting out).

5.2.2 Computation times

In Figure 8 we also compare the computational times for the pure normal movement and splitting ALE approach (light grey and black in the figure). In Figure 8 the errors at time $T = 1$ (both in the L^2 and H^1 norms, left and right, respectively) are plotted against the CPU times for different spatial refinements (denoted by different markers: $\square$, $\circ$, $\times$) and different stepsizes (which are $\tau = 0.05, 0.025, 0.01, 0.005,\ 0.0025,\ 0.001$, corresponding to the markers). The CPU times include all computations: normal evolution of the surface, surface matrix assemblies, solution of the linear system, and also the computation of the ALE mesh in the ALE case. A computational trade-off can be clearly observed, (see, for instance the two rightmost lines in each plot), that the same errors can be achieved by the ALE method on a much coarser mesh (having only quarter as many nodes as the non-ALE mesh), at a computational cost reduction of factor 10. Again, in the region with smaller stepsizes, the spatial discretization error is dominating (hence the curves are flatting out).

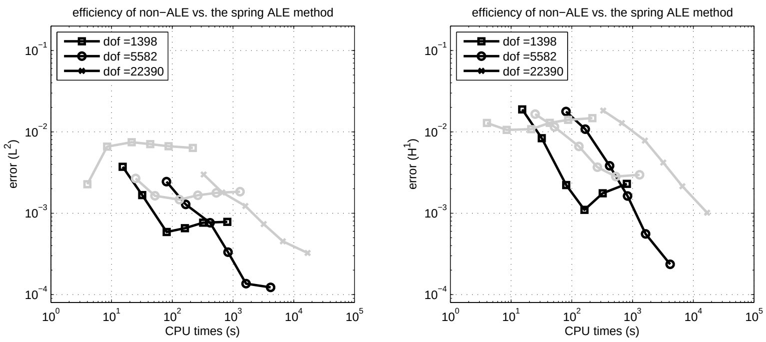

Figure 8: L^2 and H^1 norms of the errors for different spatial refinements and time stepsizes plotted against the CPU times (in seconds) for the normal movement and splitting ALE map (light grey and black, respectively)

5.3 Meshes with angle conditions

We also report on a somewhat simplified version of the force function proposed in Section 4.1 in order to construct acute or nonobtuse meshes.

We consider here a stationary example of a torus. We constructed a triangulation using DistMesh [PS04], seen left in Figure 9, which is not acute, the largest angle of the mesh is above 100° (see the first entry in rightmost graph in Figure 9).

As the surface is stationary here, we have the modified DAE system

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\,\mathbf{x}(t) &= kF(\mathbf{x}(t)) + k_\alpha F_\alpha(\mathbf{x}(t)) - D(\mathbf{x}(t))^T\boldsymbol{\lambda}(t),\\ d(\mathbf{x}(t),t) &= 0,\end{aligned}$$

with an additional force function F_α, which is used to eliminate non-acute angles. It is defined analogously as the function F, but using a length function $f_\alpha(e)$ based on the angle α_e opposite to the edge e, instead of the formula in (9). If the angle α_e is larger than a user given tolerance, the desired length of e is set to a value based on the law of cosines. As the important force is now F_α we set its spring constant $k_\alpha = 4k$. The original force function F (with $p = 0.1$ in (9)) is only kept for smoothing reasons. We note here, that these parameters are not claimed to be used universally.

The numerical solution of the above DAE system yields meshes with favourable angle properties. We set here the angle tolerance to 85° and integrate the system over 25 steps. The maximum angle in each time step can be seen on the right in Figure 9, it can be observed that the angles quickly drop around the desired value, while the finally obtained acute mesh can be seen in the middle of Figure 9.

6 Conclusion

An efficient algorithm has been proposed to compute arbitrary Lagrangian Eulerian maps for closed evolving surfaces, without any a priori knowledge on the

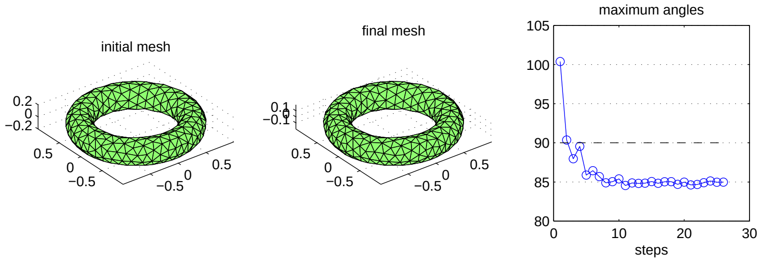

Figure 9: Acute triangulation of a torus

surface evolution. To the knowledge of the author, such an algorithm was not presented before in the literature.

The algorithm is based on the fast solution of the a constrained (DAE) system, which obtains a tangential velocity based a spring analogy (which equidistributes the grid points). Both the quality of the evolved meshes and the efficiency of the algorithm is demonstrated on various numerical experiments chosen from the literature.

Various generalisations of the proposed method is discussed. Among them, an algorithm of ALE maps with angle conditions is also described in detail and illustrated as well. Such methods are of interest for qualitative results, such as discrete maximum principles [FMSV16, FMSV17, KKK17].

Acknowledgement

We thank Christian Lubich for our many discussions on differential algebraic problems, and for his invaluable comments and remarks. We thank Hanna Walach for her helpful suggestions regarding the manuscript. We also thank Christian Power for our early discussions on the topic.

This work was supported by the Deutsche Forschungsgemeinschaft (DFG) through SFB 1173.

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Appendix C. High-order evolving surface finite element method for parabolic problems on evolving surfaces

IMA Journal of Numerical Analysis (2018) **38**, 430–459
doi: 10.1093/imanum/drx013
Advance Access publication on March 19, 2017

High-order evolving surface finite element method for parabolic problems on evolving surfaces

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[Received on 23 June 2016; revised on 20 October 2016]

High-order spatial discretizations and full discretizations of parabolic partial differential equations on evolving surfaces are studied. We prove convergence of the high-order evolving surface finite element method by showing high-order versions of geometric approximation errors and perturbation error estimates and by the careful error analysis of a modified Ritz map. Furthermore, convergence of full discretizations using backward difference formulae and implicit Runge–Kutta methods are also shown.

Keywords: parabolic problems; evolving surfaces; high-order ESFEM; Ritz map; convergence; BDF and Runge–Kutta methods.

1. Introduction

Numerical methods for partial differential equations (PDEs) on stationary and evolving surfaces and for coupled bulk–surface PDEs have been under intensive research in recent years. Surface finite element methods are all based on the fundamental article of Dziuk (1988), further developed for evolving surface parabolic problems by Dziuk & Elliott (2007, 2013b).

High-order versions of various finite element methods for problems on a *stationary surface* have received attention in a number of publications previously. We give a brief overview of this literature here:

- The surface finite element method of Dziuk (1988) was extended to higher-order finite elements on stationary surfaces by Demlow (2009). Some further important results for higher-order surface finite elements were shown by Elliott & Ranner (2013).
- Discontinuous Galerkin methods for elliptic surface problems were analysed by Dedner *et al.* (2013) and then extended to high-order discontinuous Galerkin methods in Antonietti *et al.* (2015).
- Recently, *unfitted* (also called trace or cut) finite element methods have been investigated intensively (see, e.g., Olshanskii *et al.*, 2009; Burman *et al.*, 2015; Reusken, 2015). A higher-order version of the trace finite element method was analysed by Grande & Reusken (2014).

However, to our knowledge, there are no articles showing convergence of the *high-order* evolving surface finite element method for parabolic partial differential equations on *evolving surfaces*.

In this article, we extend the H^1- and L^2-norm convergence results of Dziuk & Elliott (2007, 2013b) to *high-order* evolving surface finite elements applied to parabolic problems on evolving surfaces with prescribed velocity. Furthermore, convergence results for full discretizations using Runge–Kutta methods, based on Dziuk *et al.* (2012), and using backward difference formulae (BDF), based on Lubich *et al.* (2013), are also shown.

To prove high-order convergence of the spatial discretization, three main groups of errors have to be analysed:

- *Geometric errors*, resulting from the appropriate approximation of the smooth surface. Many of these results carry over from Demlow (2009) by careful investigation of time dependencies, while others are extended from Dziuk *et al.* (2012), Mansour (2013) and Lubich *et al.* (2013).
- *Perturbation errors of the bilinear forms*, whose higher-order version can be shown by carefully using the core ideas of Dziuk & Elliott (2013b).
- High-order estimates for the *errors of a modified Ritz map*, which was defined in Lubich & Mansour (2015). These projection error bounds rely on the nontrivial combination of the mentioned geometric error bounds and on the well-known Aubin–Nitsche trick.

We further show convergence results for full discretizations using Runge–Kutta and BDF methods. The error estimates, based on *energy estimates*, for Runge–Kutta methods shown in Dziuk *et al.* (2012) and for BDF methods in Lubich *et al.* (2013) are applicable without any modifications, because the semidiscrete problem can be written in a matrix–vector formulation (cf. Dziuk *et al.*, 2012), where the matrices have exactly the same properties as in the linear finite element case. Therefore, the fully discrete convergence results transfer to high-order evolving surface finite elements using the mentioned error estimates of the Ritz map.

The implementation of the high-order method is also a nontrivial task. The matrix assembly of the time-dependent mass and stiffness matrices is based on the usual reference element technique. Similar to isoparametric finite elements, the approximating surface is parameterized over the reference element, and therefore all the computations are done there.

It was pointed out by Grande & Reusken (2014) that the approach of Demlow (2009) requires *explicit knowledge of the exact signed distance function to the surface* Γ. However, the signed distance function is used only in the analysis, but it is not required for the numerical computations away from the initial time level. This is possible since the high-order element is uniquely determined by its elements, and by using the usual reference element technique. It is used only for generating the initial surface approximation.

Here, we consider only linear parabolic PDEs on evolving surfaces; however, we believe our techniques and results carry over to other cases, such as to the Cahn–Hilliard equation (Elliott & Ranner, 2015), to wave equations (Lubich & Mansour, 2015), to ALE methods (Elliott & Venkataraman, 2014; Kovács & Power Guerra, 2014), nonlinear problems (Kovács & Power Guerra, 2016) and to evolving versions of bulk–surface problems (Elliott & Ranner, 2013). For more details, see a remark later on. Furthermore, while, in this article, we consider only evolving surfaces with prescribed velocity, many of the high-order geometric estimates of this article are essential for the numerical analysis of parabolic problems where the *surface velocity depends on the solution* (cf. Kovács *et al.*, 2016).

The article is organized in the following way. In Section 2, we recall the basics of linear parabolic problems on evolving surfaces together with some notation based on Dziuk & Elliott (2007). Section 3 deals with the description of higher-order evolving surface finite elements based on Demlow (2009). Section 4 contains the time discretizations and states the main results of this article: semidiscrete and fully discrete convergence estimates. In Section 5, we turn to the estimates of geometric errors and geometric perturbation estimates. In Section 6, the errors in the generalized Ritz map and its material derivatives are estimated. Section 7 contains the proof of the main results. Section 8 briefly describes the implementation of the high-order evolving surface finite elements. In Section 9, we present some numerical experiments illustrating our theoretical results.

2. The problem

Let us consider a smooth-evolving compact surface, given by a smooth signed distance function d, $\Gamma(t) = \{x \in \mathbb{R}^{m+1} \mid d(x,t) = 0\} \subset \mathbb{R}^{m+1}$ ($m \leq 3$), $0 \leq t \leq T$, which moves with a given smooth velocity v. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of the function u, where ∇_Γ is the tangential gradient given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \mathrm{n}\mathrm{n}$, with unit normal n. We are sharing the setting of Dziuk & Elliott (2007) and Dziuk & Elliott (2013b).

We consider the following linear problem on the above surface, for $u = u(x,t)$:

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u &= f \qquad \text{on } \Gamma(t), \\ u(\cdot,0) &= u_0 \qquad \text{on } \Gamma(0), \end{aligned} \tag{2.1}$$

where for simplicity we set $f = 0$, but all our results hold with a nonvanishing inhomogeneity as well.

Let us briefly recall some important concepts used later on, whereas, in general, for basic formulae we refer to Dziuk & Elliott (2013a). An important tool is the Green's formula on closed surfaces,

$$\int_{\Gamma(t)} \nabla_{\Gamma(t)} z \cdot \nabla_{\Gamma(t)} \phi = -\int_{\Gamma(t)} (\Delta_{\Gamma(t)} z)\phi.$$

We use Sobolev spaces on surfaces: for a smooth surface $\Gamma(t)$, for fixed $t \in [0,T]$ or for the space–time manifold given by $\mathscr{G}_T = \cup_{t\in[0,T]}\Gamma(t) \times \{t\}$, we define

$$\begin{aligned} H^1(\Gamma(t)) &= \left\{\eta \in L^2(\Gamma(t)) \mid \nabla_{\Gamma(t)}\eta \in L^2(\Gamma(t))^{m+1}\right\}, \\ H^1(\mathscr{G}_T) &= \left\{\eta \in L^2(\mathscr{G}_T) \mid \nabla_{\Gamma(t)}\eta \in L^2(\Gamma(t))^{m+1}, \partial^\bullet\eta \in L^2(\Gamma(t))\right\} \end{aligned}$$

and analogously for higher-order versions $H^k(\Gamma(t))$ and $H^k(\mathscr{G}_T)$ for $k \in \mathbb{N}$ (cf. Dziuk & Elliott, 2007, Section 2.1).

The variational formulation of this problem reads as follows: find $u \in H^1(\mathscr{G}_T)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)}\varphi = \int_{\Gamma(t)} u\partial^\bullet\varphi \tag{2.2}$$

holds for almost every $t \in (0,T)$ for every $\varphi(\cdot,t) \in H^1(\Gamma(t))$ with $\partial^\bullet\varphi(\cdot,t) \in L^2(\Gamma(t))$ and $u(\cdot,0) = u_0$ holds. For suitable u_0, existence and uniqueness results for (2.2) were obtained in Dziuk & Elliott (2007, Theorem 4.4).

3. High-order evolving surface finite elements

We define the high-order evolving surface finite element method (ESFEM) applied to our problem following Dziuk (1988), Dziuk & Elliott (2007) and Demlow (2009). We use simplicial elements and continuous piecewise polynomial basis functions of degree k.

3.1 *Basic notions*

The smooth *initial* surface $\Gamma(0)$ is approximated by a k-order interpolating discrete surface constructed in (a) below. This high-order approximation surface is then *evolved* in time by the *a priori* known surface velocity v, detailed in (b). The construction presented here is from Demlow (2009, Section 2).

(a) Let $\Gamma_h^1(0)$ be a polyhedron having triangular elements (denoted by E) with vertices lying on the initial surface $\Gamma(0)$, forming a quasiuniform triangulation $\mathscr{T}_h^1(0)$, with mesh parameter h, and unit outward normal n_h^1.

Next, for $k \geq 2$, we define $\Gamma_h^k(0)$, the k-order polynomial approximations to $\Gamma(0)$. For a given fixed element $E \in \mathscr{T}_h^1(0)$, with nodes (numbered locally) $x_1, x_2, \ldots, x_{n_k}$, the corresponding Lagrange basis functions of degree k on E are denoted by $\tilde{\chi}_1^k, \tilde{\chi}_2^k, \ldots, \tilde{\chi}_{n_k}^k$. For arbitrary $x \in E$, we define the discrete projection

$$p^k(x,0) = \sum_{j=1}^{n_k} p(x^j,0)\tilde{\chi}_j^k(x,0), \qquad \text{with} \qquad \Gamma(0) \ni p(x^j,0) = x^j - d(x^j,0)\mathrm{n}(p(x^j,0)).$$

This definition yields a continuous piecewise polynomial map on $\Gamma_h^1(0)$. Then the k-order approximation surface is defined by

$$\Gamma_h^k(0) = \{p^k(x,0) \mid x \in \Gamma_h^1(0)\}.$$

The vertices of $\Gamma_h^k(0)$ are denoted by $a_i(0)$, $i = 1, 2, \ldots, N$ (here numbered globally). Note that these vertices are sitting on the exact surface. The high-order triangulation is denoted by $\mathscr{T}_h^k(0)$. Further small details can be found in Demlow (2009).

(b) The surface approximation $\Gamma_h^k(t)$ at time $t \in [0,T]$ is given by evolving the nodes of the initial triangulation by the velocity v along the space-time manifold. For instance, the nodes $a_i(t)$ are determined by the ordinary differential equation (ODE) system

$$\frac{\mathrm{d}}{\mathrm{d}t}a_i(t) = v(a_i(t),t).$$

Then the base triangulation and the basis functions move along as well. Hence, for $0 \leq t \leq T$, the nodes $(a_i(t))_{i=1}^N$ define an approximation of $\Gamma(t)$ of degree k,

$$\Gamma_h^k(t) = \{p^k(x,t) \mid x \in \Gamma_h^1(t)\},$$

defined analogously as for $t = 0$. Therefore, the discrete surface $\Gamma_h^k(t)$ remains an interpolation of $\Gamma(t)$ for all times. The high-order triangulation at t is denoted by $\mathscr{T}_h^k(t)$.

REMARK 3.1 It is highly important to note here that computing $p(\cdot,t)$ or $p^k(\cdot,t)$, for $t > 0$, is never used during the numerical computations, hence explicit knowledge of the distance function is avoided (except for $t = 0$). Instead, the positions of the nodes appearing in an element of $\Gamma_h^k(t)$ are used to construct a reference mapping over the reference triangle E_0. In fact, the nodes (locally numbered here) $a_1(t), a_2(t), \ldots, a_{n_k}(t)$ of the high-order element uniquely determine an approximating surface of degree k (being an image of a polynomial of m variables over E_0). The ESFEM matrices are then assembled using the reference element and the reference mapping.

The discrete tangential gradient on the discrete surface $\Gamma_h^k(t)$ is given by

$$\nabla_{\Gamma_h^k(t)}\phi := \nabla\phi - \nabla\phi \cdot \mathrm{n}_h^k \mathrm{n}_h^k,$$

understood in an elementwise sense, with n_h^k denoting the normal to $\Gamma_h^k(t)$.

For every $t \in [0, T]$ and for the k-order approximation surface $\Gamma_h^k(t)$, we define the finite element subspace of order k, denoted by $S_h^k(t)$, spanned by the continuous evolving basis functions χ_j of piecewise degree k, satisfying $\chi_j(a_i(t), t) = \delta_{ij}$ for all $i, j = 1, 2, \dots, N$:

$$\begin{aligned} S_h^k(t) &= \{\phi \in C(\Gamma_h^k(t)) \mid \phi = \tilde{\phi} \circ p^k(\cdot, t)^{-1} \text{ where } \tilde{\phi} \in \tilde{S}_h^k(t)\} \\ &= \operatorname{span}\big\{\chi_1(\cdot, t), \chi_2(\cdot, t), \dots, \chi_N(\cdot, t)\big\}, \end{aligned}$$

where

$$\chi_j(\cdot, t) = \tilde{\chi}_j(p^k(\cdot, t)^{-1}, t) \qquad (j = 1, 2, \dots, N),$$

with $\tilde{\chi}_j(\cdot, t)$ being the degree k polynomial basis function over the base triangulation Γ_h^1, spanning the space $\tilde{S}_h^k(t) = \{\tilde{\phi} \in C(\Gamma_h^1(t)) \mid \tilde{\phi}$ is a piecewise polynomial of degree $k\}$. By construction $S_h^k(t)$ is an isoparametric finite element space.

We interpolate the surface velocity on the discrete surface using the basis functions and denote it by V_h. Then the discrete material derivative is given by

$$\partial_h^\bullet \phi_h = \partial_t \phi_h + V_h \cdot \nabla \phi_h \qquad (\phi_h \in S_h^k(t)).$$

The key *transport property* derived by Dziuk & Elliott (2007, Proposition 5.4) is

$$\partial_h^\bullet \chi_j = 0 \qquad \text{for} \quad j = 1, 2, \dots, N. \tag{3.1}$$

This result translates from the linear ESFEM case by the original proof of Dziuk & Elliott (2007, Section 5.2) using barycentric coordinates and the chain rule.

The spatially discrete problem (for a fixed degree k) then reads as follows: find $U_h \in S_h^k(t)$, with $\partial_h^\bullet U_h \in S_h^k(t)$ and temporally continuous, such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma_h^k(t)} U_h \phi_h + \int_{\Gamma_h^k(t)} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h = \int_{\Gamma_h^k(t)} U_h \partial_h^\bullet \phi_h \qquad (\forall \phi_h \in S_h^k(t) \text{ with } \partial_h^\bullet \phi_h \in S_h^k(t)), \tag{3.2}$$

with the initial condition $U_h^0 \in S_h^k(0)$ being a suitable approximation to u_0.

Later on we will always work with a high-order approximation surface $\Gamma_h^k(t)$ and with the corresponding high-order evolving surface finite element space $S_h^k(t)$ (with $2 \leq k \in \mathbb{N}$); therefore from now on, we drop the upper index k, unless we would like to emphasize it or it is not clear from the context.

3.2 *The ODE system*

Similarly to Dziuk *et al.* (2012), the ODE form of the above problem (3.2) can be derived by setting

$$U_h(\,\cdot\,, t) = \sum_{j=1}^N \alpha_j(t) \chi_j(\,\cdot\,, t)$$

in the semidiscrete problem, and by testing with $\phi_h = \chi_j$ $(j = 1, 2, \dots, N)$, and using the transport property (3.1).

The spatially semidiscrete problem (3.2) is equivalent to the following ODE system for the vector $\alpha(t) = (\alpha_j(t))_{j=1}^N \in \mathbb{R}^N$, collecting the nodal values of $U_h(\cdot, t)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(M(t)\alpha(t)) + A(t)\alpha(t) &= 0, \\ \alpha(0) &= \alpha_0, \end{aligned} \tag{3.3}$$

where the evolving mass matrix $M(t)$ and the stiffness matrix $A(t)$ are defined as

$$M(t)|_{kj} = \int_{\Gamma_h(t)} \chi_j \chi_k \quad \text{and} \quad A(t)|_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h} \chi_j \cdot \nabla_{\Gamma_h} \chi_k,$$

for $j, k = 1, 2, \ldots, N$.

3.3 *Lifts*

For the error analysis we need to compare functions on different surfaces. This is conveniently done by the *lift operator*, which was introduced in Dziuk (1988) and further investigated in Dziuk & Elliott (2007, 2013b). The lift operator maps a function on the discrete surface onto a function on the exact surface.

Let $\Gamma_h(t)$ be a k-order approximation to the exact surface $\Gamma(t)$. Using the oriented distance function d (cf. Dziuk & Elliott, 2007, Section 2.1), the lift of a continuous function $\eta_h \colon \Gamma_h(t) \to \mathbb{R}$ is defined as

$$\eta_h^l(p, t) := \eta_h(x, t), \qquad x \in \Gamma(t),$$

where for every $x \in \Gamma_h(t)$ the point $p = p(x, t) \in \Gamma(t)$ is uniquely defined via

$$p = x - \mathrm{n}(p, t)\, d(x, t). \tag{3.4}$$

By η^{-l} we denote the function on $\Gamma_h(t)$ whose lift is η.

In particular, we will often use the space of lifted basis functions

$$(S_h(t))^l = (S_h^k(t))^l = \left\{\phi_h^l \mid \phi_h \in S_h^k(t)\right\}.$$

4. Convergence estimates

4.1 *Convergence of the semidiscretization*

We now formulate the convergence theorem for the semidiscretization using high-order evolving surface finite elements. This result is the higher-order extension of Dziuk & Elliott (2013b, Theorem 4.4).

THEOREM 4.1 Consider the ESFEM of order k as a space discretization of the parabolic problem (2.1). Let u be a sufficiently smooth solution of the problem, and assume that the initial value satisfies

$$\|u_h^0 - u(\cdot, 0)\|_{L^2(\Gamma(0))} \le C_0 h^{k+1}.$$

Then there exists $h_0 > 0$, such that for mesh size $h \leq h_0$, the following error estimate holds, for $t \leq T$:

$$\|u_h(\cdot,t) - u(\cdot,t)\|_{L^2(\Gamma(t))} + h\left(\int_0^t \|\nabla_{\Gamma(s)}(u_h(\cdot,s) - u(\cdot,s))\|^2_{L^2(\Gamma(s))}\,\mathrm{d}s\right)^{1/2} \leq Ch^{k+1}.$$

The constant C is independent of h and t but depends on T.

The proof of this result is postponed to a later section, after we have shown some preparatory results.

4.2 *Time discretization: BDF*

We apply a p-step BDF for $p \leq 5$ as a discretization of the ODE system (3.3), coming from the ESFEM space discretization of the parabolic evolving surface PDE.

We briefly recall the p-step BDF method applied to system (3.3) with step size $\tau > 0$:

$$\frac{1}{\tau}\sum_{j=0}^{p} \delta_j M(t_{n-j})\alpha_{n-j} + A(t_n)\alpha_n = 0 \qquad (n \geq p), \tag{4.1}$$

where the coefficients of the method are given by $\delta(\zeta) = \sum_{j=0}^{p} \delta_j \zeta^j = \sum_{\ell=1}^{p} \frac{1}{\ell}(1-\zeta)^\ell$, while the starting values are $\alpha_0, \alpha_1, \ldots, \alpha_{p-1}$. The method is known to be 0-stable for $p \leq 6$ and have order p (for more details, see Hairer & Wanner, 1996, Chapter V).

In the following result, we compare the fully discrete solution

$$U_h^n = \sum_{j=1}^{N} \alpha_j^n \chi_j(\,\cdot\,, t_n),$$

obtained by solving (4.1) and the Ritz map $\widetilde{\mathscr{P}}_h : H^1(\Gamma(t)) \to S_h^k(t)$ $(t \in [0,T])$ of the sufficiently smooth solution u. The precise definition of the Ritz map is given later.

The $H_h^{-1}(\Gamma_h(t))$-norm of the ESFEM function R_h is defined as

$$\|R_h(\cdot,t)\|_{H_h^{-1}(\Gamma_h(t))} = \sup_{0 \neq \phi_h \in S_h^k(t)} \frac{m_h(R_h(\cdot,t),\phi_h)}{\|\phi_h\|_{H^1(\Gamma_h(t))}}\,.$$

The following error bound was shown in Lubich *et al.* (2013) for BDF methods up to order 5 (see also Mansour, 2013).

THEOREM 4.2 (Lubich *et al.*, 2013, Theorem 5.1) Consider the parabolic problem (2.1), having a sufficiently smooth solution for $0 \leq t \leq T$. Couple the k-order ESFEM as space discretization with time discretization by a p-step backward difference formula with $p \leq 5$. Assume that the Ritz map of the solution has continuous discrete material derivatives up to order $p+1$. Then there exists $\tau_0 > 0$, independent of h, such that for $\tau \leq \tau_0$, for the error $E_h^n = U_h^n - \widetilde{\mathscr{P}}_h u(\cdot,t_n)$ the following estimate holds for $t_n = n\tau \leq T$:

$$\|E_h^n\|_{L^2(\Gamma_h(t_n))} + \left(\tau\sum_{j=1}^{n} \|\nabla_{\Gamma_h(t_j)} E_h^j\|^2_{L^2(\Gamma_h(t_j))}\right)^{1/2}$$

$$\leq C\tilde{\beta}_{h,p}\tau^p + \left(\tau\sum_{j=1}^{n}\|R_h(\cdot,t_j)\|^2_{H_h^{-1}(\Gamma_h(t_j))}\right)^{1/2} + C\max_{0\leq i\leq p-1}\|E_h^i\|_{L^2(\Gamma_h(t_i))},$$

where R_h is the high-order ESFEM residual. The constant C is independent of h, n and τ but depends on T. Furthermore,

$$\tilde{\beta}^2_{h,p} = \int_0^T \sum_{\ell=1}^{p+1}\|(\partial_h^\bullet)^{(\ell)}(\widetilde{\mathscr{P}}_h u)(\cdot,t)\|_{L^2(\Gamma_h(t))}\,\mathrm{d}t.$$

REMARK 4.3 The most important technical tools in the proof of Theorem 4.2, and also in the proof of the BDF stability result in Lubich *et al.* (2013, Lemma 4.1), are the ODE formulation (3.3) and the key estimates first shown in Dziuk *et al.* (2012, Lemma 4.1), where the following estimates are shown: there exist $\mu, \kappa > 0$ such that, for $w, z \in \mathbb{R}^N$,

$$w^{\mathrm{T}}\big(M(s) - M(t)\big)z \leq (e^{\mu(s-t)} - 1)\|w\|_{M(t)}\|z\|_{M(t)}, \quad w^{\mathrm{T}}\big(A(s) - A(t)\big)z \leq (e^{\kappa(s-t)} - 1)\|w\|_{A(t)}\|z\|_{A(t)}.$$

The proof of these bounds involves only basic properties of the mass and stiffness matrices $M(t)$ and $A(t)$; hence it is independent of the order of the basis functions. Hence, the high-order ESFEM versions of these two inequalities also hold. Therefore, the original results of Lubich *et al.* (2013) hold here as well.

Later on, when suitable results are at hand, we give some remarks on the smoothness assumption of the Ritz map.

4.3 *Convergence of the full discretization*

We are now in a position to formulate one of the main results of this article, which yields optimal-order error bounds for high-order finite element semidiscretization coupled to BDF methods up to order 5 applied to an evolving surface PDE.

THEOREM 4.4 (k-order ESFEM and BDF-p) Consider the ESFEM of order k as space discretization of the parabolic problem (2.1), coupled to the time discretization by a p-step backward difference formula with $p \leq 5$. Let u be a sufficiently smooth solution of the problem and assume that the starting values satisfy (with $\mathscr{P}_h u = (\widetilde{\mathscr{P}}_h u)^l$)

$$\max_{0\leq i\leq p-1}\|u_h^i - (\mathscr{P}_h u)(\cdot,t_i)\|_{L^2(\Gamma(t_i))} \leq C_0 h^{k+1}.$$

Then there exist $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leq h_0$ and $\tau \leq \tau_0$, the following error estimate holds for $t_n = n\tau \leq T$:

$$\|u_h^n - u(\cdot,t_n)\|_{L^2(\Gamma(t_n))} + h\left(\tau\sum_{j=p}^{n}\|\nabla_{\Gamma(t_j)}(u_h^j - u(\cdot,t_j))\|^2_{L^2(\Gamma(t_j))}\right)^{1/2} \leq C\big(\tau^p + h^{k+1}\big).$$

The constant C is independent of h, τ and n but depends on T.

The proof of this result is also postponed to a later section, after we have shown some preparatory results.

REMARK 4.5 We remark here that an analogous fully discrete convergence result is readily available for algebraically stable implicit Runge–Kutta methods (such as the Radau IIA methods), since the Runge–Kutta analogue of Theorem 4.2 has been proved in Dziuk *et al.* (2012). The combination of this result with our high-order semidiscrete error bounds (Theorem 7.1) proves the Runge–Kutta analogue of Theorem 4.4.

5. Geometric estimates

In this section, we present further notation and some technical lemmas that will be used later on in the proofs leading to the convergence result. These estimates are high-order analogues of some previous results proved in Dziuk (1988), Dziuk & Elliott (2007), Demlow (2009), Mansour (2013) and Dziuk & Elliott (2013b).

5.1 *Geometric approximation results*

In the following, we state and prove estimates for the errors resulting from the geometric surface approximation. Most of these estimates hold for a sufficiently small h, which here means for $h \leq h_0$ with a sufficiently small $h_0 > 0$.

LEMMA 5.1 (Equivalence of norms, Dziuk, 1988; Demlow, 2009) Let $\eta_h : \Gamma_h(t) \to \mathbb{R}$ with lift $\eta_h^l : \Gamma(t) \to \mathbb{R}$. Then, for a sufficiently small h, the discrete and continuous L^p and Sobolev norms are equivalent, independently of the mesh size h.

For instance, there is a constant $c > 0$ such that for all h sufficiently small,

$$c^{-1}\|\eta_h\|_{L^2(\Gamma_h(t))} \leq \|\eta_h^l\|_{L^2(\Gamma(t))} \leq c\|\eta_h\|_{L^2(\Gamma_h(t))},$$
$$c^{-1}\|\eta_h\|_{H^1(\Gamma_h(t))} \leq \|\eta_h^l\|_{H^1(\Gamma(t))} \leq c\|\eta_h\|_{H^1(\Gamma_h(t))}.$$

We now turn to the study of some geometric concepts and their errors. By δ_h we denote the quotient between the continuous and discrete surface measures, $\mathrm{d}A$ and $\mathrm{d}A_h$, defined as $\delta_h \mathrm{d}A_h = \mathrm{d}A$. Further, we recall that Pr and Pr_h are the projections onto the tangent spaces of $\Gamma(t)$ and $\Gamma_h(t)$, respectively. We further set, from Dziuk & Elliott (2013b),

$$Q_h = \frac{1}{\delta_h}(\mathrm{Id} - d\mathscr{H})\mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr}(\mathrm{Id} - d\mathscr{H}), \tag{5.1}$$

where $\mathscr{H}$ ($\mathscr{H}_{ij} = \partial_{x_j}\mathrm{n}_i$) is the (extended) Weingarten map. Using this notation and (3.4), in the proof of Dziuk & Elliott (2013b, Lemma 5.5), it is shown that

$$\nabla_{\Gamma_h}\phi_h(x) \cdot \nabla_{\Gamma_h}\phi_h(x) = \delta_h Q_h \nabla_\Gamma \phi_h^l(p) \cdot \nabla_\Gamma \phi_h^l(p). \tag{5.2}$$

For these quantities we show some results analogous to their linear ESFEM version showed in Dziuk & Elliott (2013b, Lemma 5.4), Demlow (2009, Proposition 2.3) or Mansour (2013, Lemma 6.1).

Later on the following estimates will play a key technical role.

LEMMA 5.2 For $\Gamma_h^k(t)$ and $\Gamma(t)$ as above, for a sufficiently small h, we have the following geometric approximation estimates:

$$
\begin{aligned}
&\|d\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1}, && \|1-\delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1},\\
&\|\mathrm{n}-\mathrm{n}_h\|_{L^\infty(\Gamma_h(t))} \leq ch^k, && \text{and}\\
&\|\mathrm{Id}-\delta_h Q_h\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1}, && \|(\partial_h^\bullet)^{(\ell)} d\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1},\\
&\|(\partial_h^\bullet)^{(\ell)}\delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1}, && \|\mathrm{Pr}((\partial_h^\bullet)^{(\ell)} Q_h)\mathrm{Pr}\|_{L^\infty(\Gamma_h)} \leq ch^{k+1},
\end{aligned}
$$

with constants depending only on $\mathcal{G}_T$ but not on h or t.

The first three bounds were shown in Demlow (2009, Proposition 2.3) for the stationary case. Noting that the constants depend only on $\mathcal{G}_T$ these inequalities are shown.

The last four bounds are simply the higher-order extensions of the corresponding estimates of Mansour (2013, Lemma 6.1). They can be proved in the exact same way using the bounds of the first three estimates. These proofs are included in the Appendix.

5.2 *Interpolation estimates for evolving surface finite elements*

The following result gives estimates for the error in the interpolation. Our setting follows that of Demlow (2009, Section 2.5).

Let us assume that the surface $\Gamma(t)$ is approximated by the interpolation surface $\Gamma_h^k(t)$. Then for any $w \in H^{k+1}(\Gamma(t))$, there is a unique k-order surface finite element interpolation $\widetilde{I}_h^k w \in S_h^k(t)$; furthermore we set $(\widetilde{I}_h^k w)^l = I_h^k w$.

LEMMA 5.3 (Demlow, 2009, Proposition 2.7) Let $w : \mathcal{G}_T \to \mathbb{R}$ such that $w \in H^{k+1}(\Gamma(t))$ for $0 \leq t \leq T$. There exists a constant $c > 0$ depending on $\mathcal{G}_T$, but independent of h and t, such that for $0 \leq t \leq T$ and for a sufficiently small h,

$$\|w - I_h^k w\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma(w - I_h^k w)\|_{L^2(\Gamma(t))} \leq ch^{k+1}\|w\|_{H^{k+1}(\Gamma(t))}.$$

We distinguish the special case of a linear surface finite element interpolation on $\Gamma_h^k(t)$. For $w \in H^2(\Gamma(t))$, the linear surface finite element interpolant is denoted by $I_h^{(1)} w$, and it satisfies, with $c > 0$,

$$\|w - I_h^{(1)} w\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma(w - I_h^{(1)} w)\|_{L^2(\Gamma(t))} \leq ch^2\|w\|_{H^2(\Gamma(t))}.$$

Note that for $I_h^{(1)}$ the underlying approximating surface $\Gamma_h^k(t)$ is still of high order. For $k = 1$, I_h^1 and $I_h^{(1)}$ simply coincide. The upper k indices are again dropped later on.

5.3 *Velocity of lifted material points and material derivatives*

Following Dziuk & Elliott (2013b) and Lubich & Mansour (2015) we define the velocity of the lifted material points, denoted by v_h, and the corresponding discrete material derivative $\partial_h^\bullet$.

For arbitrary $y(t) = p(x(t), t) \in \Gamma(t)$, with $x(t) \in \Gamma_h(t)$, cf. (3.4), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = v_h(y(t), t) = \partial_t p(x(t), t) + V_h(x(t), t) \cdot \nabla p(x(t), t); \tag{5.3}$$

hence for $y = p(x, t)$ (see Dziuk & Elliott, 2013b),

$$v_h(y, t) = (\mathrm{Pr} - d\mathscr{H})(x, t) V_h(x, t) - \partial_t\, \mathrm{d}(x, t)\mathrm{n}(x, t) - \mathrm{d}(x, t)\partial_t \mathrm{n}(x, t).$$

Following Lubich & Mansour (2015, Section 7.3), we note that $-\partial_t d(x, t)\mathrm{n}(x, t)$ is the normal component of $v(p, t)$ and that the other terms are tangent to $\Gamma(t)$; hence

$$v - v_h \quad \text{is a tangent vector.} \tag{5.4}$$

It is also important to note that $v_h \neq V_h^l$ (cf. Dziuk & Elliott, 2013b).

The discrete material derivative of the lifted points on $\Gamma(t)$ reads

$$\partial_h^\bullet \varphi_h = \partial_t \varphi_h + v_h \cdot \varphi_h \qquad (\varphi_h \in (S_h(t))^l).$$

The lifted basis functions also satisfy the transport property

$$\partial_h^\bullet \chi_j^l = 0 \qquad (j = 1, 2, \ldots, N).$$

To prove error estimates for higher-order material derivatives of the Ritz map, we need high-order bounds for the error between the continuous velocity v and the discrete velocity v_h. We generalize here Dziuk & Elliott (2013b, Lemma 5.6) and Lubich & Mansour (2015, Lemma 7.3) to the high-order case.

LEMMA 5.4 For $\ell \geq 0$, there exists a constant $c_\ell > 0$ depending on $\mathscr{G}_T$, but independent of t and h, such that, for a sufficiently small h,

$$\|(\partial_h^\bullet)^{(\ell)}(v - v_h)\|_{L^\infty(\Gamma(t))} + h\|\nabla_\Gamma(\partial_h^\bullet)^{(\ell)}(v - v_h)\|_{L^\infty(\Gamma(t))} \leq c_\ell h^{k+1}.$$

Proof. We follow the steps of the original proofs from Dziuk & Elliott (2013b) and Lubich & Mansour (2015).

(a) For $\ell = 0$. Using the definition (5.3) and the fact $V_h = \widetilde{I}_h v$, we have

$$|v(p, t) - v_h(p, t)| = |\mathrm{Pr}(v - I_h v)(p, t) + d(\mathscr{H} I_h v(p, t) + \partial_t \mathrm{n})| \leq ch^{k+1},$$

where we have used the interpolation estimates, Lemma 5.2 and the boundedness of the other terms.

For the gradient estimate we use the fact that $\nabla_\Gamma d = \nabla_\Gamma \partial_h^\bullet d = 0$ and the geometric bounds of Lemma 5.2:

$$\begin{aligned} |\nabla_\Gamma(v - v_h)| &\leq c|v - I_h v| + c|\nabla_\Gamma(v - I_h v)| \\ &\qquad + |(\nabla_\Gamma d)(\mathscr{H} I_h v + \partial_t \mathrm{n})| + |d(\nabla_\Gamma(\mathscr{H} I_h v + \partial_t \mathrm{n}))| \\ &\leq ch^k. \end{aligned}$$

(b) For $\ell = 1$, we use the transport property and again Lemma 5.2:

$$\begin{aligned}|\partial_h^\bullet(v - v_h)| &\leq |(\partial_h^\bullet \mathrm{Pr})(v - I_h v)| + |\mathrm{Pr}(\partial_h^\bullet v - I_h \partial_h^\bullet v)| \\ &\quad + |(\partial_h^\bullet d)(\mathscr{H} I_h v + \partial_t \mathrm{n})| + |d(\partial_h^\bullet(\mathscr{H} I_h v + \partial_t \mathrm{n}))| \\ &\leq c h^{k+1}.\end{aligned}$$

Again using $\nabla_\Gamma d = \nabla_\Gamma \partial_h^\bullet d = 0$ and the geometric bounds of Lemma 5.2,

$$\begin{aligned}|\nabla_\Gamma \partial_h^\bullet(v - v_h)| &\leq c|v - I_h v| + c|\nabla_\Gamma(v - I_h v)| \\ &\quad + c|\partial_h^\bullet(v - I_h v)| + c|\nabla_\Gamma(\partial_h^\bullet v - I_h \partial_h^\bullet v)| \\ &\quad + |(\nabla_\Gamma \partial_h^\bullet d)(\mathscr{H} I_h v + \partial_t \mathrm{n})| + |d(\nabla_\Gamma \partial_h^\bullet(\mathscr{H} I_h v + \partial_t \mathrm{n}))| \\ &\leq c h^k.\end{aligned}$$

(c) For $\ell > 1$, the proof uses similar arguments. $\square$

5.4 *Bilinear forms and their estimates*

We use the time-dependent bilinear forms defined in Dziuk & Elliott (2013b): for arbitrary $z, \varphi \in H^1(\Gamma(t))$ and for their discrete analogues for $Z_h, \phi_h \in S_h(t)$,

$$\begin{aligned}
m(t; z, \varphi) &= \int_{\Gamma(t)} z\varphi, & m_h(t; Z_h, \phi_h) &= \int_{\Gamma_h(t)} Z_h \phi_h, \\
a(t; z, \varphi) &= \int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a_h(t; Z_h, \phi_h) &= \int_{\Gamma_h(t)} \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\
g(t; v; z, \varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot v) z\varphi, & g_h(t; V_h; Z_h, \phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot V_h) Z_h \phi_h, \\
b(t; v; z, \varphi) &= \int_{\Gamma(t)} \mathscr{B}(v) \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & b_h(t; V_h; Z_h, \phi_h) &= \int_{\Gamma_h(t)} \mathscr{B}_h(V_h) \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h,
\end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense, and with the tensors given by

$$\begin{aligned}\mathscr{B}(v)|_{ij} &= \delta_{ij}(\nabla_\Gamma \cdot v) - \big((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i\big), \\ \mathscr{B}_h(V_h)|_{ij} &= \delta_{ij}(\nabla_{\Gamma_h} \cdot V_h) - \big((\nabla_{\Gamma_h})_i (V_h)_j + (\nabla_{\Gamma_h})_j (V_h)_i\big),\end{aligned}$$

for $i, j = 1, 2, \ldots, m+1$. For more details, see Dziuk & Elliott (2013b, Lemma 2.1) (and the references in the proof) or Dziuk & Elliott (2013a, Lemma 5.2).

We will omit the time dependency of the bilinear forms if it is clear from the context.

5.4.1 *Discrete material derivative and transport properties.* The following transport equations, especially the one with discrete material derivatives on the continuous surface, are of great importance during the defect estimates later on.

LEMMA 5.5 (Dziuk & Elliott, 2013b, Lemma 4.2) Consider $\Gamma(t)$ as the lift of the discrete surface $\Gamma_h^k(t)$ (i.e., $\Gamma(t)$ can be decomposed into curved elements that are lifts of the elements of $\Gamma_h^k(t)$), moving with the velocity v_h from (5.3). Then for any $z, \varphi, \partial_h^\bullet z, \partial_h^\bullet \varphi \in H^1(\Gamma(t))$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} m(z,\varphi) = m(\partial_h^\bullet z, \varphi) + m(z, \partial_h^\bullet \varphi) + g(v_h; z, \varphi),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} a(z,\varphi) = a(\partial_h^\bullet z, \varphi) + a(z, \partial_h^\bullet \varphi) + b(v_h; z, \varphi).$$

The same formulae hold for $\Gamma(t)$ when $\partial_h^\bullet$ and v_h are replaced with $\partial^\bullet$ and v, respectively.

Similarly, in the discrete case, for arbitrary $z_h, \phi_h, \partial_h^\bullet z_h, \partial_h^\bullet \phi_h \in S_h(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} m_h(z_h,\phi_h) = m_h(\partial_h^\bullet z_h, \phi_h) + m_h(z_h, \partial_h^\bullet \phi_h) + g_h(V_h; z_h, \phi_h),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} a_h(z_h,\phi_h) = a_h(\partial_h^\bullet z_h, \phi_h) + a_h(z_h, \partial_h^\bullet \phi_h) + b_h(V_h; z_h, \phi_h).$$

5.4.2 *Geometric perturbation errors.* The following estimates are the most important technical results of this paper. Later on, they will play a crucial role in the defect estimates. We note here that these results extend the first-order ESFEM theory of Dziuk & Elliott (2013b) (for the first three inequalities) and Lubich & Mansour (2015) (for the last inequality) to the higher-order ESFEM case.

The high-order version of the inequalities for time-independent bilinear forms $a(\cdot,\cdot)$ and $m(\cdot,\cdot)$ were shown in Elliott & Ranner (2013, Lemma 6.2). The following results generalizes these for the time-dependent case.

LEMMA 5.6 For any $Z_h, \phi_h \in S_h^k(t)$, and for their lifts $Z_h^l, \phi_h^l \in H^1(\Gamma(t))$, we have the following bounds, with a sufficiently small h:

$$\begin{aligned}
\left|m(Z_h^l,\phi_h^l) - m_h(Z_h,\phi_h)\right| &\le c h^{k+1} \|Z_h^l\|_{L^2(\Gamma(t))} \|\phi_h^l\|_{L^2(\Gamma(t))},\\
\left|a(Z_h^l,\phi_h^l) - a_h(Z_h,\phi_h)\right| &\le c h^{k+1} \|\nabla_\Gamma Z_h^l\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \phi_h^l\|_{L^2(\Gamma(t))},\\
\left|g(v_h; Z_h^l,\phi_h^l) - g_h(V_h; Z_h,\phi_h)\right| &\le c h^{k+1} \|Z_h^l\|_{L^2(\Gamma(t))} \|\phi_h^l\|_{L^2(\Gamma(t))},\\
\left|b(v_h; Z_h^l,\phi_h^l) - b_h(V_h; Z_h,\phi_h)\right| &\le c h^{k+1} \|\nabla_\Gamma Z_h^l\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \phi_h^l\|_{L^2(\Gamma(t))},
\end{aligned}$$

where the constants $c > 0$ are independent of h and t, but depend on $\mathscr{G}_T$.

Proof. The proof of the first two estimates is a high-order generalization of Dziuk & Elliott (2013b, Lemma 5.5), while the proof of the last two estimates is a high-order extension of Lubich & Mansour (2015, Lemma 7.5). Their proofs follow these references. Both parts use the geometric estimates shown in Lemma 5.2.

To show the first inequality, we estimate

$$\left|m(Z_h^l,\phi_h^l) - m_h(Z_h,\phi_h)\right| = \left| \int_{\Gamma(t)} Z_h^l \phi_h^l \mathrm{d}A - \int_{\Gamma_h(t)} Z_h \phi_h \,\mathrm{d}A_h \right|$$

$$= \left| \int_{\Gamma(t)} (1 - \delta_h^{-1}) Z_h^l \phi_h^l \, \mathrm{d}A \right|$$
$$\leq ch^{k+1} \|Z_h^l\|_{L^2(\Gamma(t))} \|\phi_h^l\|_{L^2(\Gamma(t))}.$$

For the second inequality, similarly we have

$$\begin{aligned}\left|a(Z_h^l, \phi_h^l) - a_h(Z_h, \phi_h)\right| &= \left| \int_{\Gamma(t)} \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l \mathrm{d}A - \int_{\Gamma_h(t)} \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h \, \mathrm{d}A_h \right| \\ &= \left| \int_{\Gamma(t)} (\mathrm{Id} - \delta_h Q_h) \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l \, \mathrm{d}A \right| \\ &\leq ch^{k+1} \|\nabla_\Gamma Z_h^l\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \phi_h^l\|_{L^2(\Gamma(t))}.\end{aligned}$$

For the third estimate we start by taking the time derivative of the equality $m(Z_h^l, \phi_h^l) = m_h(Z_h, \phi_h \delta_h)$, using the first transport property from Lemma 5.5 to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} m(Z_h^l, \phi_h^l) &= m(\partial_h^\bullet Z_h^l, \phi_h^l) + m(Z_h^l, \partial_h^\bullet \phi_h^l) + g(v_h; Z_h^l, \phi_h^l) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} m_h(Z_h, \phi_h \delta_h) \\ &= m_h(\partial_h^\bullet Z_h, \phi_h \delta_h) + m_h(Z_h, (\partial_h^\bullet \phi_h)\delta_h) \\ &\quad + g_h(V_h; Z_h, \phi_h \delta_h) + m_h(Z_h, (\partial_h^\bullet \delta_h)\phi_h).\end{aligned}$$

Hence, using $\partial_h^\bullet w_h^l = (\partial_h^\bullet w_h)^l$,

$$\begin{aligned}g(v_h; Z_h^l, \phi_h^l) - g_h(V_h; Z_h, \phi_h \delta_h) =&\ m((\partial_h^\bullet Z_h)^l, \phi_h^l) - m_h(\partial_h^\bullet Z_h, \phi_h \delta_h) \\ &+ m(Z_h^l, (\partial_h^\bullet \phi_h)^l) - m_h(Z_h, (\partial_h^\bullet \phi_h)\delta_h) + m_h(Z_h, (\partial_h^\bullet \delta_h)\phi_h) \\ =&\ m_h(Z_h, (\partial_h^\bullet \delta_h)\phi_h).\end{aligned}$$

Hence, together with Lemma 5.2, the bound

$$\begin{aligned}|g(v_h; Z_h^l, \phi_h^l) - g_h(V_h; Z_h, \phi_h)| &\leq |g_h(V_h; Z_h, \phi_h(\delta_h - 1))| + |m_h(Z_h, (\partial_h^\bullet \delta_h)\phi_h)| \\ &\leq c \left(\|\partial_h^\bullet \delta_h\|_{L^\infty(\Gamma_h(t))} + \|1 - \delta_h\|_{L^\infty(\Gamma_h(t))} \right) \|Z_h^l\|_{L^2(\Gamma(t))} \|\phi_h^l\|_{L^2(\Gamma(t))}\end{aligned}$$

finishes the proof of the third estimate.

For the last inequality we take a similar approach, using the second transport property from Lemma 5.5 and the relation (5.2) to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} a_h(Z_h, \phi_h) &= a_h(\partial_h^\bullet Z_h, \phi_h) + a_h(Z_h, \partial_h^\bullet \phi_h) + b_h(V_h; Z_h, \phi_h) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} Q_h^l \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l\end{aligned}$$

$$= \int_{\Gamma(t)} Q_h^l \nabla_\Gamma \partial_h^\bullet Z_h^l \cdot \nabla_\Gamma \phi_h^l + \int_{\Gamma(t)} Q_h^l \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \partial_h^\bullet \phi_h^l$$
$$+ \int_{\Gamma(t)} (\partial_h^\bullet Q_h^l) \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l + \int_{\Gamma(t)} \mathscr{B}(v_h) Q_h^l \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l.$$

Again, using $\partial_h^\bullet w_h^l = (\partial_h^\bullet w_h)^l$, together with (5.2) and the geometric estimates of Lemma 5.2 we obtain

$$|b_h(V_h; Z_h, \phi_h) - b(v_h; Z_h^l, \phi_h^l)| = \Big| \int_{\Gamma(t)} (\partial_h^\bullet Q_h^l) \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l + \int_{\Gamma(t)} \mathscr{B}(v_h) \big(Q_h^l - \mathrm{Id}\big) \nabla_\Gamma Z_h^l \cdot \nabla_\Gamma \phi_h^l \Big|$$
$$\leq ch^{k+1} \|\nabla_\Gamma Z_h^l\|_{L^2(\Gamma(t))} \|\nabla_\Gamma \phi_h^l\|_{L^2(\Gamma(t))},$$

completing the proof. □

6. Generalized Ritz map and higher-order error bounds

We recall the generalized Ritz map for evolving surface PDEs from Lubich & Mansour (2015).

Definition 6.1 (Ritz map) For any given $z \in H^1(\Gamma(t))$, there is a unique $\widetilde{\mathscr{P}}_h z \in S_h^k(t)$ such that for all $\phi_h \in S_h^k(t)$, with the corresponding lift $\varphi_h = \phi_h^l$, we have

$$a_h^*(\widetilde{\mathscr{P}}_h z, \phi_h) = a^*(z, \varphi_h), \tag{6.1}$$

where we let $a^* = a + m$ and $a_h^* = a_h + m_h$, so that the forms a and a_h are positive definite. Then $\mathscr{P}_h z \in (S_h^k(t))^l$ is defined as the lift of $\widetilde{\mathscr{P}}_h z$, i.e., $\mathscr{P}_h z = (\widetilde{\mathscr{P}}_h z)^l$.

We note here that originally in Lubich & Mansour (2015, Definition 8.1) an extra term appeared involving $\partial^\bullet z$ and the surface velocity, which is not needed for the parabolic case. The Ritz map above is still well defined.

Galerkin orthogonality does not hold in this case, just up to a small defect.

Lemma 6.2 (Galerkin orthogonality up to a small defect) For any $z \in H^1(\Gamma(t))$ and for all $\varphi_h \in (S_h^k(t))^l$ and for a sufficiently small h,

$$|a^*(z - \mathscr{P}_h z, \varphi_h)| \leq ch^{k+1} \|\mathscr{P}_h z\|_{H^1(\Gamma(t))} \|\varphi_h\|_{H^1(\Gamma(t))}, \tag{6.2}$$

where c is independent of ξ, h and t.

Proof. Using the definition of the Ritz map and Lemma 5.6, we estimate

$$|a^*(z - \mathscr{P}_h z, \varphi_h)| = |a_h^*(\widetilde{\mathscr{P}}_h z, \phi_h) - a^*(\mathscr{P}_h z, \varphi_h)| \leq ch^{k+1} \|\mathscr{P}_h z\|_{H^1(\Gamma(t))} \|\varphi_h\|_{H^1(\Gamma(t))}.$$

□

6.1 *Errors in the Ritz map*

Now we prove higher-order error estimates for the Ritz map (6.1) and also for its material derivatives; the analogous results for the first-order ESFEM case can be found in Dziuk & Elliott (2013b, Section 6) or in Mansour (2013, Section 7).

6.1.1 *Error bounds for the Ritz map.*

THEOREM 6.3 Let $z : \mathscr{G}_T \to \mathbb{R}$ with $z \in H^{k+1}(\Gamma(t))$ for every $0 \leq t \leq T$. Then the error in the Ritz map satisfies the bound, for $0 \leq t \leq T$ and for $h \leq h_0$ with a sufficiently small h_0,

$$\|z - \mathscr{P}_h z\|_{L^2(\Gamma(t))} + h\|z - \mathscr{P}_h z\|_{H^1(\Gamma(t))} \leq ch^{k+1}\|z\|_{H^{k+1}(\Gamma(t))},$$

where the constant $c > 0$ is independent of h and t.

Proof. To ease the presentation, we suppress all time arguments t appearing in the norms within the proof (except for some special occasions).

(a) We first prove the gradient estimate. Starting with the definition of the $H^1(\Gamma(t))$-norm, then using the estimate (6.2), we have

$$\begin{aligned}
\|z - \mathscr{P}_h z\|^2_{H^1(\Gamma)} &= a^*(z - \mathscr{P}_h z, z - \mathscr{P}_h z) \\
&= a^*(z - \mathscr{P}_h z, z - I_h z) + a^*(z - \mathscr{P}_h z, I_h z - \mathscr{P}_h z) \\
&\leq \|z - \mathscr{P}_h z\|_{H^1(\Gamma)}\|z - I_h z\|_{H^1(\Gamma)} + ch^{k+1}\|\mathscr{P}_h z\|_{H^1(\Gamma)}\|I_h z - \mathscr{P}_h z\|_{H^1(\Gamma)} \\
&\leq ch^k\|z - \mathscr{P}_h z\|_{H^1(\Gamma)}\|z\|_{H^{k+1}(\Gamma)} \\
&\quad + ch^{k+1}\left(2\|z - \mathscr{P}_h z\|^2_{H^1(\Gamma)} + \|z\|^2_{H^1(\Gamma)} + ch^{2k}\|z\|^2_{H^{k+1}(\Gamma)}\right),
\end{aligned}$$

using interpolation error estimate, and for the second term, we used the estimate

$$\begin{aligned}
\|\mathscr{P}_h z\|_{H^1(\Gamma)}\|I_h z - \mathscr{P}_h z\|_{H^1(\Gamma)} &\leq \left(\|\mathscr{P}_h z - z\|_{H^1(\Gamma)} + \|z\|_{H^1(\Gamma)}\right)\left(\|I_h z - z\|_{H^1(\Gamma)} + \|z - \mathscr{P}_h z\|_{H^1(\Gamma)}\right) \\
&\leq 2\|z - \mathscr{P}_h z\|^2_{H^1(\Gamma)} + \|z\|^2_{H^{k+1}(\Gamma)} + ch^{2k}\|z\|^2_{H^{k+1}(\Gamma)}.
\end{aligned}$$

Now using Young's inequality, and for a sufficiently small (but t independent) $h \leq h_0$, we have the gradient estimate

$$\|z - \mathscr{P}_h z\|_{H^1(\Gamma(t))} \leq ch^k\|z\|_{H^{k+1}(\Gamma(t))}.$$

(b) The L^2-estimate follows from the Aubin–Nitsche trick. Let us consider the problem

$$-\Delta_{\Gamma(t)} w + w = z - \mathscr{P}_h z \qquad \text{on} \quad \Gamma(t);$$

then the usual elliptic theory (see, e.g., Dziuk & Elliott, 2013a, Section 3.1; Aubin, 1998) yields the following: the solution $w \in H^2(\Gamma(t))$ satisfies the bound, with $c > 0$ independent of t,

$$\|w\|_{H^2(\Gamma(t))} \leq c\|z - \mathscr{P}_h z\|_{L^2(\Gamma(t))}.$$

By testing the elliptic weak problem with $z - \mathscr{P}_h z$, using (6.2) again, and using the linear finite element interpolation $I_h^{(1)}$ on $\Gamma_h^k(t)$, we obtain

$$\begin{aligned}\|z-\mathscr{P}_hz\|_{L^2(\Gamma)}^2 &= a^*(z-\mathscr{P}_hz,w)\\ &= a^*(z-\mathscr{P}_hz,w-I_h^{(1)}w)+a^*(z-\mathscr{P}_hz,I_h^{(1)}w)\\ &\le \|z-\mathscr{P}_hz\|_{H^1(\Gamma)}\|w-I_h^{(1)}w\|_{H^1(\Gamma)}+ch^{k+1}\|\mathscr{P}_hz\|_{H^1(\Gamma)}\|I_h^{(1)}w\|_{H^1(\Gamma)}\\ &\le ch^k\|z\|_{H^{k+1}(\Gamma)}ch\|w\|_{H^2(\Gamma)}+ch^{k+1}\|\mathscr{P}_hz\|_{H^1(\Gamma)}\|I_h^{(1)}w\|_{H^1(\Gamma)},\end{aligned}$$

where for the second term we have now used

$$\begin{aligned}\|\mathscr{P}_hz\|_{H^1(\Gamma)}\|I_h^{(1)}w\|_{H^1(\Gamma)} &\le (\|z-\mathscr{P}_hz\|_{H^1(\Gamma)}+\|z\|_{H^1(\Gamma)})(\|w-I_h^{(1)}w\|_{H^1(\Gamma)}+\|w\|_{H^1(\Gamma)})\\ &\le (1+ch^k)\|z\|_{H^{k+1}(\Gamma)}(1+ch)\|w\|_{H^2(\Gamma)}.\end{aligned}$$

Then the combination of the gradient estimate for the Ritz map and the interpolation error yields

$$\|z-\mathscr{P}_hz\|_{L^2(\Gamma(t))}\frac{1}{c}\|w\|_{H^2(\Gamma(t))}\le\|z-\mathscr{P}_hz\|_{L^2(\Gamma(t))}^2\le ch^{k+1}\|z\|_{H^{k+1}(\Gamma(t))}\|w\|_{H^2(\Gamma(t))},$$

which completes the proof. □

6.1.2 *Error bounds for the material derivatives of the Ritz map.* Since, in general, $\partial_h^\bullet\mathscr{P}_hz=\mathscr{P}_h\partial_h^\bullet z$ does not hold, we need the following result.

THEOREM 6.4 The error in the material derivatives of the Ritz map, for any $\ell\in\mathbb{N}$, satisfies the following bounds, for $0\le t\le T$ and for $h\le h_0$ with a sufficiently small h_0:

$$\|(\partial_h^\bullet)^{(\ell)}(z-\mathscr{P}_hz)\|_{L^2(\Gamma(t))}+h\|\nabla_\Gamma(\partial_h^\bullet)^{(\ell)}(z-\mathscr{P}_hz)\|_{L^2(\Gamma(t))}\le c_\ell h^{k+1}\sum_{j=0}^{\ell}\|(\partial^\bullet)^{(j)}z\|_{H^{k+1}(\Gamma(t))},$$

where the constant $c_\ell>0$ is independent of h and t.

Proof. The proof is a modification of Mansour (2013, Theorem 7.3). Again, to ease the presentation, we suppress the t argument of the surfaces norms (except for some special occasions).

For $\ell=1$: (a) We start by taking the time derivative of the definition of the Ritz map (6.1), use the transport properties Lemma 5.5 and apply the definition of the Ritz map once more; we arrive at

$$\begin{aligned}a^*(\partial_h^\bullet z,\varphi_h) &= -b(v_h;z,\varphi_h)-g(v_h;z,\varphi_h)\\ &\quad+a_h^*(\partial_h^\bullet\widetilde{\mathscr{P}}_hz,\phi_h)+b_h(V_h;\widetilde{\mathscr{P}}_hz,\phi_h)+g_h(V_h;\widetilde{\mathscr{P}}_hz,\phi_h).\end{aligned}$$

Then we obtain

$$\begin{aligned}a^*(\partial_h^\bullet z-\partial_h^\bullet\mathscr{P}_hz,\varphi_h) &= -b(v_h;z-\mathscr{P}_hz,\varphi_h)-g(v_h;z-\mathscr{P}_hz,\varphi_h)\\ &\quad+F_1(\varphi_h),\end{aligned}\tag{6.3}$$

where

$$F_1(\varphi_h) = \big(a_h^*(\partial_h^\bullet \widetilde{\mathscr{P}}_h z, \phi_h) - a^*(\partial_h^\bullet \mathscr{P}_h z, \varphi_h)\big)$$
$$+ \big(b_h(V_h; \widetilde{\mathscr{P}}_h z, \phi_h) - b(v_h; \mathscr{P}_h z, \varphi_h)\big)$$
$$+ \big(g_h(V_h; \widetilde{\mathscr{P}}_h z, \phi_h) - g(v_h; \mathscr{P}_h z, \varphi_h)\big).$$

Using the geometric estimates of Lemma 5.6, F_1 can be estimated as

$$|F_1(\varphi_h)| \le ch^{k+1} \left(\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma(t))} + \|\mathscr{P}_h z\|_{H^1(\Gamma(t))}\right) \|\varphi_h\|_{H^1(\Gamma(t))}. \tag{6.4}$$

The velocity error estimate Lemma 5.4 yields

$$\|\partial_h^\bullet z\|_{H^1(\Gamma)} \le \|\partial^\bullet z\|_{H^1(\Gamma)} + ch^k \|z\|_{H^2(\Gamma)}.$$

Then using $\partial_h^\bullet \mathscr{P}_h z$ as a test function in (6.3), and using the error estimates of the Ritz map, together with the estimates above, with $h \le h_0$, we have

$$\begin{aligned}
\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}^2 =&\ a^*(\partial_h^\bullet \mathscr{P}_h z, \partial_h^\bullet \mathscr{P}_h z)\\
&= b(v_h; z - \mathscr{P}_h z, \partial_h^\bullet \mathscr{P}_h z) + g(v_h; z - \mathscr{P}_h z, \partial_h^\bullet \mathscr{P}_h z) + a^*(\partial_h^\bullet z, \partial_h^\bullet \mathscr{P}_h z) - F_1(\partial_h^\bullet \mathscr{P}_h z)\\
&\le\ ch^k \|z\|_{H^{k+1}(\Gamma)} \|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} + \|\partial_h^\bullet z\|_{H^1(\Gamma)} \|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}\\
&\quad + ch^{k+1}\big(\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} + \|\mathscr{P}_h z\|_{H^1(\Gamma)}\big)\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}\\
&\le ch^k \|z\|_{H^{k+1}(\Gamma)} \|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} + (\|\partial^\bullet z\|_{H^1(\Gamma)} + ch^k \|z\|_{H^2(\Gamma)})\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}\\
&\quad + ch^{k+1}\big(\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} + \|z - \mathscr{P}_h z\|_{H^1(\Gamma)} + \|z\|_{H^1(\Gamma)}\big)\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}\\
&\le \big(ch^k \|z\|_{H^{k+1}(\Gamma)} + \|\partial^\bullet z\|_{H^1(\Gamma)}\big) \|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} + ch^{k+1} \|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}^2,
\end{aligned}$$

absorption using an $h \le h_0$, with a sufficiently small $h_0 > 0$, and dividing through yields

$$\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} \le c\|\partial^\bullet z\|_{H^1(\Gamma)} + ch^k \|z\|_{H^{k+1}(\Gamma)}.$$

Combining all the previous estimates and using Young's inequality, the Cauchy–Schwarz inequality and Theorem 6.3, for a sufficiently small $h \le h_0$, we obtain

$$\begin{aligned}
a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, \varphi_h) \le&\ c\|z - \mathscr{P}_h z\|_{H^1(\Gamma)} \|\varphi_h\|_{H^1(\Gamma)}\\
&+ ch^{k+1}\big(\|\partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} + \|\mathscr{P}_h z\|_{H^1(\Gamma)}\big)\|\varphi_h\|_{H^1(\Gamma)}\\
\le&\ ch^k \|z\|_{H^{k+1}(\Gamma)} \|\varphi_h\|_{H^1(\Gamma)}\\
&+ ch^{k+1}\big(\|\partial^\bullet z\|_{H^1(\Gamma)} + (1 + ch^k)\|z\|_{H^{k+1}(\Gamma)}\big)\|\varphi_h\|_{H^1(\Gamma)}\\
\le&\ ch^k \left(\|z\|_{H^{k+1}(\Gamma)} + h\|\partial^\bullet z\|_{H^1(\Gamma)}\right) \|\varphi_h\|_{H^1(\Gamma)}.
\end{aligned}$$

Then, similarly to the previous proof, we have

$$\begin{aligned}\|\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}^2 &\leq a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, \partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z) \\ &= a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, \partial_h^\bullet z - I_h \partial^\bullet z) + a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, I_h \partial^\bullet z - \partial_h^\bullet \mathscr{P}_h z) \\ &\leq \|\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)} \|\partial_h^\bullet z - I_h \partial^\bullet z\|_{H^1(\Gamma)} \\ &\quad + ch^k \left(\|z\|_{H^{k+1}(\Gamma)} + h\|\partial^\bullet z\|_{H^1(\Gamma)}\right) \|I_h \partial^\bullet z - \partial_h^\bullet \mathscr{P}_h z\|_{H^1(\Gamma)}.\end{aligned}$$

Finally, the interpolation estimates, Young's inequality and absorption using a sufficiently small $h \leq h_0$, yields the gradient estimate.

(b) The L^2 estimate again follows from the Aubin–Nitsche trick. Let us now consider the problem

$$-\Delta_{\Gamma(t)} w + w = \partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z \qquad \text{on} \quad \Gamma(t),$$

together with the usual elliptic estimate, for the solution $w \in H^2(\Gamma(t))$,

$$\|w\|_{H^2(\Gamma(t))} \leq c\|\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z\|_{L^2(\Gamma(t))};$$

again, c is independent of t and h.

Following the proof of Dziuk & Elliott (2013b, Theorem 6.2), let us first bound

$$\begin{aligned}-b(v_h; z - \mathscr{P}_h z, I_h^{(1)} w) &= b(v_h; z - \mathscr{P}_h z, w - I_h^{(1)} w) - b(v_h; z - \mathscr{P}_h z, w) \\ &\leq ch^k \|z\|_{H^{k+1}(\Gamma)}\, ch\|w\|_{H^2(\Gamma)} - b(v_h; z - \mathscr{P}_h z, w) \\ &= ch^{k+1} \|z\|_{H^{k+1}(\Gamma)} \|w\|_{H^2(\Gamma)} + b(v; z - \mathscr{P}_h z, w) \\ &\quad + b(v; z - \mathscr{P}_h z, w) - b(v_h; z - \mathscr{P}_h z, w),\end{aligned}$$

where again $I_h^{(1)}$ denotes the linear finite element interpolation operator on $\Gamma_h^k(t)$.

The pair in the last line can be estimated, using Lemma 5.4, by

$$\begin{aligned}b(v; z - \mathscr{P}_h z, w) - b(v_h; z - \mathscr{P}_h z, w) &\leq \int_{\Gamma(t)} |\mathscr{B}(v) - \mathscr{B}(v_h)||\nabla_\Gamma (z - \mathscr{P}_h z)||\nabla_\Gamma w| \\ &\leq ch^{2k} \|z\|_{H^{k+1}(\Gamma)} \|w\|_{H^1(\Gamma)}.\end{aligned}$$

Finally, for the remaining term, the proof of Dziuk & Elliott (2013b, Theorem 6.2) yields

$$b(v; z - \mathscr{P}_h z, w) \geq -c\|z - \mathscr{P}_h z\|_{L^2(\Gamma)} \|w\|_{H^2(\Gamma)} \geq -ch^{k+1} \|z\|_{H^{k+1}(\Gamma)} \|w\|_{H^2(\Gamma)}.$$

For the other bilinear form in (6.3), we have

$$g(v_h; z - \mathscr{P}_h z, \varphi_h) \leq ch^{k+1} \|z\|_{H^{k+1}(\Gamma)} \|w\|_{H^1(\Gamma)}.$$

The combination of all these estimates with (6.4) yields

$$a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, I_h w) \leq ch^{k+1} \|z\|_{H^{k+1}(\Gamma)} \|w\|_{H^2(\Gamma)}.$$

By testing the above elliptic weak problem with $z - \mathscr{P}_h z$, and using the above bound and the gradient estimate from (a), we obtain

$$\begin{aligned}
\|\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z\|_{L^2(\Gamma)}^2 &= a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, w) \\
&= a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, w - I_h^{(1)} w) + a^*(\partial_h^\bullet z - \partial_h^\bullet \mathscr{P}_h z, I_h^{(1)} w) \\
&\leq ch^k \left(\|z\|_{H^{k+1}(\Gamma)} + h\|\partial^\bullet z\|_{H^1(\Gamma)}\right) \, ch\|w\|_{H^2(\Gamma(t))} + ch^{k+1}\|z\|_{H^{k+1}(\Gamma)}\|w\|_{H^2(\Gamma)} \\
&\leq ch^{k+1} \left(\|z\|_{H^{k+1}(\Gamma)} + h\|\partial^\bullet z\|_{H^1(\Gamma)}\right) \|w\|_{H^2(\Gamma)}.
\end{aligned}$$

For $\ell > 1$, the proof is analogous. □

7. Error bounds for the semidiscretization and full discretization

7.1 *Convergence proof for the semidiscretization*

By combining the error estimates in the Ritz map and in its material derivatives and the geometric results of Section 5, we prove convergence of the high-order ESFEM semidiscretization.

Proof of Theorem 4.1. The result is simply shown by repeating the arguments of Dziuk & Elliott (2013b, Section 7) for our setting but using the high-order versions for all results: geometric estimates Lemma 5.2, perturbation estimates of bilinear forms Lemma 5.6 and Ritz map error estimates Theorems 6.3 and 6.4. □

7.2 *Convergence proof for the full discretization*

7.2.1 *Bound of the semidiscrete residual.* We follow the approach of Mansour (2013, Section 8.1) and Lubich *et al.* (2013, Section 5) by defining the ESFEM residual $R_h(\cdot,t) = \sum_{j=1}^N r_j(t)\chi_j(\cdot,t) \in S_h^k(t)$ as

$$\int_{\Gamma_h^k(t)} R_h \phi_h = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h^k(t)} \widetilde{\mathscr{P}}_h u \phi_h + \int_{\Gamma_h^k(t)} \nabla_{\Gamma_h}(\widetilde{\mathscr{P}}_h u)\cdot\nabla_{\Gamma_h}\phi_h - \int_{\Gamma_h^k(t)} (\widetilde{\mathscr{P}}_h u)\partial_h^\bullet \phi_h, \tag{7.1}$$

where $\phi_h \in S_h^k(t)$, and $\widetilde{\mathscr{P}}_h u(\cdot,t)$ is the Ritz map of the smooth solution u.

We now show the optimal-order H_h^{-1}-norm estimate of the residual R_h.

THEOREM 7.1 Let the solution u of the parabolic problem be sufficiently smooth. Then there exist $C > 0$ and $h_0 > 0$, such that, for all $h \leq h_0$ and $t \in [0,T]$, the finite element residual R_h of the Ritz map is bounded as

$$\|R_h\|_{H_h^{-1}(\Gamma_h(t))} \leq Ch^{k+1}.$$

Proof. (a) We start by applying the discrete transport property to the residual equation (7.1):

$$\begin{aligned}
m_h(R_h,\phi_h) &= \frac{\mathrm{d}}{\mathrm{d}t} m_h(\widetilde{\mathscr{P}}_h u, \phi_h) + a_h(\widetilde{\mathscr{P}}_h u, \phi_h) - m_h(\widetilde{\mathscr{P}}_h u, \partial_h^\bullet \phi_h) \\
&= m_h(\partial_h^\bullet \widetilde{\mathscr{P}}_h u, \phi_h) + a_h(\widetilde{\mathscr{P}}_h u, \phi_h) + g_h(V_h; \widetilde{\mathscr{P}}_h u, \phi_h).
\end{aligned}$$

(b) We continue by the transport property with discrete material derivatives from Lemma 5.5 but for the weak form, with $\varphi := \varphi_h = (\phi_h)^l$:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} m(u, \varphi_h) + a(u, \varphi_h) - m(u, \partial^{\bullet} \varphi_h) \\ &= m(\partial_h^{\bullet} u, \varphi_h) + a(u, \varphi_h) + g(v_h; u, \varphi_h) + m(u, \partial_h^{\bullet} \varphi_h - \partial^{\bullet} \varphi_h). \end{aligned}$$

(c) Subtraction of the two equations, using the definition of the Ritz map (6.1) and using that

$$\partial_h^{\bullet} \varphi_h - \partial^{\bullet} \varphi_h = (v_h - v) \cdot \nabla_{\Gamma} \varphi_h$$

holds, we obtain

$$\begin{aligned} m_h(R_h, \phi_h) = m_h(\partial_h^{\bullet} \widetilde{\mathscr{P}}_h u, \phi_h) &- m(\partial_h^{\bullet} u, \varphi_h) \\ &+ g_h(V_h; \widetilde{\mathscr{P}}_h u, \phi_h) - g(v_h; u, \varphi_h) \\ &+ m(u, \varphi_h) - m_h(\widetilde{\mathscr{P}}_h u, \phi_h) \\ &+ m(u, (v_h - v) \cdot \nabla_{\Gamma} \varphi_h). \end{aligned}$$

All the pairs can be easily estimated separately as $ch^{k+1} \|\varphi_h\|_{H^1(\Gamma(t))}$ by combining the geometric perturbation estimates of Lemma 5.6, the velocity estimate of Lemma 5.4 and the error estimates of the Ritz map from Theorems 6.3 and 6.4. The proof is finished using the definition of the H_h^{-1}-norm and the equivalence of norms Lemma 5.1. □

7.2.2 *Proof of Theorem* 4.4. Using the error estimate for the BDF methods Theorem 4.2 and using the bounds for the semidiscrete residual Theorem 7.1, we give here a proof for the fully discrete error estimates of Theorem 4.4.

Proof of Theorem 4.4. The global error is decomposed into two parts,

$$u_h^n - u(\cdot, t_n) = \big(u_h^n - (\mathscr{P}_h u)(\cdot, t_n)\big) + \big((\mathscr{P}_h u)(\cdot, t_n) - u(\cdot, t_n)\big),$$

and then the terms are estimated separately by results from above.

The first term is estimated, analogously to Thomée (2006) or exactly as in Lubich *et al.* (2013) and Mansour (2013) as follows. The vectors collecting the nodal values of the error $u_h^n - (\mathscr{P}_h u)(\cdot, t_n)$ satisfy the fully discrete problem (4.1) perturbed by the semidiscrete residual (7.1) (cf. Mansour, 2013). Then applying results for BDF methods Theorem 4.2, together with the residual bound Theorem 7.1 (and by the assumption on the initial value approximation), gives the desired bound $\mathscr{O}(h^{k+1} + \tau^p)$.

The second term is directly estimated by the error estimates for the Ritz map and for its material derivatives, Theorems 6.3 and 6.4. □

REMARK 7.2 In Remark 4.5, we noted that the analogous fully discrete results can be shown for algebraically stable implicit Runge–Kutta methods. To be more precise, for the Runge–Kutta analogue, instead of Theorem 4.2 one has to use the error bounds of Dziuk *et al.* (2012, Theorem 5.1), but otherwise the proof above remains the same.

REMARK 7.3 The statement in the introduction on various extensions is supported by the following facts. The first two points in the introduction, geometric errors and perturbation errors of the bilinear forms (Sections 5.1 and 5.4), and the basic high-order evolving surface finite element setting (Section 3.1), are independent of the considered problem. The velocity estimates (Section 5.3) depend only on the surface evolution as well. Furthermore, these general high-order results could be used in analogous ways, as their linear counterparts in the cited articles; or missing ones can be easily shown based on the ideas presented here (e.g., involving the additional term for ALE maps (Elliott & Venkataraman, 2014; Kovács & Power Guerra, 2014).

The proof of the error bounds of a modified Ritz map (although perhaps defined slightly differently; see, e.g., wave equations Lubich & Mansour, 2015, or quasilinear problems Kovács & Power Guerra, 2016), rely on these geometric approximation results and some general techniques, such as the almost Galerkin orthogonality and the Aubin–Nitsche trick. Naturally, the estimates for the semidiscrete residual greatly depend on the problem itself, but in the above-mentioned articles, similar (if not the same) ideas and techniques were used.

Finally, convergence of time discretizations, proved by *energy estimates*, carry over to high-order discretizations of various problems, from Dziuk *et al.* (2012), Lubich *et al.* (2013), Kovács & Power Guerra (2014), Lubich & Mansour (2015) and Kovács & Power Guerra (2016).

REMARK 7.4 Theorem 4.2 requires sufficient temporal regularity of the Ritz map.

By having sufficient regularity of the solution and the surface evolution on $[0, T]$, using Theorem 6.4 and equivalence of norms, we obtain

$$\begin{aligned}\|(\partial_h^\bullet)^{(\ell)}(\widetilde{\mathscr{P}}_h u)(\cdot,t)\|_{L^2(\Gamma_h(t))} &\leq c\|(\partial^\bullet)^{(\ell)}(u-\mathscr{P}_h u)(\cdot,t)\|_{L^2(\Gamma(t))} + c\|(\partial^\bullet)^{(\ell)}u\|_{L^2(\Gamma(t))}\\ &\leq c(c_\ell h^{k+1}+1)\sum_{j=0}^{\ell}\|(\partial^\bullet)^{(j)}u\|_{H^{k+1}(\Gamma(t))}.\end{aligned}$$

Here, the $H^{k+1}(\Gamma(t))$-norm on the right-hand side could be replaced by $H^2(\Gamma(t))$ (also the power in h^{k+1} would be reduced to 2) by modifiying the proofs of Theorems 6.3 and 6.4, using the linear interpolation $I_h^{(1)}$ on $\Gamma_h^k(t)$ instead of I_h.

Alternatively, a weaker condition can be obtained in the following way. By having sufficient regularity of the solution at $t = 0$ and having smooth evolution of the surface, by repeating the proof of Dziuk *et al.* (2012, Theorem 9.1) for the Ritz map instead of the semidiscrete solution, the following estimate is obtained:

$$\sup_{t\in[0,T]}\|(\partial_h^\bullet)^{(\ell)}\widetilde{\mathscr{P}}_h u(\cdot,t)\|_{L^2(\Gamma_h^k(t))} + \int_0^T\|\nabla_{\Gamma_h}\big((\partial_h^\bullet)^{(\ell)}\widetilde{\mathscr{P}}_h u(\cdot,s)\big)\|_{L^2(\Gamma_h^k(s))}\mathrm{d}s \leq c\sum_{j=1}^{\ell}\|(\partial_h^\bullet)^{(j)}u(\cdot,0)\|_{L^2(\Gamma_h^k(0))}.$$

The result could even be obtained directly by using Dziuk *et al.* (2012, Theorem 9.1) and modifying our results using the semidiscrete solution instead of the Ritz map.

8. Implementation

The implementation of the high-order ESFEM code follows the typical method of finite element mass matrix, stiffness matrix and load vector assembly, mixed with techniques from isoparametric FEM theory. This was also used for linear ESFEM (see Dziuk & Elliott, 2007, Section 7.2).

Similarly to the linear case, a curved element E_h of the k-order interpolation surface $\Gamma_h^k(t)$ is parameterized over the reference triangle E_0, chosen to be the unit simplex. Then the polynomial map of degree k between E_h and E_0 is used to compute the local matrices. All the computations are done on the reference element, using the Dunavant quadrature rule (see Dunavant, 1985). Then the local values are summed to their correct places in the global matrices.

In a typical case, the surface is evolved by solving a series of ODEs, hence only the initial mesh is created based on $\Gamma(0)$. Naturally, the problem of velocity-based grid distortion is still present. Possible ways to overcome this are methods using the DeTurck trick (see Elliott & Fritz, 2016) or using ALE finite elements (see Elliott & Venkataraman, 2014; Kovács & Power Guerra, 2014).

9. Numerical experiments

We performed various numerical experiments with quadratic approximation of the surface $\Gamma(t)$ and using quadratic ESFEM to illustrate our theoretical results.

9.1 *Example 1: Parabolic problem on a stationary surface*

Let us briefly report on numerical experiments for parabolic problems on a stationary surface, as a benchmark problem. Let $\Gamma \subset \mathbb{R}^3$ be the unit sphere, and let us consider the parabolic surface PDE

$$\partial_t u - \Delta_\Gamma u = f,$$

with given initial value and inhomogeneity f chosen such that the solution is $u(x,t) = e^{-6t}x_1x_2$.

Let $(\mathcal{T}_k)_{k=1,2,\dots,n}$ and $(\tau_k)_{k=1,2,\dots,n}$ be series of meshes and time steps, respectively, such that $2h_k = h_{k-1}$ and $2\tau_k = \tau_{k-1}$, with $h_1 = \sqrt{2}$ and $\tau_1 = 0.2$. For each mesh $\mathcal{T}_k$ with corresponding step size τ_k, we numerically solve the surface PDE using second-order ESFEM combined with third-order BDF methods. Then, by e_k we denote the error corresponding to the mesh $\mathcal{T}_k$ and step size τ_k. We then compute the errors between the lifted numerical solution and the exact solution using the following norm and seminorm:

$$L^\infty(L^2): \qquad \max_{1\le n\le N} \|u_h^n - u(\cdot,t_n)\|_{L^2(\Gamma(t_n))},$$

$$L^2(H^1): \qquad \left(\tau\sum_{n=1}^{N} \|\nabla_{\Gamma(t_n)}\big(u_h^n - u(\cdot,t_n)\big)\|^2_{L^2(\Gamma(t_n))}\right)^{1/2}.$$

Using the above norms, the experimental order of convergence rates (EOCs) are computed by

$$\mathrm{EOC}_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)} \qquad (k = 2,3,\dots,n).$$

In Table 1, we report on the EOCs for the second-order ESFEM coupled with the BDF3 method; theoretically, we expect EOC≈ 3 in the $L^\infty(L^2)$-norm, and EOC≈ 2 in the $L^2(H^1)$-seminorm.

In Figs 1 and 2, we report on the errors

$$\|u(\cdot,N\tau) - u_h^N\|_{L^2(\Gamma)} \quad \text{and} \quad \|\nabla_\Gamma(u(\cdot,N\tau) - u_h^N)\|_{L^2(\Gamma)},$$

TABLE 1 *Errors and EOCs in the $L^\infty(L^2)$- and $L^2(H^1)$-norms for the stationary problem*

Level	dof	$L^\infty(L^2)$	EOC	$L^2(H^1)$	EOC
1	6	$5.3113 \cdot 10^{-3}$	—	$8.0694 \cdot 10^{-3}$	—
2	18	$2.9257 \cdot 10^{-3}$	0.9511	$4.3162 \cdot 10^{-3}$	0.99805
3	66	$9.2303 \cdot 10^{-4}$	1.7122	$1.9050 \cdot 10^{-3}$	1.2139
4	258	$1.7285 \cdot 10^{-4}$	2.4338	$5.4403 \cdot 10^{-4}$	1.8207
5	1026	$2.6463 \cdot 10^{-5}$	2.7124	$1.2265 \cdot 10^{-4}$	2.1529
6	4098	$3.6845 \cdot 10^{-6}$	2.8457	$2.4772 \cdot 10^{-5}$	2.3088

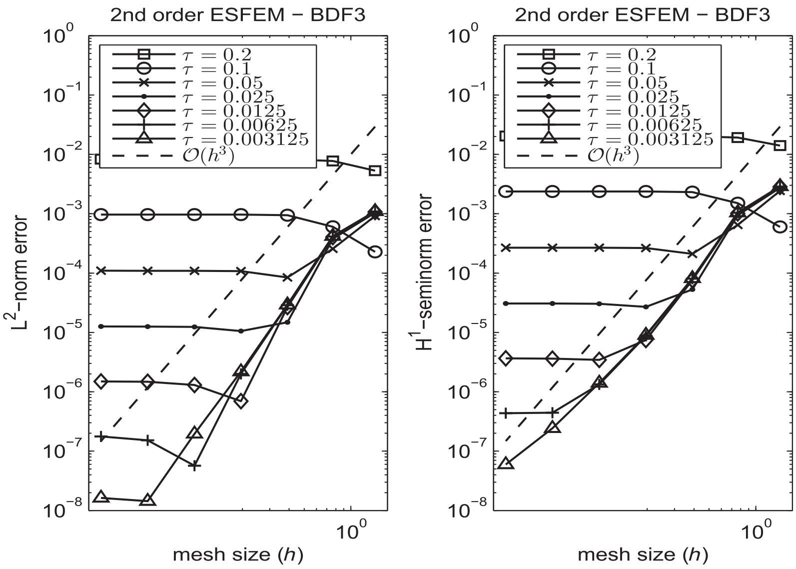

FIG. 1. Spatial convergence of the BDF3 / quadratic SFEM discretization for the stationary surface PDE.

at time $N\tau = 1$. The logarithmic plots show the errors against the mesh width h (in Fig. 1) and against time-step size τ (in Fig. 2).

The different lines correspond to different time-step sizes and to different mesh refinements, respectively in Figs 1 and 2. In both figures we can observe two regions. In Fig. 1, a region where the spatial discretization error dominates, matching the convergence rates of our theoretical results, and a region, with fine meshes, where the time discretization error dominates (the error curves flatten out). In Fig. 2, the same description applies but with reversed roles. First, the time discretization error dominates, while for smaller step sizes the spatial error dominates. The convergence in space (Fig. 1) and in time (Fig. 2) can both be observed to be nicely in agreement with the theoretical results (note the reference lines).

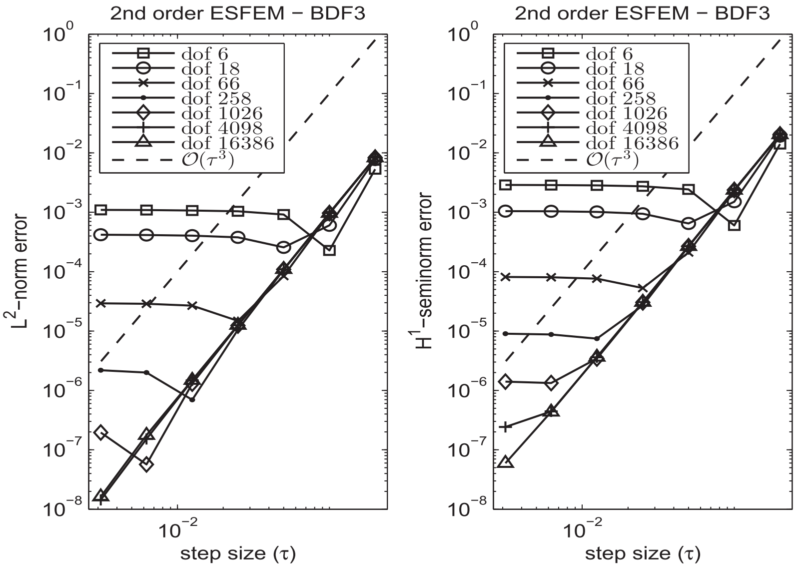

FIG. 2. Temporal convergence of the BDF3/quadratic SFEM discretization for the stationary surface PDE.

TABLE 2 *Errors and EOCs in the $L^\infty(L^2)$- and $L^2(H^1)$-norms for the evolving surface problem*

Level	dof	$L^\infty(L^2)$	EOC	$L^2(H^1)$	EOC
1	6	$5.7898 \cdot 10^{-3}$	—	$8.4446 \cdot 10^{-3}$	—
2	18	$9.5840 \cdot 10^{-4}$	2.8688	$1.8173 \cdot 10^{-3}$	2.4504
3	66	$4.0725 \cdot 10^{-4}$	1.2704	$1.6549 \cdot 10^{-3}$	0.13895
4	258	$9.1096 \cdot 10^{-5}$	2.1755	$5.4513 \cdot 10^{-4}$	1.6133
5	1026	$1.3847 \cdot 10^{-5}$	2.7226	$1.2774 \cdot 10^{-4}$	2.0971
6	4098	$1.9534 \cdot 10^{-6}$	2.8267	$2.6135 \cdot 10^{-5}$	2.2901

9.2 *Example 2: Evolving surface parabolic problem*

In the following experiment, we consider the parabolic problem (2.1) on the evolving surface given by

$$\Gamma(t) = \left\{x \in \mathbb{R}^3 \;\middle|\; a(t)^{-1}x_1^2 + x_2^2 + x_3^2 - 1 = 0\right\},$$

where $a(t) = 1 + \frac{1}{4}\sin(2\pi t)$ (see, e.g., Dziuk & Elliott, 2007; Dziuk *et al.*, 2012; Mansour, 2013), with given initial value and inhomogeneity f chosen such that the solution is $u(x,t) = e^{-6t}x_1x_2$.

Similarly to the stationary surface case, we again report on the experimental orders of convergence and similar spatial and temporal convergence plots. They are all produced exactly as described above.

The EOCs for the evolving surface problem solved with BDF method of order 3 and evolving surface finite elements of second order can be seen in Table 2.

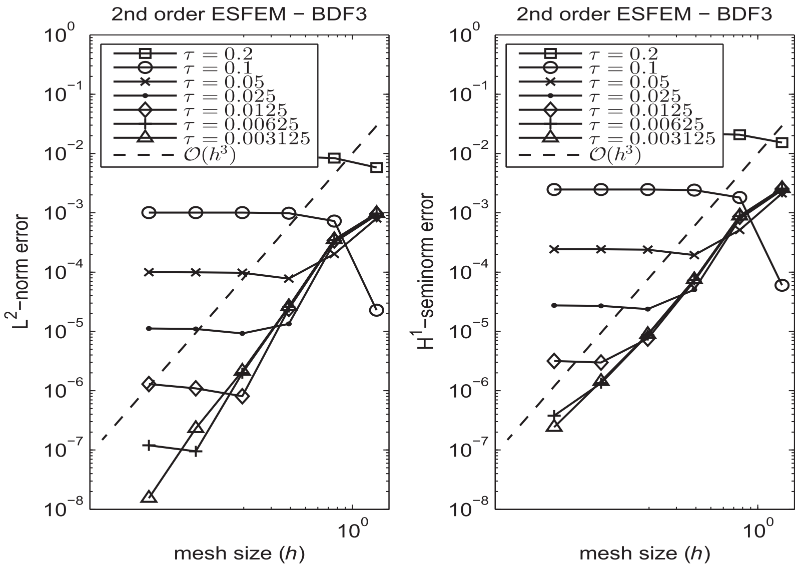

FIG. 3. Spatial convergence of the BDF3/quadratic ESFEM discretization for the evolving surface PDE.

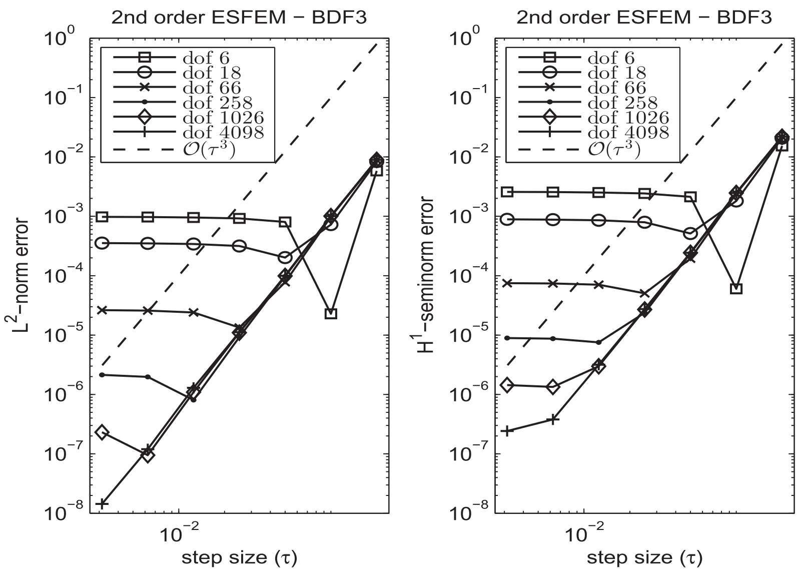

FIG. 4. Temporal convergence of the BDF3/quadratic ESFEM discretization for the evolving surface PDE.

The errors at time $N\tau = 1$ in different norms can be seen in the following plots: the different lines again correspond to different time-step sizes and to different mesh refinements in Figs 3 and 4, respectively.

Acknowledgements

The author is grateful for the valuable discussions with Prof. Christian Lubich during the preparation of the manuscript. We would also like to thank the anonymous referees for their extremely constructive suggestions and comments.

Funding

Deutsche Forschungsgemeinschaft, SFB 1173 to B.K.

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Appendix. Proof of the geometric approximation results of Lemma 5.2

For clarity we recall Lemma 5.2.

LEMMA A.1 For $\Gamma_h^k(t)$ and $\Gamma(t)$ as above, for $h \leq h_0$ with a sufficiently small $h_0 > 0$, we have the geometric approximation estimates:

$$\begin{aligned}
&\|d\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1}, && \|1-\delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1},\\
&\|\mathrm{n}-\mathrm{n}_h\|_{L^\infty(\Gamma_h(t))} \leq ch^k, && \text{and}\\
&\|\mathrm{Id}-\delta_h Q_h\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1}, && \|(\partial_h^\bullet)^{(\ell)} d\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1},\\
&\|(\partial_h^\bullet)^{(\ell)}\delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^{k+1}, && \|\mathrm{Pr}((\partial_h^\bullet)^{(\ell)} Q_h)\mathrm{Pr}\|_{L^\infty(\Gamma_h)} \leq ch^{k+1},
\end{aligned}$$

with constants depending only on $\mathcal{G}_T$, but not on h or t.

Proof of Lemma 5.2. The proofs follow the ideas of Dziuk & Elliott (2007, Lemma 5.1) and Mansour (2013, Lemma 6.1), in combination with the ideas and techniques of the proof of Demlow (2009, Proposition 2.3).

Let $E = E(t) \subset \Gamma_h^k(t)$ be an element of the discrete surface. By I_h we denote the k-order interpolation operator of Section 5.2.

(i) Since the nodes of E lie on the exact surface $\Gamma(t)$ we have that the interpolate $\widetilde{I}_h d$ vanishes on E. Then by using standard interpolation estimates (from Lemma 5.3 or from Brenner & Scott, 2007) we obtain

$$\|d\|_{L^\infty(E)} = \|d - \widetilde{I}_h d\|_{L^\infty(E)} \leq ch^{k+1}\|d\|_{W^{k+1,\infty}(E)} \leq ch^{k+1}\|d\|_{C^{k,1}(\mathcal{G}_T)}.$$

Higher-order norm estimates are shown analogously:

$$\|p - p^k\|_{W^{i,\infty}} \leq ch^{k+1-i}. \tag{A.1}$$

(iii) For the normal vector estimate,

$$\begin{aligned}
|\mathrm{n}(\hat{x}) - \mathrm{n}_h^k(\hat{x})| &\leq |\mathrm{n}(p^k(\tilde{x})) - \mathrm{n}(p(\tilde{x}))| + |\mathrm{n}(p(\tilde{x})) - \mathrm{n}_h^k(p^k(\tilde{x}))|\\
&\leq c(\mathcal{G}_T)h^{k+1} + c(\mathcal{G}_T)h^k,
\end{aligned} \tag{A.2}$$

where for the last estimate we used the smoothness of $\mathscr{G}_T$ and the above bound on d, and the bounds (A.1) and the Gram–Schmidt orthonormalization algorithm (cf. Demlow, 2009).

(ii) The second estimate is shown by recalling, from Demlow (2009, (2.10)), that (for a fixed $t \in [0,T]$)

$$\delta_h^k(x) = \mathrm{n}(x) \cdot \mathrm{n}_h^k(x) \prod_{j=1}^{m} (1 - d(x,t)K_j(x)) \qquad (x \in \Gamma_h^k(t)), \tag{A.3}$$

where $K_j(x) = \kappa_j(p(x,t))/1 + d(x,t)\kappa_j(p(x,t))$ with κ_j being the principle curvatures; cf. Demlow (2009). Then, following the proof of Demlow (2009, Proposition 4.1), using $\|d\|_{L^\infty} = \mathscr{O}(h^{k+1})$ and (A.2), we obtain

$$\begin{aligned} |1 - \delta_h^k| &\le c(\mathscr{G}_T)h^{k+1} + c(\mathscr{G}_T)|1 - \mathrm{n} \cdot \mathrm{n}_h^k| \\ &\le c(\mathscr{G}_T)h^{k+1} + c(\mathscr{G}_T)|\mathrm{n} - \mathrm{n}_h^k|^2 \le c(\mathscr{G}_T)h^{k+1}. \end{aligned} \tag{A.4}$$

(iv) To show the fourth estimate, we use the idea of the linear ESFEM case. Using the previous estimates and (5.1) the definition of Q_h,

$$|\mathrm{Id} - \delta_h Q_h| \le |\mathrm{Pr} - \mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr}| + ch^{k+1}.$$

Then, for an arbitrary unit vector z,

$$|(\mathrm{Pr} - \mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr})z| = |z \cdot (\mathrm{n}_h - (\mathrm{n}_h \cdot \mathrm{n})\mathrm{n})(\mathrm{n}_h - (\mathrm{n}_h \cdot \mathrm{n})\mathrm{n})| \le ch^{2k},$$

where the estimate follows, using (A.2), from

$$\begin{aligned} |\mathrm{n}_h - (\mathrm{n}_h \cdot \mathrm{n})\mathrm{n}| &\le |(\mathrm{n} \cdot \mathrm{n})\mathrm{n}_h - (\mathrm{n}_h \cdot \mathrm{n})\mathrm{n}_h| + |(\mathrm{n}_h \cdot \mathrm{n})\mathrm{n}_h - (\mathrm{n}_h \cdot \mathrm{n})\mathrm{n}| \\ &\le |((\mathrm{n} - \mathrm{n}_h) \cdot \mathrm{n})\mathrm{n}_h| + |(\mathrm{n}_h \cdot \mathrm{n})(\mathrm{n}_h - \mathrm{n})| \le ch^k. \end{aligned}$$

See also the proof of Demlow (2009, Proposition 4.1).

The proofs of the estimates with material derivatives are similar to their nondifferentiated versions; we follow the ideas of Mansour (2013).

(v) Again, since $(\partial_h^\bullet)^{(\ell)} d$ vanishes at the nodes of E, hence the interpolant $\widetilde{I}_h(\partial_h^\bullet)^{(\ell)} d$ vanishes on E completely. Again by interpolation estimates we obtain

$$\|(\partial_h^\bullet)^{(\ell)} d\|_{L^\infty(E)} = \|(\partial_h^\bullet)^{(\ell)} d - \widetilde{I}_h(\partial_h^\bullet)^{(\ell)} d\|_{L^\infty(E)} \le ch^{k+1}\|(\partial_h^\bullet)^{(\ell)} d\|_{W^{k+1,\infty}(E)} \le ch^{k+1}\|(\partial_h^\bullet)^{(\ell)} d\|_{C^{k,1}(\mathscr{G}_T)}.$$

(vi) The sixth estimate is shown by taking the material derivative of (A.3) and by a similar argument to (A.4),

$$\|(\partial_h^\bullet)^{(\ell)} \delta_h\|_{L^\infty(\Gamma_h(t))} \le c(\mathscr{G}_T)h^{k+1} + c(\mathscr{G}_T)|\partial_h^\bullet(\mathrm{n} - \mathrm{n}_h^k)|^2 \le c(\mathscr{G}_T)h^{k+1},$$

where the last inequality follows by the chain rule and since the combination of (A.1) and (A.2), together with $\mathrm{n} = \nabla d$, yields $|\partial_h^\bullet(\mathrm{n} - \mathrm{n}_h^k)| = \mathscr{O}(h^k)$.

(vii) Let us show the last estimate for $\ell = 1$. The higher-order version follows by a similar argument recursively. It is clear from the definition of Q_h (see (5.1)) that with a smooth remainder function R we can write

$$Q_h = \frac{1}{\delta_h} \mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr} + dR(\delta_h, \mathrm{Pr}, \mathrm{Pr}_h, \mathscr{H}).$$

Now using the facts $\|d\|, \|\partial_h^\bullet d\| = \mathscr{O}(h^{k+1})$, $\delta_h = 1 + \mathscr{O}(h^{k+1})$ and $\|\partial_h^\bullet \delta_h\| = \mathscr{O}(h^{k+1})$ we bound

$$\mathrm{Pr}(\partial_h^\bullet Q_h)\mathrm{Pr} = \mathrm{Pr}\partial_h^\bullet(\mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr})\mathrm{Pr} + \mathscr{O}(h^{k+1}).$$

The first term here is estimated separately. Using $\partial_h^\bullet \mathrm{n} \cdot \mathrm{n} = 0$, in Mansour (2013, equation 6.4), it is shown that

$$\mathrm{Pr}\partial_h^\bullet(\mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr})\mathrm{Pr} = \mathrm{Pr}\partial_h^\bullet(\mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr} - \mathrm{Pr})\mathrm{Pr} = -\mathrm{Pr}\partial_h^\bullet(\mathrm{Pr}\, \mathrm{n}_h\mathrm{n}_h^{\mathrm{T}}\mathrm{Pr})\mathrm{Pr}.$$

Since the projections are bounded, we need only the bounds

$$|\mathrm{Pr}\, \mathrm{n}_h| = |\mathrm{n}_h - (\mathrm{n} \cdot \mathrm{n}_h)\mathrm{n}| \leq ch^k,$$
$$|\partial_h^\bullet(\mathrm{Pr}\, \mathrm{n}_h)| = |\partial_h^\bullet(\mathrm{n}_h - (\mathrm{n} \cdot \mathrm{n}_h)\mathrm{n})| \leq ch^k,$$

where the first inequality has been shown above, while the second follows by the chain rule and by $|\partial_h^\bullet(\mathrm{n} - \mathrm{n}_h^k)| = \mathscr{O}(h^k)$ proved above, together with the boundedness of $\partial_h^\bullet \mathrm{n}$ and $\partial_h^\bullet \mathrm{n}_h$ (which follows from velocity approximation and the smoothness of $\mathscr{G}_T$ for the first term, while for the latter by the same arguments and an additional triangle inequality). $\square$

Appendix D. Maximum norm stability and error estimates for the evolving surface finite element method

Received: 7 December 2016 | Revised: 29 May 2017 | Accepted: 8 August 2017

DOI: 10.1002/num.22212

RESEARCH ARTICLE

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Maximum norm stability and error estimates for the evolving surface finite element method

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Funding information
Deutscher Akademischer Austausch Dienst (DAAD)

We show convergence in the natural L^∞ and $W^{1,\infty}$ norm for a semidiscretization with linear finite elements of a linear parabolic partial differential equations on evolving surfaces. To prove this, we show error estimates for a Ritz map, error estimates for the material derivative of a Ritz map and a weak discrete maximum principle.

KEYWORDS
evolving surfaces, volving surface finite element method, parabolic partial differential equation, weak discrete maximum principle, weighted norm

1 | INTRODUCTION

Many important problems can be modeled by partial differential equations (PDEs) on evolving surfaces. Examples for such equations are given in material sciences, fluid mechanics, and biophysics [1–3].

We consider the heat equation on closed evolving surfaces, in the case where the surface velocity v is explicitly given, derived in [4]. Dziuk and Elliott [4] introduced the evolving surface finite element method (ESFEM) to spatially approximate such problems. Error estimates for the semidiscretization with piecewise linear finite elements in the L^2 and H^1 norm are given in [4, 5]. Convergence results for time discretizations, as well as full discretizations, have been shown in [6, 7].

The aim of this work is to give error bounds for the semidiscretization with linear finite elements in the L^∞ and $W^{1,\infty}$ norm. The authors are not aware of any other maximum norm convergence results for *evolving surface* PDEs.

Such estimates are of particular interest for nonlinear parabolic PDEs on evolving surfaces, and in the case when the velocity v is not explicitly given, but depends on the solution u. Example of such problems are given in [1, 8–11] and the references therein. The first convergence results for such coupled problems have been recently shown in [12]. The treatment of such general equations is beyond the scope of this paper.

Our convergence proof for the semidiscretization of the linear heat equation on evolving surfaces relies on three main results.

- We give error bounds in the L^∞ and $W^{1,\infty}$ norms for a suitable *time-dependent* Ritz map (also used in [13], which is not the same as the one in [5]). The proofs of these results are based on Nitsche's weighted norm technique [14].
- As the surface evolves in time the Ritz map is time dependent, hence it does not commute with the time derivative. We therefore need the essential novel results: the L^∞ and $W^{1,\infty}$ norm error bounds in the *material derivatives* of the Ritz map. Up to our knowledge, such maximum norm estimates have not been shown in the literature until now.
- We extend the weak finite element maximum principle, which is originally due to Schatz et al. [15] for Euclidean domains, to the evolving surface case. In [15], they use basic properties of the semigroup corresponding to the linear heat equation on a bounded domain. As there is no semigroup theory for the linear heat equation on evolving surfaces, we are going to use a different approach.

The proven convergence result has optimal order, in the sense of powers of the mesh size, however, contains a nonoptimal logarithmic factor. We expect that the results presented here may be improved to have optimal logarithmic factors, shown using more involved proof techniques generalized from the Euclidean domains, see for instance [16–18] and especially the proof of the logarithm-free discrete maximum principle proved in [19]. However, such logarithmically optimal bounds are not in the scope of the present work, as such a refined analysis would easily double the length of the paper.

In a recent preprint of Kröner [20], L^∞ estimates—of order $O(|\log(h)|h + \tau^{1/2})$ —are shown for full discretizations of parabolic PDEs on stationary surfaces. The results of that paper are obtained using different proof techniques.

The layout of the paper is as follows. We begin in Section 2 by introducing the problem along with some notation. In the first three subsections of Section 3, we quickly develop the ESFEM, and recall basic results and estimates. In the following three subsections, we introduce a surface version of Nitsche's weighted norms, and define an L^2-projection. In Section 4, we give error bound in the maximum norm for our Ritz map and for its material derivative. In Section 5, we derive a weak ESFEM maximum principle. In Section 6, we give error bounds for the semidiscretization of the linear heat equation on evolving surfaces in the L^∞ and $W^{1,\infty}$ norm. In Section 7, we present the results of a numerical experiment. We gather technical details for calculations with our weight functions in Appendix B.

2 | A PARABOLIC PROBLEM ON EVOLVING SURFACES

Let us consider a smooth evolving closed hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$ (our main focus is on the case $m = 2$, but some of our results hold for more general cases), $0 \leq t \leq T$, which moves with a given smooth velocity v. More precise we assume that there exists a smooth dynamical system $\Phi : \Gamma_0 \times [0,T] \to \mathbb{R}^{m+1}$, such that for each $t \in [0,T]$ the map $\Phi_t = \Phi(\ \cdot\ , t)$ is an embedding. We define $\Gamma(t) = \Phi_t(\Gamma_0)$ and define the velocity v via the equation $\partial_t\Phi(x,t) = v(\Phi(x,t),t)$. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of the function u. The tangential gradient is given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu\nu$, where ν is the unit normal and finally we define the Laplace–Beltrami operator via $\Delta_\Gamma u = \nabla_\Gamma \cdot \nabla_\Gamma u$. This article shares the setting of Dziuk and Elliott [4, 5, 21].

We consider the following linear problem derived in [4, Section 3]:

$$\begin{cases} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u = f & \text{on } \Gamma(t), \\ u(\cdot, 0) = u_0 & \text{on } \Gamma(0). \end{cases} \tag{1}$$

We use Sobolev spaces on surfaces: For a sufficiently smooth surface Γ and $1 \leq p \leq \infty$, we define

$$W^{1,p}(\Gamma) = \left\{ \eta \in L^p(\Gamma) | \nabla_\Gamma \eta \in L^p(\Gamma)^{m+1} \right\},$$

and analogously $W^{k,p}(\Gamma)$ for $k \in \mathbb{N}$ [4, Section 2.1]. We set $H^k(\Gamma) = W^{k,2}(\Gamma)$. Finally, $\mathcal{G}_T$ denotes the space-time manifold, that is, $\mathcal{G}_T = \cup_{t \in [0,T]} \Gamma(t) \times \{t\}$.

If $f = 0$, then a weak formulation of this problem reads as follows.

Definition 2.1 (weak solution, [4] Definition 4.1)**.** A function $u \in H^1(\mathcal{G}_T)$ is called a weak solution of (1), if for almost every $t \in [0, T]$

$$\frac{d}{dt} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \nabla_{\Gamma(t)} u \cdot \nabla_{\Gamma(t)} \varphi = \int_{\Gamma(t)} u \partial^\bullet \varphi,$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(\cdot, 0) = u_0$.

For suitable f and u_0 existence and uniqueness results, for the strong and the weak problem, were obtained in [4, Section 4].

Throughout this article, we assume that f and u_0 a such regular that $u \in W^{3,\infty}(\mathcal{G}_T)$. Furthermore, we set for simplicity reasons in all sections $f = 0$, as the extension of our results to the inhomogeneous case are straightforward.

3 | PRELIMINARIES

We give a summary of this section. In Section 3.1, we introduce the ESFEM, which is due to Dziuk and Elliott [4]. In Section 3.2, we recall the lifting process, which originates in Dziuk [22]. In Section 3.3, we collect important results from Dziuk and Elliott [5] and sometimes state them in a slightly more general fashion. In Section 3.4, we introduce weighted norms, which are due to Nitsche [14], and give connections to the L^∞ norm. In Section 3.5, we give interpolation estimates in the L^2, L^∞, and weighted norms and further give some special interpolation estimates in weighted norms. The latter two were first stated in Nitsche [14]. In Section 3.6, we introduce an L^2-projection, give a stability bound in L^p norms and finish with a error estimate with respect to a different weight function. The basic reference for this is Douglaset al. [23] and Schatz et al. [15].

3.1 | Semidiscretization with the ESFEM

The smooth surface $\Gamma(t)$ is approximated by a triangulated one denoted by $\Gamma_h(t)$, whose vertices $(a_j(t))_{j=1}^N = (\Phi(a_j(0), t))_{j=1}^N$ are sitting on the surface for all time, such that

$$\Gamma_h(t) = \bigcup_{E(t) \in \mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ are forming an admissible triangulation $\mathcal{T}_h(t)$, with h denoting the maximum diameter. Admissible triangulations were introduced in [4, Section 5.1]: Every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by ch with $c > 0$, and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$. The discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)} \phi = \nabla \phi - \nabla \phi \cdot \nu_h \nu_h,$$

understood in a piecewise sense, with ν_h denoting the normal to $\Gamma_h(t)$ (see [4]).

For every $t \in [0, T]$, we define the finite element space $S_h(t)$ spanned by the continuous, piecewise linear evolving basis functions χ_j, satisfying

$$\chi_j(a_i(t), t) = \delta_{ij} \quad \text{for all } i, j = 1, 2, \dots, N,$$

therefore, $S_h(t) = \operatorname{span}\{\chi_1(\,\cdot\,, t), \chi_2(\,\cdot\,, t), \dots, \chi_N(\,\cdot\,, t)\}$.

The continuous dynamical system Φ is

$$\text{interpolated by } \Phi_h : \Gamma_h(0) \times [0, T] \to \mathbb{R}^{m+1},$$
$$\Phi_h(\cdot, t) = \sum_{j=1}^{N} \Phi(a_j(0), t)\chi_j(\cdot, t), \tag{2}$$

the discrete dynamical system of the interpolating discrete surface $\Gamma_h(t)$.

This defines a discrete surface velocity V_h via the ordinary differential equation (ODE) $\partial_t \Phi_h(\cdot, t) = V_h(\Phi_h(\cdot, t), t)$. Then, the discrete material derivative is given by

$$\partial_h^\bullet \phi_h = \partial_t \phi_h + V_h \cdot \nabla \phi_h \qquad (\phi_h \in S_h(t)).$$

The key *transport property* derived in [4, Proposition 5.4], is the following

$$\partial_h^\bullet \chi_k = 0 \qquad \text{for} \quad k = 1, 2, \dots, N. \tag{3}$$

The spatially discrete problem for evolving surfaces is: Find a $U_h \in S_h(t)$ with $\partial_h^\bullet U_h \in S_h(t)$ and temporally smooth such that, for every $\phi_h \in S_h(t)$ with $\partial_h^\bullet \phi_h \in S_h(t)$,

$$\frac{d}{dt} \int_{\Gamma_h(t)} U_h \phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h} U_h \cdot \nabla_{\Gamma_h} \phi_h = \int_{\Gamma_h(t)} U_h \partial_h^\bullet \phi_h, \tag{4}$$

with the initial condition $U_h(\cdot, 0) = U_h^0 \in S_h(0)$ being a sufficient approximation to u_0.

3.2 | Lifts

In the following, we recall the so-called *lift operator*, which was introduced in [22] and further investigated in [4, 5]. The lift operator projects a finite element function on the discrete surface onto a function on the smooth surface.

Using the *oriented distance function* d ([4, Section 2.1]), for a continuous function $\eta_h : \Gamma_h(t) \to \mathbb{R}$ its lift is defined as

$$\eta_h^l(x^l, t) = \eta_h(x, t), \qquad x \in \Gamma_h(t),$$

where for every $x \in \Gamma_h(t)$ the value $x^l = x^l(x, t) \in \Gamma(t)$ is uniquely defined via the equation

$$x = x^l + \nu(x^l, t)d(x, t).$$

This notation for x^l will also be used later on. By η^{-l}, we mean the function whose lift is η, and by E_h^l we mean the lift of the triangle E_h. These lifted triangles, or curved elements, form a *curved triangulation* of the smooth surface $\Gamma(t)$.

The following pointwise estimate was shown in the proof of Lemma 3 from Dziuk [22]:

$$\frac{1}{c}\,|\nabla_\Gamma \eta_h^l(x^l)| \leq \ |\nabla_{\Gamma_h}\eta_h(x)| \leq c\,|\nabla_\Gamma \eta_h^l(x^l)|. \tag{5}$$

We now recall some notions using the lifting process from [4, 22]. We have the lifted finite element space

$$S_h^l(t) = \left\{\varphi_h = \phi_h^l \mid \phi_h \in S_h(t)\right\}.$$

By δ_h, we denote the quotient between the continuous and discrete surface measures, dA and dA_h, defined as $\delta_h dA_h = dA$. For these quantities, we recall some results from [4, Lemma 5.1].

Lemma 3.1 *For sufficiently small h, we have the estimates*

$$\|d\|_{L^\infty(\Gamma_h(t))} \leq ch^2, \qquad \|1-\delta_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2,$$

with constants independent of t and h.

3.3 | Geometric estimates and bilinear forms

The definitions and results of this subsection are independent of the surface dimension m.

Let us denote by $\Phi_h^l : \Gamma_0 \times [0,T] \to \mathbb{R}^{m+1}$ the lift of Φ_h from Equation (2), that is, for $x \in \Gamma_h(t)$ with lift $x^l \in \Gamma(t)$

$$\Phi_h^l(x^l,t) = \Phi_h(x,t) \qquad (t \in [0,T]).$$

We then define the velocity v_h via the formula $\partial_t \Phi_h^l(x,t) = v_h(\Phi_h^l(x,t),t)$. Hence, the discrete material derivative on $\Gamma(t)$ is given by

$$\partial_h^\bullet u = \partial_t u + v_h \cdot \nabla u,$$

which satisfies the following relations, cf. [5]:

$$\partial^\bullet u = \partial_h^\bullet u + (v_h - v)\cdot \nabla_\Gamma u, \tag{6}$$

$$\|v - v_h\|_{L^\infty(\Gamma(t))} + h\|v - v_h\|_{W^\infty(\Gamma(t))} \leq ch^2\|v\|_{W^{2,\infty}(\Gamma(t))}. \tag{7}$$

We use the time-dependent bilinear forms defined in [5, Section 3.3]: for $z,\varphi \in H^1(\Gamma(t))$ and $Z_h, \phi_h \in H^1(\Gamma_h(t))$:

$$\begin{aligned}
a(t;z,\varphi) &= \int_{\Gamma(t)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a_h(t;Z_h,\phi_h) &= \sum_{E\in\mathcal{T}_h}\int_E \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h,\\
m(t;z,\varphi) &= \int_{\Gamma(t)} z\varphi, & m_h(t;Z_h,\phi_h) &= \int_{\Gamma_h(t)} Z_h\phi_h,\\
g(t;v;z,\varphi) &= \int_{\Gamma(t)} (\nabla_\Gamma \cdot v) z\varphi, & g_h(t;V_h;Z_h,\phi_h) &= \int_{\Gamma_h(t)} (\nabla_{\Gamma_h}\cdot V_h) Z_h\phi_h,\\
b(t;v;z,\varphi) &= \int_{\Gamma(t)} \mathcal{B}(v)\nabla_\Gamma z \cdot \nabla_\Gamma\varphi, & b_h(t;V_h;Z_h,\phi_h) &= \sum_{E\in\mathcal{T}_h}\int_E \mathcal{B}_h(V_h)\nabla_{\Gamma_h} Z_h,
\end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense, and with the matrices

$$\mathcal{B}(v)_{ij} = \delta_{ij}(\nabla_\Gamma \cdot v) - ((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i),$$
$$\mathcal{B}_h(V_h)_{ij} = \delta_{ij}(\nabla_\Gamma \cdot V_h) - ((\nabla_{\Gamma_h})_i (V_h)_j + (\nabla_{\Gamma_h})_j (V_h)_i),$$

where $i,j = 1,2,\ldots,m+1$.

We will also use the following bilinear forms, for $t \in [0,T]$ on $H^1(\Gamma(t)) \times H^1(\Gamma(t))$,

$$a^*(t;\cdot,\cdot) = a(t;\cdot,\cdot) + m(t;\cdot,\cdot),$$
$$(g+b)(t;v;\cdot,\cdot) = g(t;v;\cdot,\cdot) + b(t;v;\cdot,\cdot),$$

and similarly, their discrete bilinear form counterparts for the discrete surface, a_h^* and $(g_h + b_h)$. Note that both bilinear forms a^* and a_h^* are positive definite.

If it is clear from the context, we will drop the omnipresent argument t from the bilinear forms.

The time derivatives of the bilinear forms are given in the following lemma.

Lemma 3.2 (Discrete transport property). *For $z, \varphi, \partial_h^\bullet z, \partial_h^\bullet \varphi \in H^1(\Gamma(t))$, we have*

$$\frac{d}{dt} m(z,\varphi) = m(\partial_h^\bullet z, \varphi) + m(z, \partial_h^\bullet \varphi) + g(v_h; z, \varphi),$$
$$\frac{d}{dt} a(z,\varphi) = a(\partial_h^\bullet z, \varphi) + a(z, \partial_h^\bullet \varphi) + b(v_h; z, \varphi). \tag{8}$$

Similarly for $Z_h, \phi_h, \partial_h^\bullet Z_h, \partial_h^\bullet \phi_h \in H^1(\Gamma_h(t))$, we have

$$\frac{d}{dt} m_h(Z_h,\phi_h) = m_h(\partial_h^\bullet Z_h, \phi_h) + m_h(Z_h, \partial_h^\bullet \phi_h) + g_h(V_h; Z_h, \phi_h),$$
$$\frac{d}{dt} a_h(Z_h,\phi_h) = a_h(\partial_h^\bullet Z_h, \phi_h) + a_h(Z_h, \partial_h^\bullet \phi_h) + b_h(V_h; Z_h, \phi_h). \tag{9}$$

Important and often used results are the bounds of the geometric perturbation errors in the bilinear forms.

Lemma 3.3 *For all $1 \le p, q \le \infty$, that are conjugate, $p^{-1} + q^{-1} = 1$, and for arbitrary $Z_h \in L^p(\Gamma_h(t))$ and $\phi_h \in L^q(\Gamma_h(t))$, with corresponding lifts $z_h \in L^p(\Gamma(t))$ and $\varphi_h \in L^q(\Gamma(t))$, we have the following estimates:*

$$|m(z_h,\varphi_h) - m_h(Z_h,\phi_h)| \le ch^2 \|z_h\|_{L^p(\Gamma(t))} \|\varphi_h\|_{L^q(\Gamma(t))},$$
$$|g(v_h; z_h,\varphi_h) - g_h(V_h; Z_h,\phi_h)| \le ch^2 \|z_h\|_{L^p(\Gamma(t))} \|\varphi_h\|_{L^q(\Gamma(t))}.$$

Similarly, for $Z_h \in W^{1,p}(\Gamma_h(t))$ and $\phi_h \in W^{1,q}(\Gamma_h(t))$, with lifts $z_h \in W^{1,p}(\Gamma(t))$ and $\varphi_h \in W^{1,q}(\Gamma(t))$,

$$|a(z_h,\varphi_h) - a_h(Z_h,\phi_h)| \le ch^2 \|\nabla_\Gamma z_h\|_{L^p(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^q(\Gamma(t))},$$
$$b(v_h; z_h,\varphi_h) - b_h(V_h; Z_h,\phi_h)| \le ch^2 \|\nabla_\Gamma z_h\|_{L^p(\Gamma(t))} \|\nabla_\Gamma \varphi_h\|_{L^q(\Gamma(t))}.$$

Here the constant $c > 0$ is independent from $t \in [0,T]$ and the mesh width h.

Proof These geometric estimates were established for the case $p = q = 2$ in [5, Lemma 5.5] and [13, Lemma 7.5]. To show the estimates for general p and q, the same proof applies, except the last step where we use a Hölder inequality. ∎

3.4 | Weighted norms and basic estimates

Similarly, as in the works of Nitsche [14], weighted Sobolev norms and their properties play a very important and central role. In this section, we recall some basic results for them.

Definition 3.1 (Weight function). For $\gamma > 0$ sufficiently big but independent of t and h, we set

$$\rho : [0,\infty) \to [0,\infty), \quad \rho^2 := \rho^2(h) := \gamma h^2 \,|\log h|.$$

We define a weight function $\mu = \mu(t;\ .\ ) : \Gamma(t) \to \mathbb{R}$ via the formula, for any $y \in \Gamma(t)$,

$$\mu(x) := \mu(x,y) := \ |x-y|^2 + \rho^2 \quad \forall x \in \Gamma(t). \tag{10}$$

The actual choice of γ is going to be clear from the proofs.

Definition 3.2 (Weighted norms, [14] Section 2). Let μ be a weight function and $\alpha \in \mathbb{R}$. We define the norms

$$\|u\|^2_{L^2,\alpha} = \int_\Gamma \mu^{-\alpha}|u|^2,$$
$$\|u\|^2_{H^1,\alpha} = \|u\|^2_{L^2,\alpha} + \|\nabla_\Gamma u\|^2_{L^2,\alpha}, \quad \|u\|^2_{H^2,\alpha} = \|u\|^2_{H^1,\alpha} + \|\nabla^2_\Gamma u\|^2_{L^2,\alpha}.$$

In order to show basic estimates for the weighted norms, we need the following general version of inverse estimates for finite element functions, cf. [15].

Lemma 3.4 (Inverse estimate). *There exists $c > 0$ such that for each triangle $E_h(t) \subset \Gamma_h(t)$ the following inequality holds*

$$\|\phi_h(t)\|_{W^{k,p}(E_h(t))} \le ch^{m-k-2(1/q-1/p)}\|\phi_h(t)\|_{W^{m,q}(E_h(t))} \qquad (\forall \phi_h \in S_h(t)).$$

We can now turn to the estimates of the weighted norms defined above.

Lemma 3.5 *Let* $\dim\Gamma(t) = 2$. *Let $\phi_h \in S_h(t)$ with corresponding lift $\varphi_h \in S_h^l(t)$. Then there exist constants $c > 0$ independent of t, h, and γ such that*

$$\|\varphi_h\|_{L^\infty(\Gamma(t))} \le\ ch\,|\log h|\|\varphi_h\|_{L^2,2}, \tag{11}$$
$$\|\varphi_h\|_{W^{1,\infty}(\Gamma(t))} \le\ c\gamma\,|\log h|^{1/2}\|\varphi_h\|_{H^1,1}. \tag{12}$$

Proof There is a point $y_{0,h} \in E_0 \subset \Gamma_h(t)$ such that

$$\|\phi_h\|_{W^{1,\infty}(\Gamma_h(t))} = \ |\phi_h(y_{0,h})| + \ |\nabla_{\Gamma_h}\phi_h(y_{0,h})| = \|\phi_h\|_{W^{1,\infty}(E_0)}.$$

Note that on E_0 the estimate $\mu_h(x_h) \leq c\rho^2$ holds for $h < h_0$, h_0 sufficiently small. Then, the second bound yields from using inverse inequality (Lemma 3.4) and (57. The bound (11) is proved using similar arguments. ▪

We remind here that the weighted norms and ρ depend on h, cf. Definitions 3.1 and 3.2.

Lemma 3.6 *Let* $\dim \Gamma(t) = 2$ *and let* $u : \Gamma(t) \to \mathbb{R}$ *be a sufficiently regular. For all* $\gamma > 0$ *exists* $h_0 = h_0(\gamma) > 0$ *sufficiently small and a constant* $c = c(h_0) > 0$ *such that for all* $h < h_0$, *we have*

$$\|u\|_{L^2,2} \leq \; c\rho^{-1}\|u\|_{L^\infty(\Gamma(t))}, \tag{13}$$

$$\|u\|_{H^1,1} \leq \; c\,|\log\rho|^{1/2}\|u\|_{W^{1,\infty}(\Gamma(t))}. \tag{14}$$

c is independent of t and h.

Proof For $\alpha = 1, 2$, we obviously have

$$\|u\|^2_{L^2,\alpha} \leq \; \|u\|^2_{L^\infty(\Gamma(t))}\int_{\Gamma(t)} \frac{1}{(|x-y|^2 + \gamma h^2\,|\log h|)^\alpha}dy.$$

Let $\alpha = 1$. Denote by $r = \mathrm{dist}_\Gamma(x, y)$ the intrinsic distance. As the intrinsic distance is equivalent to the (extrinsic) Euclidean distance, cf. (56), we have

$$\int_{\Gamma(t)} \frac{1}{(\,|x-y|^2 + \gamma h^2\,|\log h|)}dy \leq c\int_{\Gamma(t)} \frac{1}{(r^2 + \gamma h^2\,|\log h|)}dy.$$

We use geodesic polar coordinates, cf. Section B1, to reach

$$\int_{\Gamma(t)} \frac{1}{(r^2 + \gamma h^2\,|\log h|)}dy \leq c\int_0^R \frac{r}{(r^2 + \gamma h^2\,|\log h|)}dr.$$

The result readily follows. The case $\alpha = 2$ is shown using similar arguments. ▪

Naturally, there is a weighted version of the Cauchy–Schwarz inequality, namely we have

$$\begin{aligned} |a^*(z_h, \varphi_h)| &\leq \|z_h\|_{H^1,\alpha}\|\varphi_h\|_{H^1,-\alpha}, \\ |a_h^*(Z_h, \phi_h)| &\leq c\|z_h\|_{H^1,\alpha}\|\varphi_h\|_{H^1,-\alpha}, \end{aligned} \tag{15}$$

and similarly for the bilinear forms g and b. Furthermore, this yields a weighted version of the geometric errors of the bilinear forms (Lemma 3.3), for any dimensions.

Lemma 3.7 *Under the conditions of Lemma 3.3, the following estimates hold, with a constant* $c > 0$ *independent of t, h, and* γ,

$$|a^*(z_h^l, \phi_h^l) - a_h^*(Z_h, \phi_h)| \leq \; ch^2\|z_h^l\|_{H^1,\alpha}\|\phi_h^l\|_{H^1,-\alpha}, \tag{16}$$

$$|(g+b)(v_h; z_h^l, \phi_h^l) - (g_h + b_h)(V_h; Z_h, \phi_h)| \leq \; ch^2\|z_h^l\|_{H^1,\alpha}\|\phi_h^l\|_{H^1,-\alpha}. \tag{17}$$

Lemma 3.8 *(i) Spatial derivatives of* μ^{-1} *arebounded as*

$$|\nabla_\Gamma \mu^{-1}| \le 2\mu^{-3/2}, \qquad |\Delta_\Gamma \mu^{-1}| \le c\mu^{-2} \tag{18}$$

with $c > 0$ *independent of* t, h, *and* γ.

(ii) For arbitrary $u \in H^1(\Gamma(t))$, *the following norm inequalities hold:*

$$\|\mu^{-1}u\|_{H^{1,-1}} \le c(\|u\|_{L^{2,2}} + \|u\|_{H^{1,1}}), \tag{19}$$

$$\|\mu^{-2}u\|_{L^{2,-1}} \le \rho^{-1}\|u\|_{L^{2,2}}. \tag{20}$$

Proof (i) The first estimate follows from

$$|\nabla_\Gamma \mu^{-1}| \le |\nabla \mu^{-1}| \le \frac{2\,|x-y|}{\mu^2} \le \frac{2\sqrt{\mu}}{\mu^2}.$$

For the second inequality consider the formula,

$$\Delta_\Gamma f = \Delta \bar{f} - \nabla^2 \bar{f}(\nu,\nu) - H\nu \cdot \nabla \bar{f},$$

where $\bar{f} : U \to \mathbb{R}$ is a smooth extension of the function f to an open neighbourhood $U \subset \mathbb{R}^{m+1}$ of $\Gamma(t)$ (cf. [4]), $\nabla^2 \bar{f}$ denotes the Hessian of $\bar{f}$ and H denotes the trace of the Weingarten map of $\Gamma(t)$.

(ii) In order to show these estimates, we use the bounds (18) obtained above. ∎

3.5 | Interpolation and an improved inverse estimate

Here, we collect some results involving evolving surface finite element functions.

For a sufficiently regular function $u : \Gamma(t) \to \mathbb{R}$, we denote by $\widetilde{I}_h u \in S_h(t)$ its Lagrange interpolation on $\Gamma_h(t)$. Then, the finite element interpolation is given by $I_h u = (\widetilde{I}_h u)^l \in S_h^l(t)$, having the error estimate below, cf. [24].

Lemma 3.9 *For* $m \le 3$ *and* $p \in \{2, \infty\}$, *there exists a constant* $c > 0$ *independent of* h *and* t *such that for* $u \in W^{2,p}(\Gamma(t))$:

$$\begin{aligned} &\|u - I_h u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma(u - I_h u)\|_{L^p(\Gamma(t))} \\ &\qquad \le ch^2(\|\nabla_\Gamma^2 u\|_{L^p(\Gamma(t))} + h\|\nabla_\Gamma u\|_{L^p(\Gamma(t))}). \end{aligned}$$

The interpolation estimates hold also if weighted norms are considered.

Lemma 3.10 *Let* $m \le 3$. *There exists a constant* $c > 0$ *such that for* $u \in W^{2,\infty}(\Gamma(t))$ *the following inequality holds*

$$\|u - I_h u\|^2_{L^{2,2}} + \|u - I_h u\|^2_{H^{1,1}} \le ch^2\,|\log h|\|u\|^2_{W^{2,\infty}(\Gamma(t))}. \tag{21}$$

Proof Use Lemma 3.6 in conjunction with Lemma 3.9 for $p = \infty$. ∎

Lemma 3.11 *There exists $h_0 > 0$, $\gamma_0 > 0$ such that for all $\alpha \in \mathbb{R}$, there exists a constant $c = c(h_0, \gamma_0) > 0$ independent of t and h such that for all $\gamma > \gamma_0$ for the weight μ, compare to Equation (10), and for all $h < h_0$ the following inequalities holds:*

(i) Let $u \in H^1(\Gamma(t))$ be curved element-wise H^2, that is, for each element $E \subset \Gamma_h(t)$ with corresponding lift $E^l \subset \Gamma(t)$, we have that $u|_{E^l} \in H^2(E^l)$. Then, the interpolation $I_h u \in S_h^l(t)$ satisfies

$$\|u - I_h u\|_{L^2,\alpha} + h\|\nabla_\Gamma(u - I_h u)\|_{L^2,\alpha} \leq ch^2(\|\nabla_\Gamma^2 u\|_{L^2,\alpha} + ch\|\nabla_\Gamma u\|_{L^2,\alpha}), \tag{22}$$

where $\|\nabla_\Gamma^2 u\|_{L^2,\alpha}$ is understood curved element-wise.
(ii) For any $\varphi_h \in S_h^l(t)$, the following estimate holds:

$$\|\mu^{-1}\varphi_h - I_h(\mu^{-1}\varphi_h)\|_{H^1,-1} \leq c\left(\frac{h}{\rho} + h\right)(\|\varphi_h\|_{L^2,2} + \|\nabla_\Gamma\varphi_h\|_{L^2,1}). \tag{23}$$

Proof (i) To prove inequality (22), it suffices to show that there exists a constant $c = c(\alpha) > 0$ independent of t, h such that for each element $K \in \mathcal{T}_h(t)$ it holds

$$\int_{K^l} \mu^\alpha((w - I_h w)^2 + h\,|\nabla_\Gamma(w - I_h w)|^2) \leq ch^2 \int_{K^l} \mu^\alpha(\nabla_\Gamma^2 w|^2 + ch\,|\nabla_\Gamma w|^2),$$

where $K^l \subset \Gamma(t)$ denote the lifted curved element of K. It is easy to show that there exists $\gamma_0 = \gamma_0(h_0) > 0$ and $c = c(\gamma_0) > 0$ such that for all $\gamma > \gamma_0$ it holds

$$\max_{K \in \mathcal{T}_h}\left(\frac{\max_{x \in K^l}\mu(x, y)}{\min_{x \in K^l}\mu(x, y)}\right) \leq c.$$

A straightforward calculation finishes the proof.

(ii) For an arbitrary function $f : \Gamma_h(t) \to \mathbb{R}$, which is element-wise H^2, a short calculation, similar to the one done in Dziuk [22, Lemma 3], shows that

$$|(\nabla_\Gamma)_i(\nabla_\Gamma)_j(f^l)| \leq c(|((\nabla_{\Gamma_h})_i(\nabla_{\Gamma_h})_j f)^l| + ch\,|\nabla_\Gamma(f^l)|),$$

for a sufficiently small $h_0 > h > 0$. A straightforward calculation combined with 1 and (18) finishes the proof. ∎

Now, we show a modified version of the inverse estimate of Lemma 3.4.

Lemma 3.12 *There exists c > 0 with*

$$\|\varphi_h\|_{L^\infty(\Gamma(t))} \leq \left|\frac{1}{V}\int_{\Gamma(t)} \varphi_h(y)\mathrm{d}V(y)\right| + c\,|\log h|^{1/2}\|\nabla_\Gamma\varphi_h\|_{L^\infty(\Gamma(t))},$$

where $V = \int_{\Gamma(t)} \mathrm{d}V$.

Proof Follow the steps in Schatz et al. [15] using the Green's function from Theorem A.1. For the estimates, use geodesic polar coordinates as in Section B.1. ∎

3.6 | Estimates for an L^2-projection

This section shows some technical results for the L^2-projection, which is denoted by P_0 (in contrast with the Ritz map which will be denoted by P_1).

Definition 3.3 (L^2-projection). We define $P_0(t) : L^2(\Gamma_h(t)) \to S_h(t)$ as follows. Let $u_h \in L^2(\Gamma_h(t))$ be given. Then, there exits a unique finite element function $P_0(t)u_h \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ it holds

$$m_h(P_0(t)u_h, \phi_h) = m_h(u_h, \phi_h). \tag{24}$$

Definition 3.4 For two points $x, y \in \Gamma_h(t)$, we define its intrinsic Riemannian distance as

$$\mathrm{dist}_h(x, y) = \mathrm{dist}_{\Gamma_h(t)}(x, y) = \inf_\sigma \int_0^1 |\sigma|^2,$$

where σ ranges over all possible curves $[0, 1] \to \Gamma_h(t)$ with $\sigma(0) = x$, $\sigma(1) = y$ and where σ is piecewise smooth, that is, there exists a finite partition of $[0, 1]$ such that σ restricted on that subinterval is smooth. For two sets $A, B \subset \Gamma_h(t)$, we set

$$\mathrm{dist}_h(A, B) = \inf_{(a,b)\in A\times B} \mathrm{dist}_h(a, b).$$

The following important L^p-stability bound and exponential decay property from Douglas et al. [23, Equations (6) and (7)] holds without any serious modification.

Theorem 3.1 *For $p \in [1, \infty]$, let $u_h \in L^p(\Gamma_h(t))$. Then, there exists a constant $c > 0$ independent of h and t such that*

$$\|P_0(t)u_h\|_{L^p(\Gamma_h(t))} \le c\|u_h\|_{L^p(\Gamma_h(t))}.$$

Further there exists $c_2, c_3 > 0$ independent of h and t such that for $A_h^1(t)$ and $A_h^2(t)$ disjoint subsets of $\Gamma_h(t)$ with $supp(u_h) \subseteq A_h^1$, we have

$$\|P_0(t)u_h\|_{L^2(A_h^2(t))} \le c_2 e^{-c_3 dist_h(A_h^1, A_h^2)h^{-1}} \|u_h\|_{L^2(A_h^1(t))}. \tag{25}$$

A quick proof sketch can be found in the Appendix.

For the proof of our discrete weak maximum principle, we are going to use a different weight function then (10). Let $y : [0, T] \to \mathbb{R}^{m+1}, t \mapsto y(t)$ be a curve with the property $y(t) \in \Gamma(t)$. In the following, we write y instead of $y(t)$. We define, for any $x \in \Gamma(t)$ and mesh width h,

$$\sigma(x) := \sigma_h^y(x) := \sigma_h(x, y) := (|x - y|^2 + h^2)^{1/2}. \tag{26}$$

We gather some estimates concerning σ in the next lemma.

Lemma 3.13 *There exists a constant $c > 0$ independent of t and h such that the following estimates hold*

$$\|\partial^\bullet \sigma\|_{L^\infty(\Gamma(t))} \le c, \qquad \|\partial_h^\bullet \sigma\|_{L^\infty(\Gamma(t))} \le c, \tag{27}$$

$$\|\nabla_\Gamma \sigma\|_{L^\infty(\Gamma(t))} \le 1, \qquad |\nabla_\Gamma^2 \sigma| \le c\left(\frac{1}{\sigma}+1\right), \qquad \|\nabla_\Gamma^2(\sigma^2)\|_{L^\infty(\Gamma(t))} \le c. \tag{28}$$

The proof of this lemma is a straightforward calculation and is omitted here.

Lemma 3.14 *There exists $c>0$ such for fixed $t \in [0,T]$, $x_h \in \Gamma_h(t)$, $\sigma = \sigma^{x_h}$, $\phi_h \in S_h(t)$, and $\psi_h = P_0(\sigma^2\phi_h)$ the following inequality holds:*

$$\begin{aligned}\|\sigma^2\phi_h - \psi_h\|_{L^2(\Gamma_h(t))} &+ h\|\nabla_{\Gamma_h}(\sigma^2\phi_h - \psi_h)\|_{L^2(\Gamma_h(t))}\\ &\le ch^2(\|\phi_h\|_{L^2(\Gamma_h(t))} + \|\sigma\nabla_{\Gamma_h}\phi_h\|_{L^2(\Gamma_h(t))}).\end{aligned}$$

Proof Consider a triangle $E_h \subset \Gamma_h(t)$ and set $g_h = \tilde{I}_h(\sigma^2\phi_h)$. Use Lemma 3.13 and (58) and follow the steps in Schatz et al. [15, Lemma 1.4]. ■

4 | A RITZ MAP AND SOME ERROR ESTIMATES

Just as in the usual L^2-theory the Ritz map plays a very important role for our L^∞ -error estimates. This section is devoted to the careful L^∞ and weighted norm analysis of the errors in the Ritz map.

Definition 4.1 (Ritz map, [13]). We define $P_{h,1}(t) : H^1(\Gamma(t)) \to S_h(t)$ as follows: Let $u \in H^1(\Gamma(t))$ be given. Then, there exits a unique finite element function $P_{h,1}(t)u \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ with $\varphi_h = \phi_h^l$ it holds

$$a_h^*(P_{h,1}(t)u, \phi_h) = a^*(u, \varphi_h). \tag{29}$$

This naturally defines the Ritz map on the continuous surface:

$$P_1(t)u = (P_{h,1}(t)u)^l \in S_h^l(t).$$

Note that the Ritz map does not satisfy the Galerkin orthogonality, however it satisfies, using (16), the following estimate, cf. [13]. For all $\varphi_h \in S_h^l(t)$, we have

$$|a^*(u - P_1(t)u, \varphi_h)| \ \le\ ch^2\|P_1(t)u\|_{H^{1,\alpha}}\|\varphi_h\|_{H^{1,-\alpha}}. \tag{30}$$

In this section, we aim to bound the following errors of the Ritz map:

$$u - P_1(t)u \quad \text{and} \quad \partial_h^\bullet(u - P_1(t)u),$$

in the L^∞ and $W^{1,\infty}$ norms. Previously, H^1 and L^2 error estimates have been shown in [4, 5].

4.1 | Weighted a priori estimates

Before turning to the maximum norm error estimates, we state and prove some technical regularity results involving weighted norms.

Lemma 4.1 (Weighted a priori estimates). *For $f \in L^2(\Gamma(t))$, the problem*

$$-\Delta_{\Gamma(t)} w + w = f \qquad \text{on } \Gamma(t),$$

has a unique weak solution $w \in H^1(\Gamma(t))$. Furthermore, $w \in H^2(\Gamma(t))$ and we have the following weighted a priori estimates

$$\|w\|_{H^{1,-1}} \le c(\|f\|_{L^{2,-1}} + \|w\|_{L^2(\Gamma(t))}) \tag{31}$$

$$\|w\|_{H^{2,-1}} \le c(\|f\|_{L^{2,-1}} + \|w\|_{H^1(\Gamma(t))}), \tag{32}$$

where the constant $c > 0$ is independent of t, h, and γ.

Proof Existence and uniqueness of a weak solution follows from [25]. Using integration by parts, Young's inequality and $|\nabla_\Gamma \mu| \le \sqrt{\mu}$ a short calculation shows (31). For the details on elliptic regularity and a derivation of the a priori estimate

$$\|w\|_{H^2(\Gamma(t))} \le c\| - \Delta_\Gamma w + w\|_{L^2(\Gamma(t))}, \tag{33}$$

where $c > 0$ is independent of t, we refer to [26, Appendix A].

Because of (31) it suffices to prove (32) with $\|\nabla_\Gamma^2 w\|_{L^{2,-1}}^2$ as the left-hand side instead of $\|w\|_{H^{2,-1}}^2$. We have

$$\begin{aligned}
\mu|\nabla_\Gamma^2 w|^2 &= \sum_{i=1}^{m+1} |(x^i - y^i)\nabla_\Gamma^2 w|^2 + \rho^2|\nabla_\Gamma^2 w|^2 \\
&= \sum_{i=1}^{m+1} |\nabla_\Gamma^2((x-y)^i w) - \nabla_\Gamma (x-y)^i \otimes \nabla_\Gamma w \\
&\quad -\nabla_\Gamma w \otimes \nabla_\Gamma (x-y)^i - w\nabla_\Gamma^2 (x-y)^i|^2 + \rho^2 \, |\nabla_\Gamma^2 w|^2,
\end{aligned}$$

where $x = (x^1, \dots, x^{m+1}),\ y = (y^1, \dots, y^{m+1})$ and $(x-y)^i = x^i - y^i$. We deduce

$$\|\nabla_\Gamma^2 w\|_{L^{2,-1}} \le c \left(\sum_{i=1}^{m+1} ||\nabla_\Gamma^2((x-y)^i w)\|_{L^{2,-1}} + \|w\|_{L^2} \right).$$

Using (33) with the product rule

$$\Delta_\Gamma(fg) = g\Delta_\Gamma f + f\Delta_\Gamma g - 2\nabla_\Gamma f \cdot \nabla_\Gamma g$$

leads to (32). ∎

Lemma 4.2 *For $g \in L^2(\Gamma(t))$ the problem*

$$-\Delta_{\Gamma(t)} w + w = \mu^{-2} g.$$

has a unique weak solution $w \in H^1(\Gamma(t))$. Furthermore, $w \in H^2(\Gamma(t))$, and there exists a constant $c > 0$ independent of t and h such that

$$\|w\|^2_{H^1(\Gamma(t))} \leq c\rho^{-2} \, |\log \rho| \|g\|^2_{L^2,2}. \tag{34}$$

Proof Lemma 4.1 gives us existence, uniqueness, and regularity of w. Consider the number

$$\frac{1}{\lambda(t)} = \sup\left\{ \|f\|^2_{H^1(\Gamma(t))} | f \in H^2(\Gamma(t)), \; \| - \Delta_{\Gamma(t)} f + f\|^2_{L^2,-2} \leq 1 \right\}.$$

Inequality (34) is proven if we show

$$\frac{1}{\lambda(t)} \leq c\rho^{-2} \, |\log \rho|,$$

where c is t independent. A short calculation shows that the smallest eigenvalue $\tilde{\lambda}_{\min}(t)$ of the elliptic eigenvalue problem

$$-\Delta_{\Gamma(t)} f + f = \tilde{\lambda}\mu^{-2} f \quad \text{on } \Gamma(t)$$

is equal to $\lambda(t)$. The weighted Rayleigh quotient implies

$$\tilde{\lambda}_{\min} = \inf_{f \in H^1} \frac{\|f\|^2_{H^1}}{\|f\|^2_{L^2,2}}.$$

Hence it suffices to prove

$$\|f\|^2_{L^2,2} \leq c\rho^{-2} \, |\log(\rho)| \|f\|^2_{H^1}, \tag{35}$$

for a $f \in H^1$. With a Hölder estimate, we arrive at

$$\|f\|^2_{L^2,2} \leq \left(\int_{\Gamma(t)} \mu^{-2p} \right)^{1/p} \left(\int_{\Gamma(t)} f^{2q} \right)^{1/q} = \left(\int_{\Gamma(t)} \mu^{-2p} \right)^{1/p} \|f\|^2_{L^{2q}(\Gamma(t))},$$

where $1 < p, q < \infty$ satisfies $p^{-1} + q^{-1} = 1$. We take the choice $q = \sqrt{|\log \rho|}$. It is easy to prove the following quantitative Sobolev–Nierenberg inequality for moving surfaces:

$$\|f\|_{L^q(\Gamma(t))} \leq cq \|f\|_{H^1(\Gamma(t))},$$

where c is independent of t and q. A straightforward calculation with geodesic polar coordinates using Lemma B.3 and Lemma B.2 shows inequality (35). ∎

4.2 | Maximum norm error estimates

Before showing L^∞ and $W^{1,\infty}$ norm error estimates for the Ritz map, we show similar estimates for weighted norms. Then, by connecting the norms, use these results to obtain our original goal.

Throughout this subsection, we write $P_1 u$ instead of $P_1(t)u$.

Lemma 4.3 *There exists $h_0 > 0$ sufficiently small and $\gamma_0 > 0$ sufficiently large and a constant $c = c(h_0, \gamma_0) > 0$ such that for $u \in W^{2,\infty}(\Gamma(t))$ it holds*

$$\|u - P_1 u\|^2_{L^2,2} + \|u - P_1 u\|^2_{H^1,1} \leq ch^2\, |\log h| \|u\|^2_{W^{2,\infty}(\Gamma(t))}. \tag{36}$$

Proof **Step 1**: Our goal is to show

$$\|u - P_1 u\|^2_{H^1,1} \leq ch^2\, |\log h| \|u\|^2_{W^{2,\infty}(\Gamma(t))} + \hat{c}\|u - P_1 u\|^2_{L^2,2}. \tag{37}$$

Similarly, as in Nitsche [14, Theorem 1], for an $f \in H^1(\Gamma(t))$ we have using partial integration

$$\frac{1}{2}\int_{\Gamma(t)} (\Delta_\Gamma \mu^{-1}) f^2 = -\int_{\Gamma(t)} f \nabla_\Gamma \mu^{-1} \cdot \nabla_\Gamma f.$$

This further implies

$$\int_{\Gamma(t)} \mu^{-1}\, |\nabla_\Gamma f|^2 = \int_{\Gamma(t)} \nabla_\Gamma f \cdot \nabla_\Gamma(\mu^{-1} f) - \frac{1}{2}\int_{\Gamma(t)} (\Delta_\Gamma \mu^{-1}) f^2.$$

As $P_1 u \in H^1(\Gamma)$, we deduce using (18)

$$\begin{aligned}
\|u - P_1 u\|^2_{H^1,1} &\leq a^*(u - P_1 u, \mu^{-1}(u - P_1 u)) - \frac{1}{2}\int_{\Gamma(t)} (\Delta_\Gamma \mu^{-1})(u - P_1 u)^2 \\
&\leq a^*(u - P_1 u, \mu^{-1}(u - P_1 u)) + c\|u - P_1 u\|^2_{L^2,2}.
\end{aligned}$$

For simplicity, we set $e = u - P_1 u$, and use $I_h u = (\widetilde{I}_h u)^l$ to obtain

$$\begin{aligned}
a^*(e, \mu^{-1} e) = a^*(e, \mu^{-1}(u - I_h u)) \\
&+ a^*(e, \mu^{-1}(I_h u - P_1 u) - I_h(\mu^{-1}(I_h u - P_1 u))) \\
&+ a^*(e, I_h(\mu^{-1}(I_h u - P_1))) = I_1 + I_2 + I_3.
\end{aligned}$$

Using Lemma 3.7 15), Lemma 3.8 (19), Lemma 3.10 (21), and ε -Young inequality we estimate as

$$|I_1| \leq \varepsilon\|e\|^2_{H^1,1} + ch^2\, |\log h| \|u\|^2_{W^{2,\infty}(\Gamma(t))}.$$

For the second term use in addition Lemma 3.11 (23) and a $0 < h < h_0$ sufficiently small to get

$$|I_2| \leq \varepsilon\|e\|^2_{H^1,1} + c(h^2\, |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))} + \|e\|_{L^2,2}).$$

For the last term use in addition Lemma 3.7 (30) to obtain

$$|I_3| \le \varepsilon \|e\|^2_{H^1,1} + c(h^2 \, |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))} + \|e\|_{L^2,2}).$$

These estimates together, and absorbing $\|e\|^2_{H^1,1}$, imply (37).

Step 2: Using an Aubin–Nitsche argument, we prove that there exists $\gamma > \gamma_0 > 0$ sufficiently large such that for all $\delta > 0$ the following estimate holds

$$\|u - P_1 u\|^2_{L^2,2} \le ch^4 \|u\|^2_{W^{2,\infty}(\Gamma(t))} + \delta \|u - P_1 u\|^2_{H^1,1}. \tag{38}$$

Let $w \in H^2(\Gamma(t))$ be the weak solution of

$$-\Delta_\Gamma w + w = \mu^{-2} e.$$

Then by testing with e, we obtain

$$\|e\|^2_{L^2,2} = (a^*(e,w) - a^*(e,I_h w)) + a^*(e,I_h w) = a^*(e, w - I_h w) + a^*(e, I_h w).$$

In addition to the already mentioned lemmata in Step 1 use Lemma 4.1 (32), Lemma 3.8 (20), Lemma 4.2 (34), and a sufficiently large $\gamma > \gamma_0 > 0$ to estimate

$$|a^*(e, w - I_h w)| \le \frac{1}{4} \|e\|^2_{L^2,2} + \frac{\delta}{2} \|e\|^2_{H^1,1}.$$

For the other term, we estimate

$$|a^*(e, I_h w)| \le ch^2 \|e\|_{H^1} \|I_h w\|_{H^1} \le ch^4 \|u\|^2_{W^{2,\infty}(\Gamma(t))} + \frac{1}{4} \|e\|^2_{L^2,2}.$$

By absorption, this implies (38).

The final estimate is shown by combining (37) and (38), and choosing $\delta > 0$ such that $\hat{c}\delta < 1$. Then, an absorption finishes the proof. ∎

Theorem 4.1 *There exist constants c > 0 independent of h and t such that*

$$\begin{aligned} \|u - (P_{h,1}(t)u)^l\|_{L^\infty(\Gamma(t))} &\le ch^2 \, |\log h|^{3/2} \|u\|_{W^{2,\infty}(\Gamma(t))}, \\ \|u - (P_{h,1}(t)u)^l\|_{W^{1,\infty}(\Gamma(t))} &\le ch \, |\log h| \|u\|_{W^{2,\infty}(\Gamma(t))}, \quad (u \in W^{2,\infty}(\Gamma(t))). \end{aligned}$$

Proof Using Lemma 3.9, Lemma 3.5 (12), and Lemma 3.6 (14), we get

$$\begin{aligned} \|u - P_1 u\|_{W^{1,\infty}(\Gamma(t))} &\le \|u - I_h u\|_{W^{1,\infty}(\Gamma(t))} + c\|\tilde{I}_h u - P_{h,1} u\|_{W^{1,\infty}(\Gamma_h(t))} \\ &\le ch\|u\|_{W^{2,\infty}(\Gamma(t))} + c|\log h|^{1/2} \|\tilde{I}_h u - P_{h,1} u\|_{H^1,1} \\ &\le ch \, |\log h| \|u\|_{W^{2,\infty}(\Gamma_h(t))} + c\|u - (P_{h,1} u)^l\|_{H^1,1}. \end{aligned}$$

For the $W^{1,\infty}$ -estimate use Lemma 4.3 to estimate the weighted norms. The L^∞ -estimate is obtained in a similar way. ∎

Remark 4.1 The paper of Demlow [27] (dealing with elliptic problems on stationary surfaces) contains a related result in Corollary 4.6, however, it does not directly imply Lemma 4.3. There are two crucial differences compared to the theorem above. As there is no surface evolution in [27], the constants appearing in his proof would need to be shown being uniform in time.[1] Furthermore, Demlow uses a different Ritz map (denoted by $\tilde{u}_{hk}^{\ell}$ there): instead of using the positive definite bilinear form $a^*(\cdot,\cdot)$ in (29), he uses the original positive semidefinite bilinear form $a(\cdot,\cdot)$ and works with functions with mean value zero.

4.3 | Maximum norm material derivative error estimates

As the material derivative does not commute with the *time dependent* Ritz map, that is, $\partial_h^\bullet P_1(t)u \neq P_1(t)\partial_h^\bullet u$, we have to bound the error $\partial_h^\bullet(u - P_1(t)u)$. Again, we first show our estimates in the weighted norms, and then use these results for the L^∞ and $W^{1,\infty}$ norm error estimates. Up to the authors knowledge such a maximum norm error estimate for the material derivative of the Ritz map have not been shown in the literature before.

For this subsection, we write $P_{h,1}u$ instead of $P_{h,1}(t)u$ and further $P_1 u$ instead of $P_1(t)u$.

We first state a substitute for our weighted pseudo Galerkin inequality (30).

Lemma 4.4 *There exists a constant $c > 0$ independent of h and t such that for all $u \in W^{2,\infty}(\mathcal{G}_T)$ and $\varphi_h \in S_h^l(t)$ it holds*

$$|a^*(\partial_h^\bullet(u - P_1u), \varphi_h)| \leq c(h^2\|\partial_h^\bullet(u - P_1u)\|_{H^{1,1}} + h|\log h|^{1/2}(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{1,\infty}(\Gamma(t))}))\|\varphi_h\|_{H^{1,-1}}. \tag{39}$$

Proof The main idea is given by Dziuk and Elliott in [5]. Using (6) and Lemma 3.6 (14) it is easy to verify

$$\|\partial_h^\bullet P_1 u\|_{H^{1,1}} \leq \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{H^{1,1}} + c\,|\log h|^{1/2}(\|\partial^\bullet u\|_{W^{1,\infty}(\Gamma(t))} + h\|u\|_{W^{2,\infty}(\Gamma(t))}). \tag{40}$$

Let $\phi_h \in S_h(t)$, such that $\varphi_h = \phi_h^l$. Taking time derivative of the definition of the Ritz map (29), using the discrete transport properties (8) Lemma 3.2, and the definition of the Ritz map, we obtain

$$\begin{aligned} a^*(\partial_h^\bullet u - \partial_h^\bullet P_1 u, \varphi_h) = {} & a_h^*(\partial_h^\bullet P_{h,1}u, \phi_h) - a^*(\partial_h^\bullet P_1 u, \varphi_h) \\ & + (g_h + b_h)(V_h; u^{-l}, \phi_h) - (g + b)(v_h; u, \varphi_h) \\ & - (g_h + b_h)(V_h; u^{-l} - P_{h,1}u, \phi_h). \end{aligned} \tag{41}$$

Then estimate using Lemma 3.7 (16), (17), Lemma 4.3 (36), and the above inequality to finish the proof (cf. [21, Thm. 7.2]). ∎

Lemma 4.5 *For $k \in \{0, 1\}$, there exists $c = c(k) > 0$ independent of t and h such that for $u \in W^{3,\infty}(\mathcal{G}_T)$, the following inequalities hold*

$$\|\partial_h^\bullet u - I_h\partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))} \leq ch^{2-k}(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}), \tag{42}$$

[1] In fact, some of them is later shown to be t-independent in the Appendix.

$$\|\partial_h^\bullet u - I_h\partial^\bullet u\|_{L^2,2}^2 + \|\partial_h^\bullet u - I_h\partial^\bullet u\|_{H^1,1}^2$$
$$\leq ch^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}). \tag{43}$$

Proof Using (6), we get

$$\|\partial_h^\bullet u - I_h\partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))}$$
$$\leq \|(v - v_h)\cdot \nabla_\Gamma u\|_{W^{k,\infty}(\Gamma(t))} + \|\partial^\bullet u - I_h\partial^\bullet u\|_{W^{k,\infty}(\Gamma(t))}.$$

Use Lemma 3.9 and (7) to show the first estimate.

For the second inequality use a Hölder estimate, and (42) with Lemma 3.6 (13) and (14). ∎

Lemma 4.6 *There exists $h_0 > 0$ sufficiently small and $\gamma_0 > 0$ sufficiently large and a constant $c = c(h_0, \gamma_0) > 0$ such that for $u \in W^{3,\infty}(\mathcal{G}_T)$, the following holds*

$$\|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{L^2,2}^2 + \|\partial_h^\bullet u - \partial_h^\bullet P_1 u\|_{H^1,1}^2$$
$$\leq ch^2 |\log h|^4 (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2). \tag{44}$$

Proof This proof has a similar structure as Lemma 4.3, and as it also uses similar arguments, we only give references if new lemmata are needed. For the ease of presentation, we set $e = u - P_1 u$ and split the error as follows

$$\partial_h^\bullet e = (\partial_h^\bullet u - I_h\partial^\bullet u) + (I_h\partial^\bullet u - \partial_h^\bullet P_1 u) =: \sigma + \theta_h.$$

Step 1: Our goal is to prove

$$\|\partial_h^\bullet e\|_{H^1,1}^2 \leq ch^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2) + \hat{c}\|\partial_h^\bullet e\|_{L^2,2}^2. \tag{45}$$

We start with

$$\|\partial_h^\bullet e\|_{H^1,1}^2 \leq a^*(\partial_h^\bullet e, \mu^{-1}\partial_h^\bullet e) + c\|\partial_h^\bullet e\|_{L^2,2}^2$$

and continue with

$$\begin{aligned} a^*(\partial_h^\bullet e, \mu^{-1}\partial_h^\bullet e) = {} & a^*(\partial_h^\bullet e, \mu^{-1}\sigma) \\ & + a^*(\partial_h^\bullet e, \mu^{-1}\theta_h - I(\mu^{-1}\theta_h)) \\ & + a^*(\partial_h^\bullet e, I(\mu^{-1}\theta_h)) = I_1 + I_2 + I_3. \end{aligned}$$

We estimate the three terms separately. For the first ϵ-Young's inequality and Lemma 4.5 (43) yields

$$|I_1| \leq \varepsilon\|\partial_h^\bullet e\|_{H^1,1}^2 + ch^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2).$$

For a sufficiently small $0 < h < h_0$, we obtain

$$|I_2| \leq \varepsilon\|\partial_h^\bullet e\|_{H^1,1}^2 + c(\|\partial_h^\bullet e\|_{L^2,2}^2 + h^2 |\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2)).$$

Using Lemma 4.4 (39) and a $0 < h < h_1$ sufficiently small, we arrive at

$$|I_3| \le \varepsilon \|\partial_h^\bullet e\|_{H^{1,1}}^2 + c(\|\partial_h^\bullet e\|_{L^{2,2}}^2 + h^2 \,|\log h| (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2)).$$

These estimates together, and absorbing $\|\partial_h^\bullet e\|_{H^{1,1}}$, imply (45).

Step 2: Using again an Aubin–Nitsche like argument we show that, for any $\delta > 0$ sufficiently small, we have

$$\|\partial_h^\bullet e\|_{L^{2,2}}^2 \le \delta \|\partial_h^\bullet e\|_{H^{1,1}}^2 + ch^2 \,|\log h|^4 (\|u\|_{W^{2,\infty}(\Gamma(t))}^2 + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}^2). \tag{46}$$

Let $w \in H^2(\Gamma(t))$ be the weak solution of

$$-\Delta_\Gamma w + w = \mu^{-2} \partial_h^\bullet e.$$

Then, we have

$$\|\partial_h^\bullet e\|_{L^{2,2}} = a^*(\partial_h^\bullet e, w - I_h w) + a^*(\partial_h^\bullet e, I_h w).$$

Again let $\varepsilon > 0$ be a small number. For $\gamma > \gamma_0$ sufficiently big, we get

$$|a^*(\partial_h^\bullet e, w - I_h w)| \le \varepsilon \|\partial_h^\bullet e\|_{H^{1,1}}^2 + \delta \|\partial_h^\bullet e\|_{L^{2,2}}^2$$

Using (41) and proceeding similar like in Dziuk and Elliott [5, Theorem 6.2], by adding and subtracting terms, we get

$$\begin{aligned}
a^*(\partial_h^\bullet e, I_h w) = &-(a^*(\partial_h^\bullet P_1 u, I_h w) - a_h^*(\partial_h^\bullet P_{h,1} u, \tilde{I}_h w) \\
&+ (g+b)(v_h; u, I_h w) - (g_h + b_h)(V_h; u^{-l}, \tilde{I}_h w) \\
&+ (g_h + b_h)(V_h; u^{-l} - P_{h,1} u, \tilde{I}_h w) - (g+b)(v_h; u - P_1 u, I_h w) \\
&+ (g+b)(v_h; u - P_1 u, I_h w) - (g+b)(v; u - P_1 u, I_h w) \\
&+ (g+b)(v; u - P_1 u, I_h w) - (g+b)(v; u - P_1 u, w) \\
&+ (g+b)(v; u - P_1 u, w)) \\
= &\, J_1 + J_2 + J_3 + J_4 + J_5 + J_6.
\end{aligned}$$

Use Lemma 3.7 (17), (40), Lemma 4.3 (36), and the inequality

$$h \|I_h w\|_{H^{1,1}} \le \varepsilon \|\partial_h^\bullet e\|_{L^{2,2}},$$

for $\gamma > \gamma_1$ sufficiently big, we obtain

$$|J_1| + \cdots + |J_4| \le \delta \|\partial_h^\bullet e\|_{H^{1,1}}^2 + \varepsilon \|\partial_h^\bullet e\|_{L^{2,2}}^2 + ch^2 (\|u\|_{W^{2,\infty}}^2 + \|\partial^\bullet u\|_{W^{1,\infty}}^2).$$

With the same arguments like for $a^*(\partial_h^\bullet e, w - I_h w)$, we estimate

$$|J_5| \le \varepsilon \|\partial_h^\bullet e\|_{H^{1,1}}^2 + \delta \|\partial_h^\bullet e\|_{L^{2,2}}^2,$$

for $\gamma > \gamma_2$ sufficiently big. For $\gamma > \gamma_3$ sufficiently big, we estimate the last term as follows

$$\begin{aligned} |J_6| &\leq c\|e\|_{L^{2,1}}\|w\|_{H^{2,-1}} \\ &\leq c\|e\|_{L^\infty}|\log\rho|^{1/2}\|w\|_{H^{2,-1}} \\ &\leq ch^2|\log h|^{3/2}\|u\|_{W^{2,\infty}}\|w\|_{H^{2,-1}} \\ &\leq \varepsilon\|\partial_h^\bullet e\|_{L^{2,2}}^2 + ch^2|\log h|^4\|u\|_{W^{2,\infty}}^2. \end{aligned}$$

By absorption, these estimates together imply (46).

The final estimate is shown by combining (45) and (46), and choosing $\delta > 0$, such that $\hat{c}\delta < 1$. Then, a further absorption finishes the proof. ∎

From the weighted version of the error estimate in the material derivatives, the L^∞ norm estimate follows easily.

Theorem 4.2 (Errors in the material derivative of the Ritz projection). *Let $z \in W^{3,\infty}(\mathcal{G}_T)$. For a sufficiently small $h < h_0$ and a sufficiently big $\gamma > \gamma_0$ there exists $c = c(h_0, \gamma_0) > 0$ independent of t and h such that*

$$\|\partial_h^\bullet(z - (P_{h,1}(t)z)^l)\|_{L^\infty(\Gamma(t))} \leq ch^2|\log h|^3(\|z\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet z\|_{W^{2,\infty}(\Gamma(t))}),$$
$$\|\partial_h^\bullet(z - (P_{h,1}(t)z)^l)\|_{W^{1,\infty}(\Gamma(t))} \leq ch\,|\log h|^{5/2}(\|z\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet z\|_{W^{2,\infty}(\Gamma(t))}).$$

Proof The above results are shown by exactly following the proof of Theorem 4.1, with Lemma 4.6 (44) being the main tool. ∎

5 | MAXIMUM NORM PARABOLIC STABILITY

The purpose of this section is to derive an evolving surface finite element weak discrete maximum principle. The proof is modeled on the weak discrete maximum principle from Schatz et al. [15]. For this, we are going to need a well-known matrix formulation of (4), which is due to Dziuk and Elliott [4].

The matrix–vector formulation was first used for theoretical reasons in Dziuk et al. [6], in order to show stability and convergence of time discretizations of (4).

Using the matrix–vector formulation, we derive a discrete adjoint problem of (4), which does not arise in Schatz et al. [15]. Here, it arises in a natural way, as the ESFEM evolution operator is not self adjoint. We then deduce a corresponding a priori estimate, and finally prove our weak discrete maximum principle.

5.1 | A discrete adjoint problem

A matrix ODE version of (4) can be derived by setting

$$U_h(\,\cdot\,,t) = \sum_{j=1}^N \alpha_j(t)\chi_j(\cdot,t),$$

testing with the basis function $\phi_h = \chi_j$, where $S_h(t) = \mathrm{lin}\{\chi_j | j = 1,\dots,N\}$, and using the transport property (3).

Proposition 5.1 (ODE system). *The spatially semidiscrete problem (4) is equivalent to the following linear ODE system for the vector $\alpha(t) = (\alpha_j(t))_{j=1}^N \in \mathbb{R}^N$, collecting the nodal values of $U_h(\cdot, t)$:*

$$\begin{aligned} \frac{d}{dt}(M(t)\alpha(t)) + A(t)\alpha(t) = 0,& \\ \alpha(0) = \alpha_0,& \end{aligned} \tag{47}$$

where the evolving mass matrix M(t) and stiffness matrix A(t) are defined as

$$M(t)|_{kj} = \int_{\Gamma_h(t)} \chi_j \chi_k, \qquad A(t)|_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h} \chi_j \cdot \nabla_{\Gamma_h} \chi_k, \quad (j, k = 1, 2, \ldots, N).$$

Definition 5.1 Let $0 \le s \le t \le T$. For given initial value $w_h \in S_h(s)$ at time s, there exists unique[2] solution u_h. This defines a linear evolution operator

$$E_h(t,s) : S_h(s) \to S_h(t), \quad w_h \mapsto u_h(t).$$

We define the adjoint of $E_h(t,s)$

$$E_h(t,s)^* : S_h(t) \to S_h(s)$$

via the equation

$$m_h(t; E_h(t,s)\varphi_h(s), w_h(t)) = m_h(s; \varphi_h(s), E_h(t,s)^* w_h(t)), \tag{48}$$

where $\varphi_h(s) \in S_h(s)$ and $w_h(t) \in S_h(t)$ are some arbitrary finite element functions.

Lemma 5.1 (Adjoint problem). *Let $s \in [0,t]$, where $t \in [0,T]$ and $w_h(t) \in S_h(t)$. Then, $u_h^\star(s) = E(t,s)^* w_h(t)$ is the unique solution, for every $\phi_h \in S_h(t)$,*

$$\begin{cases} m_h(s; \partial_h^{\bullet,s} u_h^\star, \phi_h) - a_h(s; u_h^\star, \phi_h) = 0, & \text{on } \Gamma(s) \\ u_h^\star(t) = w_h(t), & \text{on } \Gamma(t). \end{cases} \tag{49}$$

where $\partial_h^{\bullet,s}$ is the discrete material derivative with respect to the backwards time s (defined as the one in Section 3.3).

Remark 5.1 The problem (49) has the structure of a backward heat equation, where s is going backward in time. Hence, we considered (49) as a PDE of parabolic type. We recall, that using Lemma 3.2 we may write Equation (4) equivalently as

$$\begin{cases} m_h(t; \partial_h^\bullet u_h + (\nabla_{\Gamma_h} \cdot V_h)u_h, \varphi_h) + a_h(t; u_h, \varphi_h) = 0, & \text{on } \Gamma(t), \\ u_h(0) = u_{0h}, & \text{on } \Gamma(0) \end{cases} \tag{50}$$

[2] Compare to Dziuk and Elliott [4].

The problems (50) and (49) differ in the following way: If the initial data for (49) is constant, then it remains so for all times. In general, this does not hold for solutions of (50). On the other hand, (50) preserves the mean value of its initial data, which is in general not true for a solution of (49).

Proof of Lemma 5.1 First, we investigate the finite element matrix representation of $\boldsymbol{E}_h(t,s)$ with respect to the standard finite element basis, which we denote by $\boldsymbol{E}_h(t,s)$. From (47), we have

$$\frac{d}{dt}(M(t)\boldsymbol{E}_h(t,0)\boldsymbol{u}_h(0)) + A(t)\boldsymbol{E}_h(t,0)\boldsymbol{u}_h(0) = 0.$$

Let $\Lambda(t,s)$ the resolvent matrix of the ODE

$$\frac{d\xi}{dt} + A(t)M(t)^{-1}\xi = 0.$$

Then, obviously it holds

$$\boldsymbol{E}_h(t,s) = M(t)^{-1}\Lambda(t,s)M(s).$$

Denote by $\boldsymbol{E}_h(t,s)^*$ the matrix representation of $\boldsymbol{E}_h(t,s)^*$. From Equation (48), it follows

$$\boldsymbol{E}_h(t,s)^* = M(s)^{-1}\boldsymbol{E}_h(t,s)^T M(t) = \Lambda(t,s)^T.$$

Now, we calculate $\frac{d\Lambda(t,s)}{ds}$. Note that $\Lambda(t,s) = \Lambda(s,t)^{-1}$ and it holds

$$\frac{d\Lambda(s,t)^{-1}}{ds} = -\Lambda(s,t)^{-1}\frac{d\Lambda(s,t)}{ds}\Lambda(s,t)^{-1}.$$

From that it easily follows

$$\frac{d\Lambda(t,s)}{ds} = \Lambda(t,s)A(s)M(s)^{-1},$$

which now implies

$$\frac{d\boldsymbol{E}_h(t,s)^*}{s} = M(s)^{-1}A(s)\boldsymbol{E}_h(t,s)^*.$$ ∎

5.2 | A discrete delta and Green's function

Let $\delta_h = \delta_h^{x_h} = \delta_h^{t,x_h} \in S_h(t)$ be a finite element discrete delta function defined as

$$m_h(t;\delta_h^{t,x_h},\varphi_h) = \varphi_h(x_h,t) \qquad (\varphi_h \in S_h(t)). \tag{51}$$

If $\delta^{x_h} : \Gamma_h(t) \to \mathbb{R}$ is a finite element function having support in the triangle E_h containing x_h, then as $\dim\Gamma_h(t) = 2$, one easily calculates $\|\delta^{x_h}\sigma^{x_h}\|_{L^2(\Gamma_h(t))} \le c$ for some constant independent of h and t. For the discrete delta function δ_h, a similar result holds.

Lemma 5.2 *There exists c > 0 independent of t and h:*

$$\|\sigma^{x_h}\delta_h^{x_h}\|_{L^2(\Gamma_h(t))} \le c \qquad (x_h \in \Gamma_h(t)).$$

The proof is a straightforward extension of the corresponding one in Schatz et al. [15] and uses the exponential decay property of the L^2-projection, cf. Theorem 3.1 (25).

Next, we define a finite element discrete Green's function as follows. Let $s \in [0, T]$. For given $u_h \in S_h(s)$, there exists a unique $\psi_h \in S_h(s)$ such that

$$a_h^*(s; \psi_h, \varphi_h) = m_h(s; u_h, \varphi_h) \quad \forall \varphi_h \in S_h(s).$$

This defines an operator

$$T_h^{*,s} : S_h(s) \to S_h(s), \quad T_h^{*,s} u_h = \psi_h.$$

We call $G_h^{s,x} = T_h^{*,s}\delta_h^{s,x}$ a discrete Green's function.

A short calculation shows that for all $0 \ne \varphi_h \in S_h(s)$ it holds

$$m_h(s; T_h^{*,s}\varphi_h, \varphi_h) > 0,$$

which implies that $G_h^{s,x}(x) > 0$. Actually, we can bound the singularity x with $c \mid \log h|$.

Lemma 5.3 *For the discrete Green's function $G_h^{s,x}$, we have the estimate*

$$G_h^{s,x}(x) \le c \mid \log h|.$$

Proof Using Lemma 3.12 with (5), we estimate as

$$\|G_h^{s,x}\|_{L^\infty(\Gamma_h(s))} \le c \mid \log h|^{1/2} \|G_h^{s,x}\|_{H^1(\Gamma_h(s))} = c \mid \log h|^{1/2} \sqrt{G_h^{s,x}(x)}. \qquad \blacksquare$$

The next lemma needs a different treatment then the one presented in Schatz et al. [15]. The reason for that is that the mass and stiffness matrix depend on time and further the stiffness matrix is singular.

Lemma 5.4 *Let be $u_h^\star$ a solution of (49). Then, we have the estimate*

$$\int_0^t ||u_h^\star\|^2_{L^2(\Gamma_h(s))} ds \le c \cdot m_h(t; T_h^{*,t} u_h^\star, u_h^\star).$$

Proof Note that Lemma 3.2 (9) reads with the matrix notation as follows: If $\mathbf{Z}_h$ and ϕ_h are the coefficient vectors of some finite element function, then we have the estimate

$$\begin{aligned} \mathbf{Z}_h^T \frac{dM(s)}{ds} \phi_h &\le c\sqrt{\mathbf{Z}_h^T M(s) \mathbf{Z}_h} \sqrt{\phi_h^T M(s) \phi_h}, \\ \mathbf{Z}_h^T \frac{dA(s)}{ds} \phi_h &\le c\sqrt{\mathbf{Z}_h^T A(s) \mathbf{Z}_h} \sqrt{\phi_h^T A(s) \phi_h}. \end{aligned} \tag{52}$$

In the following with drop the s dependency. Let $\boldsymbol{u}$ be the time-dependent coefficient vector of $u_h^\star$. Then, we have

$$0 = -M\frac{d\boldsymbol{u}}{ds} + A\boldsymbol{u} = -M\frac{d\boldsymbol{u}}{ds} + (A+M)\boldsymbol{u} - M\boldsymbol{u}.$$

Equivalently, we write this equation as

$$\begin{aligned}-&\frac{1}{2}\frac{d}{ds}[\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}]\\ &= -\boldsymbol{u}^T M\boldsymbol{u} + \boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u} - \frac{1}{2}\boldsymbol{u}^T\frac{d}{ds}[M(A+M)^{-1}M]\boldsymbol{u}.\end{aligned}$$

The last term expanded reads

$$\begin{aligned}&\frac{1}{2}\boldsymbol{u}^T\frac{d}{ds}[M(A+M)^{-1}M]\boldsymbol{u}\\ &= \boldsymbol{u}^T\frac{dM}{ds}(A+M)^{-1}M\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^T M\frac{d(A+M)^{-1}}{ds}M\boldsymbol{u} = I_1 + I_2.\end{aligned}$$

Using (52) and a Young's inequality, we estimate as

$$\begin{aligned}|I_1| &\le c\cdot\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u} + \frac{1}{2}\boldsymbol{u}^T M\boldsymbol{u}.\\ |I_2| &= \frac{1}{2}\left|\boldsymbol{u}^T M(A+M)^{-1}\frac{d(A+M)}{ds}(A+M)^{-1}M\boldsymbol{u}\right|\\ &\le c\cdot\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}.\end{aligned}$$

Putting everything together, we obtain

$$-\frac{d}{ds}[\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}] \le -\boldsymbol{u}^T M\boldsymbol{u} + c\cdot\boldsymbol{u}^T M(A+M)^{-1}M\boldsymbol{u}.$$

The claim then follows from Lemma C.1. ▪

5.3 | A weak discrete maximum principle

Proposition 5.2 *Let $U_h(x,t) \in S_h(t)$ the ESFEM solution of our linear heat problem. Then, there exists a constant $c = c(T,\nu) > 0$, which depends exponentially on T and ν such that*

$$\|U_h(t)\|_{L^\infty(\Gamma_h(t))} \le c\,|\log h|\,\|U_h(0)\|_{L^\infty(\Gamma_h(0))}.$$

Proof There exists $x_h \in \Gamma_h(t)$ such that

$$\begin{aligned}\|U_h(t)\|_{L^\infty} &= |U_h(x_h,t)| = m_h(t;U_h(t),\delta_h^{t,x_h}) = m_h(t;E(t,0)U_h^0,\delta_h^{t,x_h})\\ &= m_h(0;U_h^0,E(t,0)^*\delta_h^{t,x_h}) \le \|U_h^0\|_{L^\infty}\|E(t,0)^*\delta_h^{t,x_h}\|_{L^1}.\end{aligned}$$

The claim follows from Lemma 5.5. ▪

Lemma 5.5 *For $G_h^x(t,s) = E_h(t,s)^* \delta_h^{t,x}$, where $\delta_h^{t,x}$ is defined via (51) and $E_h(t,s)^*$ is defined via (48), it holds*

$$\|G^x(t,0)\|_{L^1(\Gamma_h(0))} \leq c\,|\log h|,$$

where the constant $c = c(T,v)$ depending exponentially on T and v such and is independent of x, h, t, and s.

Proof The proof presented here is a modification of the proof from Schatz et al. [15, Lemma 2.1]. We estimate

$$\|G_h^x(t,0)\|_{L^1(\Gamma_h(0))} \leq \|1/\sigma^x\|_{L^2(\Gamma_h(0))}\|\sigma^x G_h^x(t,0)\|_{L^2(\Gamma_h(0))}.$$

With Subsection B.1, it follows

$$\|1/\sigma^x\|^2_{L^2(\Gamma_h(0))} \leq c\,|\log h|.$$

It remains to show

$$\|\sigma^x G_h^x(t,0)\|^2_{L^2(\Gamma_h(0))} \leq c\,|\log h|.$$

In the following, we abbreviate $\sigma = \sigma^x$ and $G_h = G_h^x(t,s)$. With Equation 49 and the discrete transport property, we proceed as follows

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{ds}\|\sigma G_h\|^2_{L^2(\Gamma_h(s))} + \|\sigma\nabla_{\Gamma_h}G_h\|^2_{L^2(\Gamma_h(s))}\\
&= -m_h(s;\partial_h^{\bullet,s}G_h,\sigma^2G_h) + a_h(s;G_h,\sigma^2G_h)\\
&\quad - 2m_h(s;\sigma\nabla_{\Gamma_h}G_h, G_h\nabla_{\Gamma_h}\sigma)\\
&\quad - m_h(s;\partial_h^{\bullet,s}\sigma,\sigma G_h^2) - \frac{1}{2}m_h(s;\sigma^2G_h^2,\nabla_{\Gamma_h}\cdot V_h)\\
&= -m_h(s;\partial_h^{\bullet,s}G_h,\sigma^2G_h-\psi_h) + a_h(s;G_h,\sigma^2G_h-\psi_h)\\
&\quad - 2m_h(s;\sigma\nabla_{\Gamma_h}G_h, G_h\nabla_{\Gamma_h}\sigma)\\
&\quad - m_h(s;G_h\partial_h^{\bullet,s}\sigma,\sigma G_h) - \frac{1}{2}m_h(s;\sigma^2G_h^2,\nabla_{\Gamma_h}\cdot V_h)\\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

For the choice $\psi_h = P_0(\sigma^2 G_h)$, we have $I_1 = 0$. Using Cauchy–Schwarz inequality, Lemma 3.14 and an inverse estimate 3.4, we get

$$|I_2| \leq c(\|G_h\|^2_{L^2(\Gamma_h(s))} + \|G_h\|_{L^2(\Gamma_h(s))}\cdot\|\sigma\nabla_{\Gamma_h}G_h\|_{L^2(\Gamma_h(s))}).$$

Using Lemma 3.13 (28), we obtain

$$|I_3| \leq c\|G_h\|_{L^2(\Gamma_h(s))}\|\sigma\nabla_{\Gamma_h}G_h\|_{L^2(\Gamma_h(s))}.$$

Using Lemma 3.13 (27), we have

$$\begin{aligned}|I_4| &\leq c\|G_h\|_{L^2(\Gamma_h(s))}\|\sigma G_h\|_{L^2(\Gamma_h(s))},\\ |I_5| &\leq c\|\sigma G_h\|^2_{L^2(\Gamma_h(s))}.\end{aligned}$$

After a Young's inequality, we have

$$-\frac{d}{ds}\|\sigma G_h\|^2_{L^2(\Gamma_h(s))} + \|\sigma \nabla_{\Gamma_h} G_h\|^2_{L^2(\Gamma_h(s))} \leq c\|G_h\|^2_{L^2(\Gamma_h(s))} + c\|\sigma G_h\|^2_{L^2(\Gamma_h(s))}.$$

Lemma C.1 yields

$$\|\sigma G_h(t,0)\|^2_{L^2(\Gamma_h(0))} \leq c\left(\int_0^t ||G_h(t,s)\|^2_{L^2(\Gamma_h(s))}ds + \|\sigma^x \delta_h^x\|_{L^2(\Gamma_h(0))}\right).$$

For the first term, we get from Lemma 5.4 and Lemma 5.3 the bound

$$\int_0^t ||G_h(t,s)\|^2_{L^2(\Gamma_h(s))}ds \leq c\,|\log h|.$$

The last term is bounded according to Lemma 5.2. ∎

Remark 5.2 By using the techniques of [19] instead of [15], the logarithmic factor $|\log(h)|$ is expected to disappear, however, this would lead to a much more technical and quite lengthy proof, as already noted in the introduction.

6 | CONVERGENCE OF THE SEMIDISCRETIZATION

Theorem 6.1 *Let $\Gamma(t)$ be an evolving surface of dimension two, let the function u : $\Gamma(t) \to \mathbb{R}$ be the solution of (1) and let $u_h = U_h^l \in H^1(\Gamma(t))$ be the solution of (4). If it holds*

$$\|P_{h,1}(t)u - U_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2,$$

then there exists $h_0 > 0$ sufficiently small and $c = c(h_0) > 0$ independent of t, such that for all $0 < h < h_0$, we have the estimate

$$\begin{aligned}&\|u - u_h\|_{L^\infty(\Gamma_{(t)})} + h\|u - u_h\|_{W^{1,\infty}(\Gamma(t))}\\ &\qquad\leq ch^2\,|\log h|^4(1+t)(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}).\end{aligned}$$

Proof We use here the notations $P_{h,1}u = P_{h,1}(t)u$, $P_1 u = (P_{h,1}u)^l$ and $u_h = U_h^l$. We then split the error as follows

$$u - u_h = (u - P_1 u) + (P_{h,1}u - U_h)^l = \sigma + \theta_h^l.$$

Because of Theorem 4.1, it remains to bound θ_h. It suffices to prove an L^∞ -estimate for θ_h, as the $W^{1,\infty}$ -bounds follows by an inverse inequality.

Obviously, there exists $R_h \in S_h(t)$ such that for all $\phi_h \in S_h(t)$ it holds

$$\frac{d}{dt}\int_{\Gamma_h(t)} \theta_h\phi_h + \int_{\Gamma_h(t)} \nabla_{\Gamma_h}\theta_h \cdot \nabla_{\Gamma_h}\phi_h - \int_{\Gamma_h(t)} \theta_h\partial_h^\bullet\phi_h = \int_{\Gamma_h(t)} R_h\phi_h.$$

By the variation of constant formula, we deduce

$$\theta_h(t) = E_h(t,0)\theta_h(0) + \int_0^t E_h(t,s)R_h(s)ds.$$

With Proposition 5.2, we get

$$\|\theta_h\|_{L^\infty(\Gamma_h(t))} \leq c\ |\log h|(\|\theta_h(0)\|_{L^2(\Gamma_h(t))} +\ t\max_{s\in[0,t]}\|R_h(s)\|_{L^\infty(\Gamma_h(t))}).$$

Observe that if we denote by $\varphi_h := \phi_h^l$, then a quick calculation reveals

$$\begin{aligned} m_h(R_h,\phi_h) &= m_h(\partial_h^\bullet P_{h,1}u,\phi_h) + g_h(V_h;P_{h,1}u,\phi_h) + a_h(P_{h,1}u,\phi_h) \\ &\quad - (m(\partial_h^\bullet u,\varphi_h) + g(v_h;u,\varphi_h) + a(u,\varphi_h)) \end{aligned} \tag{53}$$

Lemma 6.1 finishes the proof. ▪

Lemma 6.1 *Assume that $R_h \in S_h(t)$ satisfies for all $\phi_h \in S_h(t)$ with $\varphi_h := \phi_h^l$ Equation (53). Then, it holds*

$$\|R_h\|_{L^\infty(\Gamma_h(t))} \leq ch^2\ |\log h|^3(\|u\|_{W^{2,\infty}(\Gamma(t))} + \|\partial^\bullet u\|_{W^{2,\infty}(\Gamma(t))}).$$

Proof Using Definition 3.3 (24), (53) and as L^∞ is the dual of L^1, we deduce

$$\|R_h\|_{L^\infty(\Gamma_h(t))} = \sup_{\substack{f_h\in L^1(\Gamma_h(t))\\ \|f_h\|_{L^1(\Gamma_h(t))}=1}} m_h(R_h,f_h) = \sup_{\substack{f_h\in L^1(\Gamma_h(t))\\ \|f_h\|_{L^1(\Gamma_h(t))}=1}} m_h(R_h,P_0f_h).$$

Now consider

$$\begin{aligned} m_h(R_h,P_0f_h) &= m_h(\partial_h^\bullet P_{h,1}u,P_0f_h) - m(\partial_h^\bullet u,P_0f_h^l) \\ &\quad + g_h(V_h;P_{h,1}u,P_0f_h) - g(v_h;u,P_0f_h^l) \\ &\quad + a_h(P_{h,1}u,P_0f_h) - a(u,P_0f_h^l) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Using Lemma 3.3 and Theorem 3.1, it is easy to see

$$\begin{aligned} |I_1| &\leq c(\|\partial_h^\bullet u - \partial_h^\bullet(P_{h,1}u)^l\|_{L^\infty(\Gamma(t))} \\ &\qquad +\ h^2(\|\partial^\bullet u\|_{L^\infty(\Gamma(t))} + h^2\|u\|_{W^{1,\infty}(\Gamma(t))}))\|f_h\|_{L^1(\Gamma_h(t))} \\ |I_2| &\leq c(\|u - P_{h,1}u^l\|_{L^\infty(\Gamma(t))} + h^2\|u\|_{L^\infty(\Gamma(t))})\|f_h\|_{L^1(\Gamma_h(t))} \\ |I_3| &\leq c(h^2\|u\|_{L^\infty(\Gamma(t))} + \|u - (P_{h,1}u)^l\|_{L^\infty(\Gamma(t))})\|f_h\|_{L^1(\Gamma_h(t))} \end{aligned}$$

Theorems 4.1 and 4.2 imply the claim. ▪

7 | NUMERICAL EXPERIMENTS

7.1 | Convergence tests

We present a numerical experiment for an evolving surface parabolic problem discretized in space by the ESFEM. As a time discretization method, we choose backward difference formula 4 with a sufficiently small time step (in all the experiments we choose $\tau = 0.001$).

As initial surface Γ_0 we choose the unit sphere $S^2 \subset \mathbb{R}^3$. The dynamical system is given by $\Phi(x, y, z, t) = (\sqrt{1 + 0.25\sin(2\pi t)}x, y, z)$, which implies the velocity $v(x, y, z, t) = (\pi\cos(2\pi t)/(4 + \sin(2\pi t))x, 0, 0)$, over the time interval $[0, 1]$. As the exact solution, we choose $u(x, y, z, t) = xye^{-6t}$. The complicated right-hand side was calculated using the computer algebra system Sage [28].

We give the errors in the following norm and seminorm

$$L^\infty(L^\infty): \qquad \max_{1\le n\le N} \|u_h^n - u(\cdot, t_n)\|_{L^\infty(\Gamma(t_n))},$$

$$L^2(W^{1,\infty}): \qquad \left(\tau \sum_{n=1}^N |\nabla_{\Gamma(t_n)}(u_h^n - u(\cdot, t_n))|^2_{L^\infty(\Gamma(t_n))}\right)^{1/2}.$$

The experimental order of convergence (EOC) is given as

$$\mathrm{EOC}_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \qquad (k = 2, 3, \ldots, n),$$

where e_k denotes the error of the kth level.

Table 1 reports on the EOCs for the ESFEM—backward Euler solution of the evolving surface PDE example detailed above. According to Theorem 6.1, the $L^\infty(L^\infty)$ error is expected to have convergence order 2, while we expect order 1 in the $L^\infty(W^{1,\infty})$ norm. The numerical experiments match with the theoretical convergence rates, while the obtained logarithmic factor is not detected numerically.

7.2 | Discrete maximum principle on evolving surfaces

We conducted a further experiment regarding discrete maximum principle on evolving surfaces. Although this is not our main interest in the present paper, but it is suitable here to explore such numerical examples. As it is well known, the homogeneous heat equation started from an initial condition $u_0 \equiv 1$ on an evolving surface may lead to nonconstant solutions. The maximum of the solution depends on the deformation of the surface, that is, on the velocity v.

As an initial surface, we have chosen the sphere with radius 1, $\Gamma(0) = \{|x| = 1\}$, and numerically solved the problem (1) with $f = 0$ and $u_0 \equiv 1$, with three different velocities. In Figure 1, we have plotted the time evolutions at times $t = 0, 0.25, 0.5, 0.75, 1$ (from top to bottom) for all three surface

TABLE 1 Errors and experimental order of convergences (EOCs) in the $L^\infty(L^\infty)$ and $L^2(W^{1,\infty})$ norms

Level	dof	$L^\infty(L^\infty)$	EOCs	$L^2(W^{1,\infty})$	EOCs
1	126	0.00918195	–	0.01921707	–
2	516	0.00308305	1.57	0.01481673	0.37
3	2070	0.00100752	1.61	0.00851267	0.80
4	8208	0.00025326	1.99	0.00399371	1.09

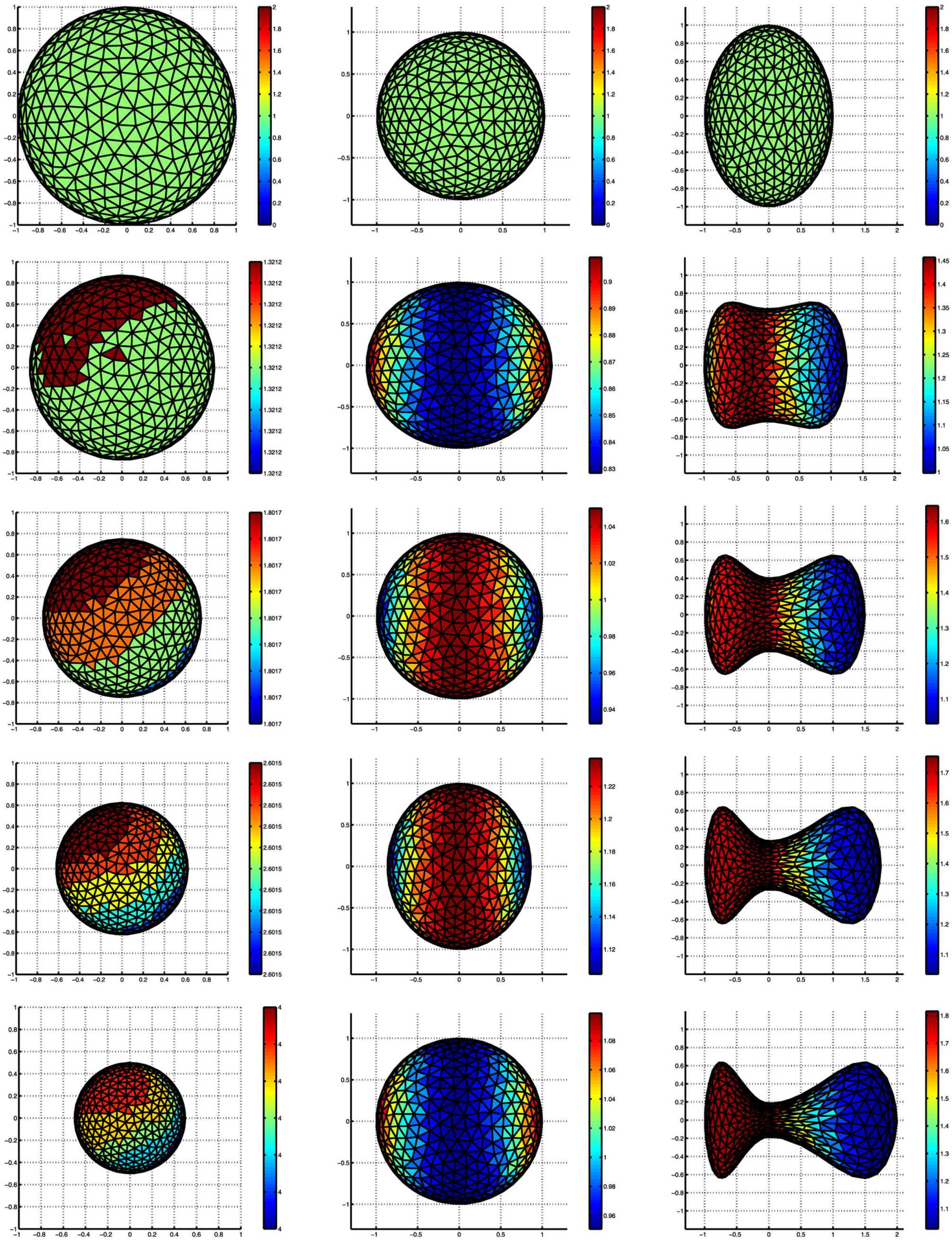

FIGURE 1 The numerical solutions (and their maximum values) for the three surface evolutions shrinking sphere, bouncing ellipsoid, and "baseball bat" (from left to right), at $t = 0, 0.25, 0.5, 0.75, 1$ (from top to bottom) [Color figure can be viewed at wileyonlinelibrary.com]

evolutions (from left to right): for a shrinking sphere $\Gamma(t) = \{|x| = 1 - t/2\}$ (left), for the bouncing ellipsoid example from Section 7.1 (middle), and the "baseball bat" like surface from Test Problem 2 of [29], or see also [21, Example 9.2] (right). The surface of the latter case is given by:

$$\begin{aligned}\Gamma(t) &= \left\{(x_1 + \max\{0, x_1\}\, t, \frac{g(x,t)x_2}{\sqrt{x_2^2+x_3^2}}, \frac{g(x,t)x_3}{\sqrt{x_2^2+x_3^2}}) \mid x \in \Gamma(0) = \mathbb{S}^2\right\},\\ g(x,t) &= e^{-2t}\sqrt{x_2^2+x_3^2} + (1-e^{-2t})((1-x_1^2)(x_1^2+0.05) + x_1^2\sqrt{1-x_1^2}).\end{aligned}$$

ACKNOWLEDGMENTS

The authors would like to thank Prof. Christian Lubich for the invaluable discussions on the topic, and for his encouragement and help during the preparation of this paper. We would also like to thank Buyang Li from The Polytechnic University of Hong Kong for our discussions on the topic. The valuable comments and remarks of an anonymous referee is acknowledged as well. The research stay of Balázs Kovács at the University of Tübingen has been funded by the Deutscher Akademischer Austausch Dienst (DAAD).

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How to cite this article: Kovács B, Power Guerra CA. Maximum norm stability and error estimates for the evolving surface finite element method. *Numer. Methods Partial Differential Eq*. 2017;00:1–37. https://doi.org/10.1002/num.22212

APPENDIX A: GREEN'S FUNCTION FOR EVOLVING SURFACES

Aubin [25, Section 4.2] proves existence of a Green's function on a closed manifold M, that is a function which satisfies in $M \times M$

$$\Delta_{Q\text{distr.}}\mathbf{G}(P,Q) = \delta_P(Q),$$

where Δ is the Laplace–Beltrami operator on M. The Green's function is unique up to a constant. For Lemma 3.12, we need that the first derivative of a Green's function can be bounded independent of t.

Theorem A.1 (Green's function). *Let $\Gamma(t)$ with $t \in [0, T]$ be an evolving surface. There exists a Green's function $\mathbf{G}(t; x, y)$ for $\Gamma(t)$. The value of $\mathbf{G}(x, y)$ depends only on the value of $dist_{\Gamma(t)}(x, y)$. $\mathbf{G}(x, y)$ satisfies the inequality*

$$|\nabla_\Gamma^x \mathbf{G}(t; x, y)| \leq c \frac{1}{dist_{\Gamma(t)}(x, y)}.$$

for some $c > 0$ independent of t.

Furthermore for all functions $\varphi \in C^2(\mathcal{G}_T)$, it holds

$$\varphi(x, t) = \frac{1}{V} \int_{\Gamma(t)} \varphi(y, t) dy - \int_{\Gamma(t)} \mathbf{G}(t; x, y) \Delta_\Gamma \varphi(y, t) dy. \tag{54}$$

Proof As noted in Aubin [25, 4.10], the distance $r = \mathrm{dist}_{\Gamma(t)}(x, y)$ is only a Lipschitzian function on $\Gamma(t)$. To use his construction, we therefore need to revise that the injectivity radius at any point $P \in \Gamma(t)$ can be bounded by below by a number independent of P and t. This follows if the Riemannian exponential map is continuous in t and from Lemma C.2. To prove that the Riemannian exponential map is continuous one carefully revises the construction of exponential map as it is given in Chavel [30, Chapter 1]. Formula (54) follows from Aubin [25, Theorem 4.13] and that the constant is independent of t is a straightforward calculation. ∎

APPENDIX B: CALCULATIONS WITH SOME WEIGHT FUNCTIONS ON EVOLVING SURFACES

B.1 | Integration with geodesic polar coordinates on evolving surfaces

Assume we have a sufficiently smooth function $f : \Gamma(t) \times \Gamma(t) \to \mathbb{R}$, where the value $f(x, y)$ depends only on the distance $r = \mathrm{dist}_{\Gamma(t)}(x, y)$. In the flat case, that is, $\Omega \subset \mathbb{R}^m$, we can use polar coordinates to show the integral

$$\int_\Omega f(x, y) dy \leq c \int_0^R r^{m-1} f(r) dr.$$

For a surface $\Gamma(t)$, this is more difficult. The purpose of this section is to derive a similar bound for

$$\int_{\Gamma(t)} f(x, y) dy.$$

Applying the well known coarea formulae to the distance function r, cf. Chavel [30, Theorem 3.13] and Morgan [31, Theorem 3.13], we obtain

$$\begin{aligned} \int_{\Gamma(t)} f(x, y) dy &= \int_0^\infty \int_{\{\mathrm{dist}_{\Gamma(t)}(x,y)=r\}} f(r) d\omega dr \\ &= \int_0^\infty \frac{\mathcal{H}^m(\{\mathrm{dist}_{\Gamma(t)}(x, y) = r\})}{r^m} f(r) r^m dr, \end{aligned}$$

where $\mathcal{H}^m$ denotes the m-dimensional Hausdorff measure. Obviously, there exists a positive number $R > 0$ independent of t and $x, y \in \Gamma(t)$ such that for all $r \geq R$ it holds

$$\mathcal{H}^m(\mathrm{dist}_{\Gamma(t)}(x, y) = r) = 0. \tag{55}$$

Lemma B.1. *There exists $c > 0$ independent of t and $p, q \in \Gamma(t)$ such that*

$$\mathcal{H}^m(\{dist_{\Gamma(t)}(p, q) = r\}) \leq c.$$

Proof For a fix point $p \in \Gamma(t)$ it is possible to use the Riemannian exponential map to flat out $\Gamma(t)$, cf. Figure 1 for an illustration on the torus. We make this argument precise. As $\Gamma(t)$ is compact we have that $\Gamma(t)$ is geodesic complete, that is, the Riemannian exponential map is defined on the whole tangent bundle.

For $r \in [0, \infty)$, let

$$S_p(r) := \left\{ v \in \mathrm{T}_p\Gamma(t) | g_p(v, v) = r^2 \right\}$$

be the sphere of radius r, where g_p is the Riemannian metric. For $v \in S_p(1)$, we have that

$$f_v : [0, \infty) \to \Gamma, \quad \lambda \mapsto \exp_p(\lambda v).$$

is a geodesic. It is well known that a geodesic is just locally length minimizing. Hence there exists a unique $\lambda_*(v) > 0$, such that $f_v|[0, \lambda_*(v)]$ is a length minimizing geodesic and for every $\varepsilon > 0$, we have that $f_v|[0, \lambda_*(v) + \varepsilon]$ is not anymore length minimizing. We define

$$W_p(t) := \left\{ w \in \mathrm{T}_p\Gamma(t) | w = \lambda \cdot v \text{ with } v \in S_p \text{ and } \lambda \in [0, \lambda_*(v)] \right\}.$$

The Hopf-Rinow theorem states that two points on a geodesic complete manifold can be joint by a length minimizing geodesic. Hence, it holds

$$\exp_p(W_p \cap S_p(r)) = \left\{ \mathrm{dist}_{\Gamma(t)}(p, q) = r \right\}.$$

Applying a general area-coarea formula, cf. [31, Theorem 3.13], shows the bound

$$\left\{ \mathrm{dist}_{\Gamma(t)}(p, q) = r \right\} \leq c r^m,$$

where c is independent of p. r can be bounded by R, cf. (55). ∎

Lemma B.2 *There exists $c > 0$ depending on $t \in [0, T]$ and $x \in \Gamma(t)$ such that*

$$\sup_{r>0} \frac{\mathcal{H}^m(\{dist_{\Gamma(t)}(x, y) = r\})}{r^m} \leq c.$$

Proof A corollary of Gray [32, Theorem 3.1.] is the following formula:

$$\mathcal{H}^m(\{\mathrm{dist}_{\Gamma(t)}(x, y) = r\}) = \omega_m r^m (1 - r^2 \int_0^1 f(\sigma r) d\sigma),$$

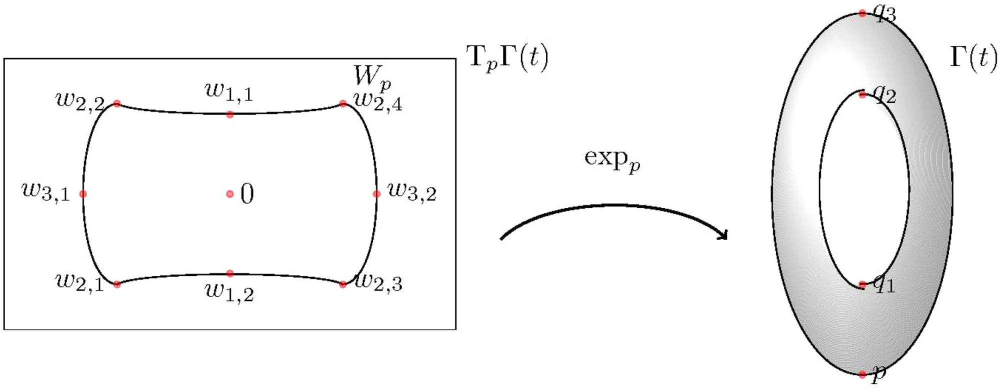

FIGURE B1 Illustration of a possible W_p for the Torus as a subset of $\mathbb{R}^3$ with induced metric. Note that the opposite boundary of W_p are identified. It holds $\exp_p(w_{i,*}) = q_i$ and $\exp_p(0) = p$ [Color figure can be viewed at wileyonlinelibrary.com]

where ω_m is the volume of the m-dimensional sphere of $\mathbb{R}^{m+1}$, where r is assumed to be smaller than the injectivity radius, that is, the biggest possible $\lambda_*(v)$ from the proof of Lemma B.1 and where f is some function depending smoothly on r and x. We easily see

$$\lim_{r\to 0} \frac{\mathcal{H}^m(\{\mathrm{dist}_{\Gamma(t)}(x,y) = r\})}{r^m} = \omega_m.$$

This implies

$$\frac{\mathcal{H}^m(\{\mathrm{dist}_{\Gamma(t)}(x,y) = r\})}{r^m} \le c$$

for all r smaller than the injectivity radius. For all larger r, we conclude with Lemma B.1. ∎

Combining Lemma B.2 with Lemma B.1, we obtain

$$\int_{\Gamma(t)} f(x,y)dy \le c\int_0^R r^m f(r)dr.$$

B.2 | Comparison of extrinsic and intrinsic distance

Lemma B.3. *There exists a constant $c > 0$ independent of t such that for all $x, y \in \Gamma(t)$ the following inequality holds*

$$c \cdot dist_{\Gamma(t)}(x,y) \le |x-y|. \tag{56}$$

Proof For simplicity, we assume that $\Gamma(t) = \Gamma_0$ for all $t \in [0,T]$. The basic idea is to find a radius $r > 0$ and two constants $c_1, c_2 > 0$ such that (56) holds with c_1 for $\mathrm{dist}_{\Gamma(t)}(x,y) \le r$ and with c_2 for $\mathrm{dist}_{\Gamma(t)}(x,y) \ge r$.

Observe that from the compactness from Γ_0 it follows that there exists $r > 0$ such that for all $\mathrm{dist}_{\Gamma(t)}(x, y) \leq r$ it holds

$$\nu(x) \cdot \nu(y) \geq \cos(\pi/6).$$

After rotation we may assume $x = 0$, $\nu(x) = e_{n+1}$ and that Γ_0 may be written as graph of a smooth function, that means that there exits $f : U(x) \to \mathbb{R}$ smooth with $U(x) \subset \mathbb{R}^n$ an open subset, such that $z = (z', w) \in \Gamma_0 \subset \mathbb{R}^m \times \mathbb{R}$ with $\mathrm{dist}_{\Gamma(t)}(z, x) \leq r$, if and only if $z' \in U(x)$ and $w = f(z')$. For $x = (0, 0)$ and $y = (y', f(y'))$ consider the path $t \mapsto (ty', f(ty'))$. We calculate

$$\mathrm{dist}_{\Gamma(t)}(x, y) \leq \int_0^1 \sqrt{1 + \mathrm{d}f_{ty'} y'}\mathrm{d}t \leq \sqrt{1 + \| f \|^2_{W^{1,\infty}}}\, |y'| \leq \sqrt{1 + \| f \|^2_{W^{1,\infty}}}\, |y - x|.$$

Now the derivatives of f are bounded by $m \cdot \tan(\pi/6)$.

To get the existence of $c_2 > 0$ observe that $\mathrm{dist}_{\Gamma(t)}$ is continuous and hence the set $\mathrm{dist}^{-1}_{\Gamma(t)} \{r > 0\}$ is compact. On this set the function $|x - y|$ does not vanish and takes it maximum and minimum. ∎

B.3 | Weight functions

Definition B.1. Let μ and $\tilde{\mu}$ be like (10) resp. (26). For given $\mu, \tilde{\mu}$ with curve $y = y(t)$, we define a curve $y_h = y_h(t) := y(t)^{-l} \in \Gamma_h(t)$. Now, we define a weight function on the discrete surface

$$\mu_h : \Gamma_h(t) \to \mathbb{R}, \quad \text{resp.} \quad \tilde{\mu}_\mathrm{h} : \Gamma_\mathrm{h}(\mathrm{t}) \to \mathbb{R},$$

via the same formula like (10) resp. (26).

Lemma B.4. *There exists a constant $h_0 = h_0(\gamma) > 0$ sufficiently small and $c = c(h_0) > 0$ independent of t and h such that for all $0 < h < h_0$ it holds*

$$\frac{1}{c}\mu \leq \mu_h^l \leq c\mu, \tag{57}$$

$$\frac{1}{c}\tilde{\mu} \leq \tilde{\mu}_h^l \leq c\tilde{\mu}. \tag{58}$$

Proof The main idea is to observe that we have the inequalities

$$|x^{-l} - y_h| \leq 2d + |x - y|,$$
$$|x - y| \leq 2d + |x^{-l} - y_h|,$$

where $d = d(t) := \max_{x \in \Gamma(t)} \mathrm{dist}_{\mathbb{R}^{n+1}}(x, \Gamma_h(t))$. ∎

APPENDIX C: MODIFIED ANALYTIC RESULTS FOR EVOLVING SURFACE PROBLEMS

Lemma C.1 (modified Gronwall inequality). *Let $c > 0$ be a positive constant, let φ, ψ, and ρ be some positive functions defined on $[t, T]$ and assume for all $s \in [t, T]$ we have the inequality*

$$-\frac{d\varphi}{ds}(s) + \psi(s) \leq c\varphi(s) + \rho(s).$$

Then it holds

$$\varphi(t) + \int_t^T \psi(s)ds \leq e^{c(T-t)}(\varphi(T) + \int_t^T \rho(s)ds).$$

Proof Calculate $-\frac{d}{ds}[\varphi e^{-c(T-s)}]$ and integrate from t to T. ▪

Lemma C.2 (modified inverse function theorem). *Let $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}^n$ be a smooth map, denote by $f(t)(x) := f(x, t)$ and assume that for all $t \in [0, T]$ the map $df(t)_0 = \frac{\partial f}{\partial x}(0, t)$ is invertible. Then there exists $r > 0$ independent of t such that*

$$f(t) : f(t)^{-1}\{B_r(0)\} \to \mathbb{R}^n, \quad x \mapsto f(x, t),$$

is a diffeomorphism onto its image and we have $B_{r/2}(0) \subset f(t)^{-1}\{B_r(0)\}$ for all t, where $B_r(0) := \{x \in \mathbb{R}^n | \; |x| \leq r\}$. The map

$$g : [0, T] \times B_r(0) \to \mathbb{R}^n, \quad (t, x) \mapsto f(t)^{-1}(x)$$

is smooth. In particular, g is smooth in t.

Proof The results follow from the compactness of $[0, T]$ and the smoothness of f. ▪

APPENDIX D: L^p-STABILITY OF AN L^2-PROJECTION ON SURFACES

We want to show Theorem 3.1. We essentially copy the proof of (23, Equations (6) and (7)) and give in one point a slightly different argument. We recall that $\Gamma_h(t) = \bigcup_{i=1}^{N'} E_i$ for some elements E_i. There exists a constant $K > 0$ which satisfies the following inequalities:

1. For each $i = 1, \ldots, N$ there exists a disc $B_i \subset E_i$ with

$$h \leq K\text{diam}(B_j),$$

   where $\text{diam}(B_j) = \sup_{(x,y)\in B_j} \text{dist}_h(x, y)$.

2. For all finite element basis function $(\chi_i)_{i=1}^N$, we have

$$\operatorname{diam}(\operatorname{supp}\chi_i) \leq Kh.$$

Proof of Theorem 3.1 For each $i = 1, \dots, N'$ set

$$u_i = \begin{cases} u & \text{on } E_i, \\ 0 & \text{else .} \end{cases}$$

Set

$$W_j = P_0 u_i = \sum_{l=1}^{N} W_{i,l}\chi_l.$$

Just like in [23, (10)–(22)] we may deduce that there exists $c, \kappa > 0$ independent of h such that

$$|W_i(x)| \leq ce^{-\kappa h^{-1}\operatorname{dist}(x,E_i)}\|u\|_{L^\infty}. \tag{59}$$

We calculate

$$\begin{aligned} |P_0 u(x)| &\leq \sum_{i=1}^{N'} |W_i(x)| \\ &\leq c\sum_{k\geq 0} e^{-\alpha k} k\|u\|_{L^\infty(\Gamma)} \\ &\leq c\|u\|_{L^\infty(\Gamma)}. \end{aligned}$$

In the second estimate, we have used that the numbers of elements E_i, which satisfy

$$kh \leq \operatorname{dist}(x, E_i) \leq (k+1)h,$$

are in $\mathcal{O}(k)$, as Γ is compact, Γ_h approximates Γ and as the triangulation is quasi-uniform.

∎

Appendix E. Error analysis for full discretisations of quasi-linear parabolic problems on evolving surfaces

Error Analysis for Full Discretizations of Quasilinear Parabolic Problems on Evolving Surfaces

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Received 31 March 2015; accepted 23 November 2015
Published online 25 February 2016 in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/num.22047

Convergence results are shown for full discretizations of quasilinear parabolic partial differential equations on evolving surfaces. As a semidiscretization in space the evolving surface finite element method is considered, using a regularity result of a generalized Ritz map, optimal order error estimates for the spatial discretization is shown. Combining this with the stability results for Runge–Kutta and backward differentiation formulae time integrators, we obtain convergence results for the fully discrete problems. Numer Methods Partial Differential Eq 32: 1200–1231, 2016

Keywords: quasilinear problems; evolving surfaces; ESFEM; Ritz map; Runge–Kutta and backward differentiation formulae methods; energy estimates

I. INTRODUCTION

In this article, we show convergence of full discretizations of quasilinear parabolic partial differential equations on evolving surfaces. As a spatial discretization we consider the evolving surface finite element method (ESFEM). The resulting system of ordinary differential equations is discretized, either with an algebraically stable Runge–Kutta (R–K) method, or with implicit or linearly implicit backward differentiation formulae (BDF).

To our knowledge [1] is the only work on error analysis for nonlinear problems on evolving surfaces. They give semidiscrete error bounds for the Cahn–Hilliard equation. The authors are not aware of fully discrete error estimates published in the literature.

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Contract grant sponsor: Deutscher Akademischer Austausch Dienst (B.K)

We show convergence results for full discretizations of quasilinear parabolic problems on evolving surfaces with prescribed velocity. We prove unconditional stability and higher-order convergence results for R–K and BDF methods. We show convergence as a full discretization when coupled with the ESFEM method as a space discretization for quasilinear problems. Similarly to the linear case the stability analysis relies on energy estimates and multiplier techniques.

First, we generalize some geometric perturbation estimates to the quasilinear setting. We define a *generalized Ritz map* for quasilinear operators, and use it to show optimal order error estimates for the spatial discretization. For deriving the optimal order L^2-error bounds of the Ritz map we will use a similar argument as Wheeler in [2], and elliptic regularity for evolving surfaces. A further important point of the analysis is the required *regularity* of the generalized Ritz map. This will be used together with the assumed Lipschitz-type estimate for the nonlinearity, analogously as in [3–5].

We show stability and convergence results for the case of stiffly accurate algebraically stable implicit R–K methods (having the Radau IIA methods in mind), and for an implicit and linearly implicit k-step BDF up to order five. These results are relying on the techniques used in [4–7]. By combining the results for the spatial semidiscretization with stability and convergence estimates we show high-order convergence bounds for the fully discrete approximation.

A starting point of the finite element approximation to (elliptic) surface partial differential equations is the paper of Dziuk [8]. Various convergence results for space discretizations of linear parabolic problems using the ESFEM were shown in [9, 10], a fully discrete scheme was analyzed in [11]. These results are surveyed in [12].

The convergence analysis of full discretizations with higher-order time integrators within the ESFEM setting for linear problems were shown: for algebraically stable R–K methods in [6]; for BDF in [7]. The ESFEM approach and convergence results were later extended to wave equations on evolving surfaces see [13].

A unified presentation of ESFEM and time discretizations for parabolic problems and wave equations can be found in [14].

A great number of real-life phenomena are modeled by nonlinear parabolic problems on evolving surfaces. Apart from general quasilinear problems on moving surfaces, see for example, 3.5 in [15], more specific applications are the nonlinear models: diffusion induced grain boundary motion [16–20]; Allen–Cahn and Cahn–Hilliard equations on evolving surfaces [1, 21–24]; modeling solid tumor growth [20, 25]; pattern formation modeled by reaction-diffusion equations [26, 27]; image processing [28]; Ginzburg–Landau model for superconductivity [29].

A number of nonlinear problems, in a general setting, were collected by Dziuk and Elliott in [9, 12, 15], also see the references therein. A great number of nonlinear problems with numerical experiments were presented in the literature, see for example, the above references, in particular [9, 15, 19, 20].

The article is organized in the following way: In Section II, we formulate our problem and detail our assumptions. In Section III, we recall the ESFEM, together with some of its important properties and estimates. We introduce the generalized Ritz map, and show optimal order error estimates for the residual, using the crucial $W^{1,\infty}$ regularity estimate mentioned above. Section IV covers the stability results and error estimates for R–K and for implicit and linearly implicit BDF methods. Section V is devoted to the error bounds of the semidiscrete residual, which then leads to error estimates for the fully discretized problem. In Section VI, we briefly discuss how our results can be extended to semilinear problems and to the case where the upper and lower bounds of the elliptic part are depending on the norm of the solution. Numerical results are presented in Section VI to illustrate our theoretical results.

II. THE PROBLEM AND ASSUMPTIONS

Let us consider a smooth evolving compact hypersurface $\Gamma(t) \subset \mathbb{R}^{m+1}$ ($m \leq 2$), $0 \leq t \leq T$, which moves with a given smooth velocity v. Let $\partial^\bullet u = \partial_t u + v \cdot \nabla u$ denote the material derivative of the function u, where ∇_Γ is the tangential gradient given by $\nabla_\Gamma u = \nabla u - \nabla u \cdot \nu\nu$, with unit normal ν. We are sharing the setting of [9, 10].

We consider the following quasilinear problem for $u = u(x, t)$:

$$\begin{cases} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \nabla_{\Gamma(t)} \cdot (\mathcal{A}(u)\nabla_{\Gamma(t)} u) = f & \text{on } \Gamma(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad u(.,0) = u_0 & \text{on } \Gamma(0), \end{cases} \tag{1}$$

where $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth function. For simplicity, we set $f = 0$, but all our results hold with a nonvanishing f as well.

Remark 2.1. The results of the article can be generalized to the case of a sufficiently smooth matrix valued diffusion coefficient $\mathcal{A}(x, t, u) : T_x\Gamma(t) \to T_x\Gamma(t)$. The proofs are similar to the ones presented here, except they are more technical and lengthy, therefore, they are not presented here.

The abstract setting of this quasilinear evolving surface partial differential equation (PDE) is a suitable combination of [4] (Section I) and [30], (Section II.C): Let $H(t)$ and $V(t)$ be real and separable Hilbert spaces (with norms $||.||_{H(t)}$, $||.||_{V(t)}$, respectively) such that $V(t)$ is densely and continuously embedded into $H(t)$, and the norm of the dual space of $V(t)$ is denoted by $||.||_{V(t)'}$. The dual space of $H(t)$ is identified with itself, and the duality $\langle .,. \rangle_t$ between $V(t)'$ and $V(t)$ coincides on $H(t) \times V(t)$ with the scalar product of $H(t)$, for all $t \in [0, T]$.

The problem casts the following nonlinear operator:

$$\langle A(u)v, w \rangle_t = \int_{\Gamma(t)} \mathcal{A}(u)\nabla_\Gamma v \cdot \nabla_\Gamma w.$$

We assume that A satisfies the following three conditions:

The bilinear form associated to the operator $A(u) : V(t) \to V(t)'$ is *elliptic* with $\mathbf{m} > 0$

$$\langle A(u)w, w \rangle_t \geq \mathbf{m}||w||^2_{V(t)} \qquad (w \in V(t)), \tag{2}$$

uniformly in $u \in V(t)$ and for all $t \in [0, T]$. It is *bounded* with $\mathcal{M} > 0$

$$|\langle A(u)v, w \rangle_t| \leq \mathcal{M}||v||_{V(t)}||w||_{V(t)} \qquad (v, w \in V(t)), \tag{3}$$

uniformly in $u \in V(t)$ and for all $t \in [0, T]$. We further assume that there is a subset $\mathcal{S}(t) \subset V(t)$ such that the following *Lipschitz–type* estimate holds: for every $\delta > 0$ there exists $L = L(\delta, (\mathcal{S}(t))_{0 \leq t \leq T})$ such that

$$\|(A(w_1) - A(w_2))u\|_{V(t)'} \leq \delta||w_1 - w_2||_{V(t)} + L||w_1 - w_2||_{H(t)}, \tag{4}$$

for $u \in \mathcal{S}(t)$, $w_1, w_2 \in V(t)$, $0 \leq t \leq T$.

The above conditions were also used to prove error estimates using energy techniques in [31] and in [3, 4], or more recently in [5].

The weak formulation uses Sobolev spaces on surfaces: For a sufficiently smooth surface Γ we define

$$H^1(\Gamma) = \left\{\eta \in L^2(\Gamma) | \nabla_\Gamma \eta \in L^2(\Gamma)^{m+1}\right\},$$

and analogously $H^k(\Gamma)$ for $k \in \mathbb{N}$ and $W^{k,p}(\Gamma)$ for $k \in \mathbb{N}, p \in [1, \infty]$, cf. [9] (Section II.A). Finally, $\mathcal{G}_T = \cup_{t\in[0,T]} \Gamma(t) \times \{t\}$ denotes the space-time manifold.

The weak problem corresponding to (1) can be formulated by choosing the setting: $V(t) = H^1(\Gamma(t))$ and $H(t) = L^2(\Gamma(t))$, and the operator:

$$\langle A(u)v, w\rangle_t = \int_{\Gamma(t)} \mathcal{A}(u) \nabla_\Gamma v \cdot \nabla_\Gamma w.$$

The coefficient function $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.

Assumption 2.1.

a. It is bounded, and Lipschitz–bounded with constant ℓ.
b. The function $\mathcal{A}(s) \geq \mathbf{m} > 0$ for arbitrary $s \in \mathbb{R}$.

Throughout the article, we use the following subspace of $V(t)$, for $r > 0$,

$$\mathcal{S}(t) := \mathcal{S}(t, r) = \left\{u \in H^2(\Gamma(t)) \,|\, ||u||_{W^{2,\infty}(\Gamma(t))} \leq r\right\},$$

that is, $W^{2,\infty}(\Gamma(t))$ functions with norm less then r.

Then, the following proposition easily follows.

Proposition 2.1. *Under Assumption 2.1 and $u \in \mathcal{S}(t)$ $(0 \leq t \leq T)$ the above operator A satisfies the conditions (2), (3) and (4) (with $\delta = 0$), they possibly depend on $\mathcal{S}(t, r)$.*

Proof. The first two conditions (2) and (3) follow from (a) and (b). Condition (4) holds, as for $u \in \mathcal{S}(t)$, $w_1, w_2 \in H^1(\Gamma(t))$ and any $z \in H^1(\Gamma(t))$, we have

$$\begin{aligned} |\langle (A(w_1) - A(w_2))u, z\rangle_t| &= \left| \int_{\Gamma(t)} (\mathcal{A}(w_1) - \mathcal{A}(w_2)) \nabla_\Gamma u \cdot \nabla_\Gamma z \right| \\ &\leq c\ell \, ||w_1 - w_2||_{L^2(\Gamma(t))} \, r \, ||z||_{H^1(\Gamma(t))}, \end{aligned}$$

hence, $L = c\ell r$, where the constant ℓ is from Assumption 2.1 (a). ∎

Definition 2.1 (Weak form). *A function $u \in H^1(\mathcal{G}_T)$ is called a weak solution of (1), if for almost every $t \in [0, T]$*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} \mathcal{A}(u) \nabla_\Gamma u \cdot \nabla_\Gamma \varphi = \int_{\Gamma(t)} u \partial^\bullet \varphi \tag{5}$$

holds for every $\varphi \in H^1(\mathcal{G}_T)$ and $u(.,0) = u_0$.

III. SPATIAL SEMIDISCRETIZATION: EVOLVING SURFACE FINITE ELEMENTS

As a spatial semidiscretization we use the ESFEM introduced by Dziuk and Elliott in [9]. We shortly recall some basic notations and definitions from [9], for more details the reader is referred to Dziuk and Elliott [8, 10, 12].

A. Basic Notations

The smooth surface $\Gamma(t)$ is approximated by a triangulated one denoted by $\Gamma_h(t)$, whose vertices $a_i(t)$, $i = 1, 2, \dots, N$, are sitting on the surface, given as

$$\Gamma_h(t) = \bigcup_{E(t)\in\mathcal{T}_h(t)} E(t).$$

We always assume that the (evolving) simplices $E(t)$ are forming an admissible triangulation $\mathcal{T}_h(t)$, with h denoting the maximum diameter. Admissible triangulations were introduced in [9] (Section 5.1): $\Gamma(t)$ is a uniform triangulation, that is, every $E(t) \in \mathcal{T}_h(t)$ satisfies that the inner radius σ_h is bounded from below by c_h with $c>0$, and $\Gamma_h(t)$ is not a global double covering of $\Gamma(t)$. Then the discrete tangential gradient on the discrete surface $\Gamma_h(t)$ is given by

$$\nabla_{\Gamma_h(t)}\phi := \nabla\phi - \nabla\phi \cdot \nu_h \nu_h,$$

understood in a piecewise sense, with ν_h denoting the normal to $\Gamma_h(t)$ (see [9]).

For every $t \in [0, T]$ we define the finite element subspace $S_h(t)$ spanned by the continuous, piecewise linear evolving basis functions χ_j, satisfying $\chi_j(a_i(t), t) = \delta_{ij}$ for all $i, j = 1, 2, \dots, N$, therefore,

$$S_h(t) = \operatorname{span}\{\chi_1(.,t), \chi_2(.,t), \dots, \chi_N(.,t)\}.$$

We interpolate the surface velocity on the discrete surface using the basis functions and denote it with V_h. Then the discrete material derivative is given by

$$\partial_h^\bullet \phi_h = \partial_t\phi_h + V_h \cdot \nabla\phi_h \qquad (\phi_h \in S_h(t)).$$

The key *transport property* derived in [9] (Proposition 5.4), is the following

$$\partial_h^\bullet \chi_k = 0 \qquad \text{for} \quad k = 1, 2, \dots, N. \tag{6}$$

The spatially discrete quasilinear problem for evolving surfaces is formulated in

Problem 3.1 *(Semidiscretization in space). Find $U_h \in S_h(t)$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} U_h\phi_h + \int_{\Gamma_h(t)} \mathcal{A}(U_h)\nabla_{\Gamma_h}U_h \cdot \nabla_{\Gamma_h}\phi_h = \int_{\Gamma_h(t)} U_h\partial_h^\bullet\phi_h, \qquad (\forall\phi_h \in S_h(t)), \tag{7}$$

with the initial condition $U_h(.,0) = U_h^0 \in S_h(0)$ being a suitable approximation to u_0.

We postpone existence and uniqueness of (7) to the next subsection.

B. The ODE System

The ODE form of the above problem can be derived by setting

$$U_h(.,t) = \sum_{j=1}^{N} \alpha_j(t)\chi_j(.,t)$$

into (7), testing with $\phi_h = \chi_j$ and using the transport property (6).

Proposition 3.1 (quasilinear ODE system). *The spatially semidiscrete problem (7) is equivalent to the following nonlinear ODE system for the vector $\alpha(t) = (\alpha_j(t)) \in \mathbb{R}^N$, collecting the nodal values of $U_h(.,t)$:*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(M(t)\alpha(t)) + A(\alpha(t))\alpha(t) = 0 \\ \alpha(0) = \alpha_0 \end{cases} \tag{8}$$

where the evolving mass matrix $M(t)$ and a nonlinear stiffness matrix $A(\alpha(t))$ are defined as

$$M(t)_{kj} = \int_{\Gamma_h(t)} \chi_j\chi_k, \qquad A(\alpha(t))_{kj} = \int_{\Gamma_h(t)} \mathcal{A}(U_h)\nabla_{\Gamma_h}\chi_j \cdot \nabla_{\Gamma_h}\chi_k,$$

for $\alpha(t)$ defining $U_h = \sum_{j=1}^{N} \alpha_j(t)\chi_j(.,t)$

The proof of this proposition is analogous to the corresponding one in [6].

Existence and uniqueness of (8) and hence of (7) can be shown as follows. As $\mathcal{A}(u)$ is Lipschitz continuous in u, we deduce that $A(\alpha)\alpha$ is Lipschitz continuous is α. Then as $M(t)$ is invertible, the existence and uniqueness of $\alpha(t)$ for the system (8) follows from the Picard–Lindelöf theorem.

Time discretizations. We briefly introduce the time discretizations applied to the above ODE system (8). However, more details can be found in Section IV.

We use algebraically stable s-stage implicit R–K methods, defined by its Butcher tableau, with step size $\tau > 0$:

$$\begin{aligned} M_{ni}\alpha_{ni} &= M_n\alpha_n + \tau\sum_{j=1}^{s} a_{ij}\dot{\alpha}_{nj}, \qquad \text{for} \quad i = 1, 2, \ldots, s, \\ M_{n+1}\alpha_{n+1} &= M_n\alpha_n + \tau\sum_{i=1}^{s} b_i\dot{\alpha}_{ni}, \\ 0 &= \dot{\alpha}_{ni} + A(\alpha_{ni})\alpha_{ni} \qquad \text{for} \quad i = 1, 2, \ldots, s, \end{aligned}$$

with $M_{ni} := M(t_n + c_i\tau)$ and $M_{n+1} := M(t_{n+1})$.

We also use k-step BDF methods with step size $\tau > 0$:

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j M(t_{n-j})\alpha_{n-j} + A(\alpha_n)\alpha_n = 0, \qquad (n \geq k),$$

where the coefficients of the method are given by $\delta(\zeta) = \sum_{\ell=1}^{k} \frac{1}{\ell}(1-\zeta)^{\ell}$. Similarly, we also consider their linearly implicit modification, using the polynomial $\gamma(\zeta) = \zeta^k - (\zeta - 1)^{k-1}$:

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j M(t_{n-j})\alpha_{n-j} + A\left(\sum_{j=1}^{k} \gamma_j \alpha_{n-j}\right)\alpha_n = 0, \qquad (n \geq k).$$

C. Discrete Sobolev Norm Estimates

Through the article, we will work with the norm and seminorm introduced in [6]. We denote these discrete Sobolev-type norms as

$$|z(t)|_{M(t)} := ||Z_h||_{L^2(\Gamma_h(t))}, \qquad |z(t)|_{\mathbf{A}(t)} := ||\nabla_{\Gamma_h} Z_h||_{L^2(\Gamma_h(t))}, \tag{10}$$

for arbitrary $z(t) \in \mathbb{R}^N$, where $Z_h(.,t) = \sum_{j=1}^{N} z_j(t)\chi_j(.,t)$, further by $M(t)$ we mean the above mass matrix and by $\mathbf{A}(t)$ we mean the linear (but time dependent) stiffness matrix:

$$\mathbf{A}(t)_{kj} = \int_{\Gamma_h(t)} \nabla_{\Gamma_h}\chi_j \cdot \nabla_{\Gamma_h}\chi_k.$$

A very important lemma in our analysis is the following:

Lemma 3.1 ([**6**] Lemma 4.1). *There are constants μ, κ (independent of h) such that*

$$z^T(M(s) - M(t))y \leq (e^{\mu(s-t)} - 1)|z|_{M(t)}|y|_{M(t)},$$
$$z^T(\mathbf{A}(s) - \mathbf{A}(t))y \leq (e^{\kappa(s-t)} - 1)|z|_{\mathbf{A}(t)}|y|_{\mathbf{A}(t)}$$

for all $y, z \in \mathbb{R}^N$ and $s, t \in [0, T]$

D. Lifting Process and Approximation Results

In the following, we recall the so called *lift operator*, which was introduced in [8] and further investigated in [9, 10]. The lift operator projects a finite element function on the discrete surface onto a function on the smooth surface.

Using the *oriented distance function d* [9] (Section 2.1), for a continuous function $\eta_h : \Gamma_h(t) \to \mathbb{R}$ its lift is define as

$$\eta_h^l(p,t) := \eta_h(x,t), \qquad x \in \Gamma(t),$$

where for every $x \in \Gamma_h(t)$ the value $p = p(x,t) \in \Gamma(t)$ is uniquely defined via $x = p + \nu(p,t)d(x,t)$. By η^{-l} we mean the function whose lift is η.

We now recall some notions using the lifting process from [8, 9] and [14]. We have the lifted finite element space

$$S_h^l(t) := \left\{\varphi_h = \phi_h^l | \phi_h \in S_h(t)\right\}.$$

By δ_h, we denote the quotient between the continuous and discrete surface measures, $\mathrm{d}A$ and $\mathrm{d}A_h$, defined as $\delta_h \mathrm{d}A_h = \mathrm{d}A$. Further, we recall that

$$\mathrm{Pr} := (\delta_{ij} - \nu_i\nu_j)_{i,j=1}^{m+1} \quad \text{and} \quad \mathrm{Pr}_h := (\delta_{ij} - \nu_{h,i}\nu_{h,j})_{i,j=1}^{m+1}$$

are the projections onto the tangent spaces of Γ and Γ_h. Further, from [10], we recall the notation

$$Q_h = \frac{1}{\delta_h}(I - d\mathcal{H})\mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr}(I - d\mathcal{H}),$$

where $\mathcal{H}$ ($\mathcal{H}_{ij} = \partial_{x_j}\nu_i$) is the (extended) Weingarten map. For these quantities we recall some results from [9] (Lemma 5.1), [10] (Lemma 5.4) and [14] (Lemma 6.1).

Lemma 3.2. *Assume that $\Gamma_h(t)$ and $\Gamma(t)$ is from the above setting, then we have the estimates:*

$$||d||_{L^\infty(\Gamma_h(t))} \le ch^2, \quad ||\nu_j||_{L^\infty(\Gamma_h(t))} \le ch, \quad ||1 - \delta_h||_{L^\infty(\Gamma_h(t))} \le ch^2,$$
$$||\partial_h^\bullet d||_{L^\infty(\Gamma_h(t))} \le ch^2, \quad ||Pr - Q_h||_{L^\infty(\Gamma_h(t))} \le ch^2, \quad ||Pr(\partial_h^\bullet Q_h)Pr||_{L^\infty(\Gamma_h(t))} \le ch^2,$$

with constants depending on $\mathcal{G}_T$, but not on t.

Lemma 3.3. *For $1 \le p \le \infty$ there exists constants $c_1, c_2 > 0$ independent of t and h such that the for all $u_h \in W^{1,p}(\Gamma_h(t))$ it holds that $u_h^l \in W^{1,p}(\Gamma(t))$ with the estimates*

$$c_1||u_h||_{W^{1,p}\Gamma_h(t))} \le ||u_h^l||_{W^{1,p}(\Gamma(t))} \le c_2||u_h||_{W^{1,p}(\Gamma_h(t))}.$$

Proof. The proofs follows easily from the relation $\nabla_{\Gamma_h} u_h = \mathrm{Pr}_h(I - d\mathcal{H})\nabla_\Gamma u_h^l$, cf. [8] (Lemma 3). ∎

E. Bilinear Forms and Their Estimates

Apart from the ξ dependence, we use the time dependent bilinear forms defined in [10]: for arbitrary $z, \varphi, \xi \in H^1(\Gamma)$, $\xi \in \mathcal{S}(t)$, and their discrete analogs for $Z_h, \phi_h, \xi_h \in S_h$:

$$m(z,\varphi) = \int_{\Gamma(t)} z\varphi,$$
$$a(\xi; z,\varphi) = \int_{\Gamma(t)} \mathcal{A}(\xi)\nabla_\Gamma z \cdot \nabla_\Gamma \varphi,$$
$$g(v; z,\varphi) = \int_{\Gamma(t)} (\nabla_\Gamma \cdot v) z\varphi,$$
$$b(\xi; v; z,\varphi) = \int_{\Gamma(t)} \mathcal{B}(\xi; v)\nabla_\Gamma z \cdot \nabla_\Gamma \varphi,$$
$$m_h(Z_h,\phi_h) = \int_{\Gamma_h(t)} Z_h\phi_h$$
$$a_h(\xi_h; Z_h,\phi_h) = \int_{\Gamma_h(t)} \mathcal{A}(\xi_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h,$$
$$g_h(V_h; Z_h,\phi_h) = \int_{\Gamma_h(t)} (\nabla_{\Gamma_h} \cdot V_h) Z_h\phi_h,$$
$$b_h(\xi_h; V_h; Z_h,\phi_h) = \int_{\Gamma_h(t)} \mathcal{B}_h(\xi_h; V_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h,$$

where the discrete tangential gradients are understood in a piecewise sense, and with the tensors given as

$$\mathcal{B}(\xi; v)_{ij} = \partial^\bullet(\mathcal{A}(\xi)) + \nabla_\Gamma \cdot v\mathcal{A}(\xi) - 2\mathcal{A}(\xi)\mathcal{D}(v),$$
$$\mathcal{B}_h(\xi_h; V_h)_{ij} = \partial_h^\bullet(\mathcal{A}(\xi_h)) + \nabla_{\Gamma_h} \cdot V_h\mathcal{A}(\xi_h) - 2\mathcal{A}(\xi_h)\mathcal{D}_h(V_h),$$

for $i, j = 1, 2, \ldots, m+1$, with

$$\mathcal{D}(v)_{ij} = \frac{1}{2}((\nabla_\Gamma)_i v_j + (\nabla_\Gamma)_j v_i),$$
$$\mathcal{D}_h(V_h)_{ij} = \frac{1}{2}((\nabla_{\Gamma_h})_i (V_h)_j + (\nabla_{\Gamma_h})_j (V_h)_i),$$

for $i, j = 1, 2, \ldots, m+1$. For more details see [10] (Lemma 2.1) (and the references in the proof), or [12] (Lemma 5.2).

We will also use the transport lemma (note that $\partial_h^\bullet z_h = \partial_t z_h + v_h \cdot \nabla z_h$ for a $z_h \in S_h^l(t)$):

Lemma 3.4. *For arbitrary $\xi_h^l \in S_h^l(t)$ and $z_h, \varphi_h, \partial_h^\bullet z_h, \partial_h^\bullet \varphi_h \in S_h^l(t)$ we have:*

$$\frac{\mathrm{d}}{\mathrm{d}t} m(z_h, \varphi_h) = m(\partial_h^\bullet z_h, \varphi_h) + m(z_h, \partial_h^\bullet \varphi_h) + g(v_h; z_h, \varphi_h),$$
$$\frac{\mathrm{d}}{\mathrm{d}t} a(\xi_h^l; z_h, \varphi_h) = a(\xi_h^l; \partial_h^\bullet z_h, \varphi_h) + a(\xi_h^l; z_h, \partial_h^\bullet \varphi_h) + b(\xi_h^l; v_h; z_h, \varphi_h),$$

where v_h is the velocity of the surface, see [10] (Definition 4.9)

Proof. This lemma can be shown analogously as [10] (Lemma 4.2), therefore, the proof is omitted. ∎

Versions of this lemma with continuous material derivatives, or discrete bilinear forms are also true.

The following estimates will play a crucial role in the proofs.

Lemma 3.5 (Geometric perturbation errors). *For any $\xi \in \mathcal{S}(t)$, and $Z_h, \phi_h \in S_h(t)$ with corresponding lifts $z_h, \varphi_h \in S_h^l(t)$ we have the following bounds*

$$|m(z_h, \varphi_h) - m_h(Z_h, \phi_h)| \leq ch^2 ||z_h||_{L^2(\Gamma(t))} ||\varphi_h||_{L^2(\Gamma(t))},$$
$$|a(\xi; z_h, \varphi_h) - a_h(\xi^{-l}; Z_h, \phi_h)| \leq ch^2 ||\nabla_\Gamma z_h||_{L^2(\Gamma(t))} ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))},$$
$$|g(v_h; z_h, \varphi_h) - g_h(V_h; Z_h, \phi_h)| \leq ch^2 ||z_h||_{L^2(\Gamma(t))} ||\varphi_h||_{L^2(\Gamma(t))},$$
$$|b(\xi; v_h; z_h, \varphi_h) - b_h(\xi^{-l}; V_h; Z_h, \phi_h)| \leq ch^2 ||\nabla_\Gamma z_h||_{L^2(\Gamma(t))} ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))}.$$

Proof. The first estimate was proved in [10] (Lemma 5.5), while the third can be found in [13] (Lemma 7.5).

The proof of the second estimate is similar to the linear case found in [12] (Lemma 4.7). Again using the notation from [12]:

$$Q_h = \frac{1}{\delta_h}(I - d\mathcal{H})\mathrm{Pr}\mathrm{Pr}_h\mathrm{Pr}(I - d\mathcal{H})$$

we obtain

$$\mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h = \delta_h \mathcal{A}(\xi^{-l}) Q_h \nabla_\Gamma z_h(p,.) \cdot \nabla_\Gamma \varphi_h(p,.). \tag{11}$$

Similarly as in [10] (Lemma 5.5), the boundedness (Proposition 2.1) and the geometric estimate $||\mathrm{Pr} - Q_h||_{L^\infty(\Gamma_h)} \leq ch^2$ provides the estimate

$$\begin{aligned}
&|a(\xi; z_h, \varphi_h) - a_h(\xi^{-l}; Z_h, \phi_h)| \\
&= \left| \int_{\Gamma(t)} \mathcal{A}(\xi)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \mathrm{d}A - \int_{\Gamma_h(t)} \mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h \mathrm{d}A_h \right| \\
&= \left| \int_{\Gamma(t)} \mathcal{A}(\xi)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \mathrm{d}A - \int_{\Gamma_h(t)} \delta_h \mathcal{A}(\xi^{-l}) Q_h \nabla_\Gamma z_h(p,.) \cdot \nabla_\Gamma \varphi_h(p,.) \mathrm{d}A_h \right| \\
&= \left| \int_{\Gamma(t)} \mathcal{A}(\xi)(\mathrm{Pr} - Q_h)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \mathrm{d}A \right| \\
&\leq \mathcal{M}ch^2 ||\nabla_\Gamma z_h||_{L^2(\Gamma(t))} ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))}.
\end{aligned}$$

To prove the fourth estimate we follow [13]: starting with the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h(t)} \mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(t)} \mathcal{A}(\xi) Q_h^l \nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h$$

then the transport lemma (Lemma 3.4 above) yields

$$\begin{aligned}
&\int_{\Gamma_h(t)} \mathcal{A}(\xi^{-l})\nabla_{\Gamma_h} \partial_h^\bullet Z_h \cdot \nabla_{\Gamma_h}\phi_h + \int_{\Gamma_h(t)} \mathcal{A}^{-l}(\xi^{-l})\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\partial_h^\bullet \phi_h \\
&\quad + \int_{\Gamma_h(t)} \mathcal{B}_h(\xi^{-l}; V_h)\nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h}\phi_h \\
&= \int_{\Gamma(t)} \mathcal{A}(\xi) Q_h^l \nabla_\Gamma \partial_h^\bullet z_h \cdot \nabla_\Gamma \varphi_h + \int_{\Gamma(t)} \mathcal{A}(\xi) Q_h^l \nabla_\Gamma z_h \cdot \nabla_\Gamma \partial_h^\bullet \varphi_h \\
&\quad + \int_{\Gamma(t)} \mathcal{B}(\xi; v_h) Q_h^l \nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h + \int_{\Gamma(t)} \mathcal{A}(\xi)\partial_h^\bullet (Q_h^l)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h.
\end{aligned}$$

Therefore, using that the lift of $\partial_h^\bullet Z_h$ is $\partial_h^\bullet z_h$, (11) and Lemma 3.2 provides

$$\begin{aligned}
&|b_h(\xi^{-l}; V_h; Z_h, \phi_h) - b(\xi; v_h; Z_h, \phi_h)| \\
&= \left| \int_{\Gamma(t)} \mathcal{A}(\xi)\partial_h^\bullet (Q_h^l)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \right| + \left| \int_{\Gamma(t)} \mathcal{B}(\xi; v_h)(Q_h^l - I)\nabla_\Gamma z_h \cdot \nabla_\Gamma \varphi_h \right| \\
&\leq ch^2 ||\nabla_\Gamma z_h||_{L^2(\Gamma(t))} ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))},
\end{aligned}$$

where the last estimates follow from Lemma 3.2, similarly as in [13] (Theorem 7.5). ■

F. Interpolation Estimates

By $I_h : H^1(\Gamma(t)) \to S_h^l(t)$ we denote the finite element interpolation operator, having the error estimate below.

Lemma 3.6. *For $m \le 3$, there exists a constant $c > 0$ independent of h and t such that for $u \in H^2(\Gamma(t))$:*

$$||u - I_h u||_{L^2(\Gamma(t))} + h||\nabla_\Gamma(u - I_h u)||_{L^2(\Gamma(t))} \le ch^2||u||_{H^2(\Gamma(t))}.$$

Furthermore, if $u \in W^{2,\infty}(\Gamma(t))$, it also satisfies

$$\|\nabla_\Gamma(u - I_h u)\|_{L^\infty(\Gamma(t))} \le ch||u||_{W^{2,\infty}(\Gamma(t))},$$

where $c > 0$ is also independent of h and t

Proof. The first inequality was shown in [8]. The dimension restriction is especially discussed in [12] (Lemma 4.3).

The analogue of the second estimate for a reference element were shown in [32] (Theorem 3.1). Denote by $E_h(t) \subset \Gamma_h(t)$ an arbitrary element and denote by $E(t) \subset \Gamma(t)$ the lift of this triangle.

$$\begin{aligned} ||\nabla_\Gamma(u - I_h u)||_{L^\infty(E(t))} &\le c||\nabla_{\Gamma_h}(u^{-l} - I_h u^{-l})||_{L^\infty(E_h(t))} \le c\frac{1}{h}||\nabla_{\mathbb{R}^2}(\hat{u} - I_h\hat{u})||_{L^\infty(E_0)} \\ &\le c\frac{1}{h}||\nabla^2_{\mathbb{R}^2}\hat{u}||_{L^\infty(E_0)} \le c\frac{1}{h}h^2||\nabla^2_{\Gamma_h}u^{-l}||_{L^\infty(E_h(t))} \le ch||u||_{W^{2,\infty}(E(t))}, \end{aligned}$$

where $E_0 \subset \mathbb{R}^2$ is the standard unit simplex, $\hat{u} : E_0 \to \mathbb{R}$ is the representation of $u^{-l}|_{E_h(t)}$ on E_0 w.r.t. a suitable affine linear transformation and $\nabla^2_{\mathbb{R}^2}\hat{u}$ denote the usual Hessian of $\hat{u}$. For the first and the last inequality we have used, that the discrete and continuous norms are equivalent. The intermediate steps uses the uniformity of the triangulation together with standard estimates for the pullback, cf. [33] or [34], (Section 10.3). ■

G. The Ritz Map for Nonlinear Problems on Evolving Surfaces

Ritz maps for quasilinear PDEs on stationary domains were investigated by Wheeler in [2]. We generalize this idea for the case of quasilinear evolving surface PDEs. We define a generalized Ritz map for quasilinear elliptic operators, for the linear case see [13].

By combining the above definitions we set the following.

Definition 3.1 (Ritz map). *For a given $z \in H^1(\Gamma(t))$ and a given function $\xi : \Gamma(t) \to \mathbb{R}$ there is a unique $\tilde{\mathcal{P}}_h z \in S_h(t)$ such that for all $\phi_h \in S_h(t)$, with the corresponding lift $\varphi_h = \phi_h^l$, we have*

$$a_h^*(\xi^{-l}; \tilde{\mathcal{P}}_h z, \phi_h) = a^*(\xi; z, \varphi_h), \tag{12}$$

where $a^* := a + m$ and $a_h^* := a_h + m_h$, to make the forms a and a_h positive definite. Then $\mathcal{P}_h z \in S_h^l(t)$ is defined as the lift of $\tilde{\mathcal{P}}_h z$, that is, $\mathcal{P}_h z = (\tilde{\mathcal{P}}_h z)^l$.

We recall here that by ξ^{-l} we mean a function (living on the discrete surface) whose lift is ξ.

Galerkin orthogonality does not hold in this case, just up to a small defect:

Lemma 3.7 (pseudo Galerkin orthogonality)**.** *For any given $\xi \in \mathcal{S}(t)$ their holds, that for every $z \in H^1(\Gamma(t))$ and $\varphi_h \in S_h^l(t)$*

$$|a^*(\xi; z - \mathcal{P}_h z, \varphi_h)| \leq ch^2 ||\mathcal{P}_h z||_{H^1(\Gamma(t))} ||\varphi_h||_{H^1(\Gamma(t))}, \tag{13}$$

where c is independent of ξ, h and t

Proof. Using the definition of the Ritz map:

$$\begin{aligned} |a^*(\xi; z - \mathcal{P}_h z, \varphi_h)| &= |a_h^*(\xi^{-l}; \tilde{\mathcal{P}}_h z, \phi_h) - a^*(\xi; \mathcal{P}_h z, \varphi_h)| \\ &\leq \mathcal{M} ch^2 ||\mathcal{P}_h z||_{H^1(\Gamma(t))} ||\varphi_h||_{H^1(\Gamma(t))}, \end{aligned}$$

where we used Lemma 3.5. ∎

H. Error Bounds for the Ritz Map and for its Material Derivatives

In this section, we prove error estimates for the Ritz map (12) and also for its material derivatives, the analogous results for the linear case can be found in [10] (Section 6), [14] (Section 7). The ξ independency of the estimates requires extra care, previous results, for example, the ones cited above, or [13] (Section 8), are not applicable.

Theorem 3.1. *The error in the Ritz map satisfies the bound, for arbitrary $\xi \in \mathcal{S}(t)$ and $0 \leq t \leq T$ and $h \leq h_0$ with sufficiently small h_0,*

$$||z - \mathcal{P}_h z||_{L^2(\Gamma(t))} + h||z - \mathcal{P}_h z||_{H^1(\Gamma(t))} \leq ch^2 ||z||_{H^2(\Gamma(t))}.$$

where the constant c is independent of ξ, h, and t (but depends on **m** *and $\mathcal{M}$)*

Proof. (a) We first prove the gradient estimate.

Starting by the ellipticity of the form a and the non-negativity of the form m, then using the estimate (13) we have:

$$\begin{aligned} \mathbf{m}||z - \mathcal{P}_h z||^2_{H^1(\Gamma(t))} &\leq a^*(\xi; z - \mathcal{P}_h z, z - \mathcal{P}_h z) \\ &= a^*(\xi; z - \mathcal{P}_h z, z - I_h z) + a^*(\xi; z - \mathcal{P}_h z, I_h z - \mathcal{P}_h z) \\ &\leq \mathcal{M}||z - \mathcal{P}_h z||_{H^1(\Gamma(t))} ||z - I_h z||_{H^1(\Gamma(t))} \\ &\quad + ch^2 ||\mathcal{P}_h z||_{H^1(\Gamma(t))} ||I_h z - \mathcal{P}_h z||_{H^1(\Gamma(t))} \\ &\leq \mathcal{M} ch ||z - \mathcal{P}_h z||_{H^1(\Gamma(t))} ||z||_{H^2(\Gamma(t))} \\ &\quad + ch^2 \Big(2||z - \mathcal{P}_h z||^2_{H^1(\Gamma(t))} + ||z||^2_{H^1(\Gamma(t))} + ch^2 ||z||^2_{H^2(\Gamma(t))} \Big), \end{aligned}$$

using the interpolation error, and for the second term we used the estimate

$$\begin{aligned} &||\mathcal{P}_h z||_{H^1(\Gamma(t))} ||I_h z - \mathcal{P}_h z||_{H^1(\Gamma(t))} \\ &\quad \leq \big(||\mathcal{P}_h z - z||_{H^1(\Gamma(t))} + ||z||_{H^1(\Gamma(t))} \big) \big(||I_h z - z||_{H^1(\Gamma(t))} + ||z - \mathcal{P}_h z||_{H^1(\Gamma(t))} \big) \\ &\quad \leq 2||z - \mathcal{P}_h z||^2_{H^1(\Gamma(t))} + ||z||^2_{H^1(\Gamma(t))} + ch^2 ||z||^2_{H^2(\Gamma(t))}. \end{aligned}$$

Now, using Young's and Cauchy–Schwarz inequality, and for sufficiently small (but ξ independent) h we have the gradient estimate

$$\|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))}^2 \leq \frac{1}{\mathbf{m}} \mathcal{M} c h^2 \|z\|_{H^2(\Gamma(t))}^2.$$

(b) The L^2-estimate follows from the Aubin–Nitsche trick. Let us consider the problem

$$-\nabla_\Gamma \cdot (\mathcal{A}(\xi)\nabla_\Gamma w) + w = z - \mathrm{P}_h z \quad \text{on } \Gamma(t),$$

then by elliptic theory, cf. Theorem A.1, we have the estimate, for the solution $w \in H^2(\Gamma(t))$

$$\|w\|_{H^2(\Gamma(t))} \leq c\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))},$$

where c is independent of t and ξ. By testing the elliptic weak problem with $z{-}\mathcal{P}\_hz$ we have

$$\begin{aligned}
\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))}^2 &= a^*(\xi; z - \mathcal{P}_h z, w) \\
&= a^*(\xi; z - \mathcal{P}_h z, w - I_h w) + a^*(\xi; z - \mathcal{P}_h z, I_h w) \\
&\leq \mathcal{M}\|z - \mathcal{P}_h z\|_{H^1(\Gamma(t))}\|w - I_h w\|_{H^1(\Gamma(t))} \\
&\quad + ch^2\|\mathcal{P}_h z\|_{H^1(\Gamma(t))}\|I_h w\|_{H^1(\Gamma(t))}.
\end{aligned}$$

Then, the estimates of the interpolation error and combination of the above results yields

$$\|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))}\frac{1}{c}\|w\|_{H^2(\Gamma(t))} \leq \|z - \mathcal{P}_h z\|_{L^2(\Gamma(t))}^2 \leq \mathcal{M}ch^2\|z\|_{H^2(\Gamma(t))}\|w\|_{H^2(\Gamma(t))},$$

which completes the proof of the first assertion. ∎

To proof error estimates for higher order material derivatives of the Ritz map, we need to control the error $(\partial_h^\bullet)^{(k)}(v - v_h)$ in the L^∞- and $W^{1,\infty}$-norm.

Lemma 3.8. *For $k \geq 0$ there exits a constant $c = c(k) > 0$ independent of t and h such that*

$$\|(\partial_h^\bullet)^{(k)}(v - v_h)\|_{L^\infty(\Gamma(t))} + h\|\nabla_\Gamma(\partial_h^\bullet)^{(k)}(v - v_h)\|_{L^\infty(\Gamma(t))} \leq ch^2$$

A proof of this lemma can be found in Mansour [14] (Lemma 6.3).

Theorem 3.2. *Assume that $\xi \in \mathcal{S}(t)$ and that in addition that for $k \geq 1$ it holds $(\partial_h^\bullet)^{(k)}(\mathcal{A}(\xi)) \in L^\infty(\mathcal{G}_T)$. The error in the material derivatives of the Ritz map satisfies the following bounds, for $0 \leq t \leq T$ and $h \leq h_0$ with sufficiently small h_0,*

$$\|(\partial_h^\bullet)^{(k)}(z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} + h\|\nabla_\Gamma(\partial_h^\bullet)^{(k)}(z - \mathcal{P}_h z)\|_{L^2(\Gamma(t))} \leq \mathcal{M}c_k h^2 \sum_{j=1}^{k} \|(\partial_h^\bullet)^{(j)} z\|_{H^2(\Gamma(t))}.$$

The constant $c_k > 0$ is independent of ξ and h (but depends on α and $\mathcal{M}$)

Proof. The proof is a modification of [14] (Theorem 7.3).

For $k = 1$: (a) We start by taking the time derivative of the definition of the Ritz map (12), use the transport properties (Lemma 3.4), and use the definition of the Ritz map once more, we arrive at

$$a^*(\xi; \partial_h^\bullet z, \varphi_h) = -b(\xi; v_h; z, \varphi_h) - g(v_h; z, \varphi_h) \\ + a_h^*(\xi^{-l}; \partial_h^\bullet \tilde{\mathcal{P}}_h z, \phi_h) + b_h(\xi^{-l}; V_h; \tilde{\mathcal{P}}_h z, \phi_h) + g_h(V_h; \tilde{\mathcal{P}}_h z, \phi_h).$$

Then, we obtain

$$a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \varphi_h) = -b(\xi; v_h; z - \mathcal{P}_h z, \varphi_h) - g(v_h; z - \mathcal{P}_h z, \varphi_h) + F_1(\varphi_h), \tag{14}$$

where

$$F_1(\varphi_h) = (a_h^*(\xi^{-l}; \partial_h^\bullet \tilde{\mathcal{P}}_h z, \phi_h) - a^*(\xi; \partial_h^\bullet \mathcal{P}_h z, \varphi_h)) \\ + (b_h(\xi^{-l}; V_h; \tilde{\mathcal{P}}_h z, \phi_h) - b(\xi; v_h; \mathcal{P}_h z, \varphi_h)) \\ + (g_h(V_h; \tilde{\mathcal{P}}_h z, \phi_h) - g(v_h; \mathcal{P}_h z, \varphi_h)).$$

Using the geometric estimates of Lemma 3.5 F_1 can be estimated as

$$|F_1(\varphi_h)| \leq c\mathcal{M}h^2(||\partial_h^\bullet \mathcal{P}_h z||_{H^1(\Gamma(t))} + ||\mathcal{P}_h z||_{H^1(\Gamma(t))})||\varphi_h||_{H^1(\Gamma(t))}.$$

Then, using $\partial_h^\bullet \mathcal{P}_h z$ as a test function in (14), and using the error estimates of the Ritz map, together with the estimates above, with $h \leq h_0$ independent of ξ, we have

$$||\partial_h^\bullet \mathcal{P}_h z||_{H^1(\Gamma(t))} \leq \mathcal{M}c||\partial^\bullet z||_{H^1(\Gamma(t))} + \mathcal{M}ch||z||_{H^2(\Gamma(t))}.$$

Combining all the previous estimates and using Young's inequality, Cauchy–Schwarz inequality, for sufficiently small (ξ independent) $h \leq h_0$, we obtain

$$a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \varphi_h) \leq \mathcal{M}ch(||z||_{H^2(\Gamma(t))} + h||\partial^\bullet z||_{H^1(\Gamma(t))})||\varphi_h||_{H^1(\Gamma(t))}.$$

Then, as in the previous proof we have

$$\begin{aligned} \mathbf{m}||\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z||_{H^1(\Gamma(t))}^2 &\leq a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z) \\ &= a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, \partial_h^\bullet z - I_h \partial^\bullet z) \\ &\quad + a^*(\xi; \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z, I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z) \\ &\leq \mathcal{M}||\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z||_{H^1(\Gamma(t))}||\partial_h^\bullet z - I_h \partial^\bullet z||_{H^1(\Gamma(t))} \\ &\quad + \mathcal{M}ch(||z||_{H^2(\Gamma(t))} + h||\partial^\bullet z||_{H^1(\Gamma(t))})||I_h \partial^\bullet z - \partial_h^\bullet \mathcal{P}_h z||_{H^1(\Gamma(t))}. \end{aligned}$$

Then the interpolation estimates, Young's inequality, absorption using $h \leq h_0$, yields the gradient estimate.

(b) The L^2-estimate again follows from the Aubin-Nitsche trick. Let us now consider the problem

$$-\nabla_\Gamma \cdot (\mathcal{A}(\xi)\nabla_\Gamma w) + w = \partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z \qquad \text{on} \quad \Gamma(t),$$

together with the elliptic estimate (cf. Theorem A.1), for the solution $w \in H^2(\Gamma(t))$

$$||w||_{H^2(\Gamma(t))} \leq c||\partial_h^\bullet z - \partial_h^\bullet \mathcal{P}_h z||_{L^2(\Gamma(t))},$$

again, c is independent of t and ξ.

Then, a similar calculation as [10] (Theorem 6.2), [14] (Theorem 7.3) provides the L^2-norm estimate.

For $k > 1$ the proof is analogous. ∎

Remark 3.1. If $\xi \in W^{k,\infty}(\Gamma(t))$ and $\mathcal{A} \in W^{k,\infty}(\mathbb{R})$ then it holds that $(\partial_h^\bullet)^{(k)}\mathcal{A}(\xi) \in L^\infty(\Gamma(t))$. For the convenience of the reader we give a proof for $k = 2$. It holds

$$(\partial_h^\bullet)^{(2)}(\mathcal{A}(\xi)) = \partial_h^\bullet(\mathcal{A}'(\xi)\partial_h^\bullet\xi) = \mathcal{A}''(\xi)(\partial_h^\bullet\xi)^2 + \mathcal{A}'(\xi)(\partial_h^\bullet)^{(2)}\xi.$$

We have the identity

$$\partial_h^\bullet\xi = \partial^\bullet\xi + (v_h - v)\cdot\nabla_\Gamma\xi.$$

For the second derivative we calculate

$$(\partial_h^\bullet)^{(2)}\xi = (\partial^\bullet)^{(2)}\xi + (v_h - v)\cdot\nabla_\Gamma\partial^\bullet\xi + \partial_h^\bullet(v_h - v)\cdot\nabla_\Gamma\xi + (v_h - v)\cdot\partial^\bullet\nabla_\Gamma\xi + \nabla_\Gamma^2\xi(v_h - v)^2.$$

Using Lemma 3.8 the claim follows.

Regularity of the Ritz Map. The following technical result will play an important role in showing optimal bounds of the semidiscrete residual.

Lemma 3.9. *For $m \leq 2$, there exists a constant $c > 0$ independent of h and t such that for a function $u \in W^{2,\infty}(\Gamma(t))$ for all $t \in [0, T]$, the following estimate holds*

$$||\nabla_\Gamma\mathcal{P}_h u||_{L^\infty(\Gamma(t))} \leq c||u||_{W^{2,\infty}(\Gamma(t))}.$$

Proof. Using the triangle inequality we start to estimate as

$$||\nabla_\Gamma\mathcal{P}_h u||_{L^\infty(\Gamma(t))} \leq ||\nabla_\Gamma(\mathcal{P}_h u - I_h u)||_{L^\infty(\Gamma(t))} + ||\nabla_\Gamma(I_h u - u)||_{L^\infty(\Gamma(t))} + ||\nabla_\Gamma u||_{L^\infty(\Gamma(t))}.$$

The last term is harmless. The second term is estimated using Lemma 3.6. For the first term, using the inverse estimate, error estimates for the Ritz map and for the interpolation operator we obtain

$$\begin{aligned}||\nabla_\Gamma(\mathcal{P}_h u - I_h u)||_{L^\infty(\Gamma(t))} &\leq ch^{-m/2}||\nabla_\Gamma(\mathcal{P}_h u - I_h u)||_{L^2(\Gamma(t))}\\ &\leq ch^{-m/2}(||\nabla_\Gamma(\mathcal{P}_h u - u)||_{L^2(\Gamma(t))} + ||\nabla_\Gamma(u - I_h u)||_{L^2(\Gamma(t))})\\ &\leq ch^{-m/2}h||u||_{H^2(\Gamma(t))} \leq c||u||_{W^{2,\infty}(\Gamma(t))}.\end{aligned}$$

∎

Remark 3.2. A stronger result holds, assuming that $u \in W^{1,\infty}(\Gamma(t))$, the bound $||\nabla_\Gamma\mathcal{P}_h u||_{L^\infty(\Gamma(t))} \leq c||u||_{W^{1,\infty}}(\Gamma(t))$ can be shown. However, the proof is technical and requires more sophisticated arguments, cf. [35]. This enables to weaken the assumption to $W^{1,\infty}$ in the definition of the $\mathcal{S}(t)$ set. We do not include these results here because of their length.

IV. TIME DISCRETIZATIONS: STABILITY

A. Runge–Kutta Methods

We consider an s-stage algebraically stable implicit R–K method for the time discretization of the ODE system (8), coming from the ESFEM space discretization of the quasilinear parabolic evolving surface PDE.

In the following, we extend the stability result for R–K methods of [6] (Lemma 7.1), to the case of quasilinear problems. Apart form the properties of the ESFEM the proof is based on the energy estimation techniques, see Lubich and Ostermann [4] (Theorem 1.1). Generally on R–K methods we refer to [36].

For the convenience of the reader we recall the method: for simplicity, we assume equidistant time steps $t_n := n\tau$, with step size τ. Our results can be straightforwardly extended to the case of nonuniform time steps. The s-stage implicit R–K method, defined by the given Butcher tableau

$$\begin{array}{c|c} (c_i) & (a_{ij}) \\ \hline & (b_i) \end{array} \qquad \text{for} \quad i, j = 1, 2, \dots, s,$$

applied to the system (8), reads as

$$\begin{aligned} M_{ni}\alpha_{ni} &= M_n\alpha_n + \tau \sum_{j=1}^{s} a_{ij}\dot{\alpha}_{nj}, \qquad \text{for} \quad i = 1, 2, \dots, s, \\ M_{n+1}\alpha_{n+1} &= M_n\alpha_n + \tau \sum_{i=1}^{s} b_i\dot{\alpha}_{ni}, \end{aligned}$$

where the internal stages satisfy

$$0 = \dot{\alpha}_{ni} + A(\alpha_{ni})\alpha_{ni} \qquad \text{for} \quad i = 1, 2, \dots, s,$$

with $M_{ni} := M(t_n + c_i\tau)$ and $M_{n+1} := M(t_{n+1})$. Here $\dot{\alpha}_{ni}$ is not a derivative but a suggestive notation.

We recall that the fully discrete solution is $U_h^n = \sum_{j=1}^{N} \alpha_{n,j}\chi_j(., t_n)$. Existence and uniqueness of the R–K solution can be obtained analogously to [37] (Theorem 7.2).

For the R–K method we make the following assumptions:

Assumption 4.1.

- *The method has stage order $q \geq 1$ and classical order $p \geq q+1$.*
- *The coefficient matrix (a_{ij}) is invertible.*
- *The method is algebraically stable, that is, $b_j > 0$ for $j = 1,2,...,s$ and the following matrix is positive semidefinite:*

$$(b_i a_{ij} + b_j a_{ji} - b_i b_j)_{i,j=1}^{s}.$$

- *The method is stiffly accurate, that is, $b_j = a_{sj}$, and $c_s = 1$ for $j = 1, 2, \dots, s$.*

Instead of (8), let us consider the following perturbed version of the equation:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(M(t)\tilde{\alpha}(t)) + A(\tilde{\alpha}(t))\tilde{\alpha}(t) = M(t)r(t) \\ \tilde{\alpha}(0) = \tilde{\alpha}_0. \end{cases} \tag{16}$$

The substitution of the true solution $\tilde{\alpha}(t)$ of the perturbed problem into the R–K method, yields the defects Δ_{ni} and δ_{ni}, by setting $e_n = \alpha_n - \tilde{\alpha}(t_n)$, $E_{ni} = \alpha_{ni} - \tilde{\alpha}(t_n + c_i\tau)$ and $\dot{E}_{ni} = \dot{\alpha}_{ni} - \dot{\tilde{\alpha}}(t_n + c_i\tau)$, then by subtraction the following *error equations* hold:

$$M_{ni}E_{ni} = M_n e_n + \tau \sum_{j=1}^{s} a_{ij}\dot{E}_{nj} - \Delta_{ni}, \qquad \text{for} \quad i = 1, 2, \ldots, s, \tag{17a}$$

$$M_{n+1}e_{n+1} = M_n e_n + \tau \sum_{i=1}^{s} b_i \dot{E}_{ni} - \delta_{n+1}, \tag{17b}$$

where the internal stages satisfy:

$$\dot{E}_{ni} + A(\alpha_{ni})E_{ni} = -(A(\alpha_{ni}) - A(\tilde{\alpha}_{ni}))\tilde{\alpha}_{ni} - M_{ni}r_{ni}, \quad \text{for} \quad i = 1, 2, \ldots, s, \tag{18}$$

with $r_{ni} := r(t_n + c_i\tau)$.

Now, we state one of the key lemmas of this paper, which provide unconditional stability for the above class of R–K methods.

Lemma 4.1. *For an s-stage implicit R–K method satisfying Assumption 4.1. If the Eq. (5) has a solution in $\mathcal{S}(t)$ for $0 \le t \le T$. Then there exists a $\tau_0 > 0$, such that for $\tau \le \tau_0$ and $t_n = n\tau \le T$, that the error e_n is bounded by*

$$\begin{aligned} |e_n|_{M_n}^2 + \tau \sum_{k=1}^{n} |e_k|_{\mathbf{A}_k}^2 \le C\Bigg(|e_0|_{M_0}^2 + \tau \sum_{k=1}^{n-1}\sum_{i=1}^{s} ||M_{ki}r_{ki}||_{*,t_{ki}}^2 + \tau \sum_{k=1}^{n} |\frac{\delta_k}{\tau}|_{M_k}^2 \\ + C\tau \sum_{k=0}^{n-1}\sum_{i=1}^{s} \Big(|M_{ki}^{-1}\Delta_{ki}|_{M_{ki}}^2 + |M_{ki}^{-1}\Delta_{ki}|_{\mathbf{A}_{ki}}^2 \Big) \Bigg), \end{aligned} \tag{19}$$

where $||w||_{*,t}^2 = w^T(\mathbf{A}(t) + M(t))^{-1}w$. The constant C is independent of h, τ, and n (but depends on $\mathbf{m}$, $\mathcal{M}$, L, μ, κ, and T)

Proof. The combination of proofs of Theorem 1.1 from [4] and of Lemma 7.1 from [6] (or [14] (Lemma 3.1) suffices, therefore it is omitted here. To be precise, the proof of this result is more closely related to [6]. Except the estimates involving the (nonlinear) internal stages, see [4].

a. We start as in the cited papers, that is, to be able to benefit from algebraic stability, we write

$$\begin{aligned} |M_{n+1}e_{n+1}|_{M_{n+1}^{-1}}^2 = {} & |M_n e_n + \tau \sum_{j=1}^{s} b_j \dot{E}_{nj}|_{M_{n+1}^{-1}}^2 \\ & - 2\Big\langle M_n e_n + \tau \sum_{j=1}^{s} b_j \dot{E}_{nj} \mid M_{n+1}^{-1} \mid \delta_{n+1} \Big\rangle + |\delta_{n+1}|_{M_{n+1}^{-1}}^2, \end{aligned}$$

and by expressing $M_n e_n$ from the R–K method (17a), for the first term, we obtain

$$\begin{aligned} |M_n e_n + \tau \sum_{j=1}^{s} b_j \dot{E}_{nj}|_{M_{n+1}^{-1}}^2 = {} & |M_n e_n|_{M_{n+1}^{-1}}^2 + 2\tau \sum_{j=1}^{s} b_j \langle \dot{E}_{nj} | M_{n+1}^{-1} | M_{nj}E_{nj} + \Delta_{nj} \rangle \\ & + \tau^2 \sum_{i=1}^{s}\sum_{j=1}^{s} (b_i b_j - b_i a_{ij} - b_j a_{ji}) \langle \dot{E}_{ni} | M_{n+1}^{-1} | \dot{E}_{nj} \rangle, \end{aligned}$$

where the last term is nonpositive by algebraic stability. The middle term is rewritten as

$$\langle \dot{E}_{nj} | M_{n+1}^{-1} | M_{nj} E_{nj} + \Delta_{nj} \rangle = \langle \dot{E}_{nj} | M_{nj}^{-1} | M_{nj} E_{nj} + \Delta_{nj} \rangle + \langle \dot{E}_{nj} | M_{n+1}^{-1} - M_{nj}^{-1} | M_{nj} E_{nj} + \Delta_{nj} \rangle. \tag{20}$$

All the terms in the above equations can be estimated identically as in the mentioned proofs, except the first term in (20).

b. To estimate this term, including the nonlinearity, we use Proposition 2.1 [i.e., the inequalities (2), (3), and (4)], like in [4]. Using (18), the internal stages, give

$$\begin{aligned}
\langle \dot{E}_{nj} | M_{nj}^{-1} | M_{nj} E_{nj} + \Delta_{nj} \rangle &= \langle \dot{E}_{nj} | E_{nj} \rangle + \langle \dot{E}_{nj} | M_{nj}^{-1} \Delta_{nj} \rangle \\
&= -\langle A(\alpha_{nj}) E_{nj} | E_{nj} \rangle_{t_{n+1}} - \langle A(\alpha_{nj}) E_{nj} | M_{nj}^{-1} | \Delta_{nj} \rangle \\
&\quad - \langle (A(\alpha_{nj}) - A(\tilde{\alpha}_{nj})) \tilde{\alpha}_{nj} | E_{nj} + M_{nj}^{-1} \Delta_{nj} \rangle \\
&\quad - \langle M_{nj} r_{nj} | E_{nj} + M_{nj}^{-1} \Delta_{nj} \rangle
\end{aligned}$$

Using the results of Proposition 2.1 and that $\tilde{\alpha}_{nj} = u(., t_n + c_j \tau)$ is assumed to be in $\mathcal{S}(t_n + c_j \tau)$, we can estimate as follows (using Cauchy–Schwarz and Young's inequality)

$$\begin{aligned}
|\langle \dot{E}_{nj} | M_{nj}^{-1} | M_{nj} E_{nj} + \Delta_{nj} \rangle| &\leq -\mathbf{m} |E_{nj}|_{\mathbf{A}_{nj}}^2 + \mathcal{M} |E_{nj}|_{\mathbf{A}_{nj}} |M_{nj}^{-1} \Delta_{nj}|_{\mathbf{A}_{nj}} \\
&\quad + L |E_{nj}|_{M_{nj}} |E_{nj} + M_{nj}^{-1} \Delta_{nj}|_{\mathbf{A}_{nj}} \\
&\quad + |\langle M_{nj} r_{nj} | E_{nj} + M_{nj}^{-1} \Delta_{nj} \rangle| \\
&\leq -\frac{\alpha}{4} |E_{nj}|_{\mathbf{A}_{nj}}^2 + C |M_{nj}^{-1} \Delta_{nj}|_{\mathbf{A}_{nj}}^2 + C |M_{nj}^{-1} \Delta_{nj}|_{M_{nj}}^2 \\
&\quad + C |E_{nj}|_{M_{nj}}^2 + C || M_{nj} r_{nj} ||_{*, t_{nj}}^2 .
\end{aligned}$$

As the right-hand side of this estimate is the same as in the cited proofs, it can be finished in the exact same way as in the mentioned references. ∎

Then, using the above stability results, the error bounds are following analogously as in [6] (Theorem 8.1) (or [14] (Theorem 5.1).

Theorem 4.1. *Consider the quasilinear parabolic problem (1), having a solution in $\mathcal{S}(t)$ for $0 \leq t \leq T$. Couple the ESFEM as space discretization with time discretization by an s-stage implicit R–K method satisfying Assumption 4.1. Assume that the Ritz map of the solution has continuous discrete material derivatives up to order q + 2. Then there exists $\tau_0 > 0$, independent of h, such that for $\tau \leq \tau_0$, for the error $\tau \leq \tau_0$ the following estimate holds for $t_n = n\tau \leq T$:*

$$\begin{aligned}
&|| E_h^n ||_{L^2(\Gamma_h(t_n))} + \left(\tau \sum_{j=1}^{n} || \nabla_{\Gamma_h(t_j)} E_h^j ||_{L^2(\Gamma_h(t_j))}^2 \right)^{\frac{1}{2}} \\
&\quad \leq C \tilde{\beta}_{h,q} \tau^{q+1} + C \left(\tau \sum_{k=0}^{n-1} \sum_{i=1}^{s} || R_h(., t_k + c_i \tau) ||_{H_h^{-1}(\Gamma_h(t_k + c_i \tau))}^2 \right)^{\frac{1}{2}} + C || E_h^0 ||_{L^2(\Gamma_h(0))},
\end{aligned}$$

where the constant C is independent of h, τ, and n (but depends on $\mathbf{m}$, $\mathcal{M}$, L, μ, κ, and T). Furthermore

$$\tilde{\beta}_{h,q}^2 = \int_0^T \sum_{\ell=1}^{q+2} ||(\partial_h^\bullet)^{(\ell)}(\mathcal{P}_h u)(.,t)||_{L^2(\Gamma_h(t))}\mathrm{d}t$$
$$+ \int_0^T \sum_{\ell=1}^{q+1} ||\nabla_{\Gamma_h(t)}(\partial_h^\bullet)^{(\ell)}(\mathcal{P}_h u)(.,t)||_{L^2(\Gamma_h(t))}\mathrm{d}t.$$

The H_h^{-1}-norm of R_h is defined as

$$||R_h(.,t)||_{H_h^{-1}(\Gamma_h(t))} := \sup_{0\neq\phi_h\in S_h(t)} \frac{\langle R_h(.,t),\phi_h\rangle_{L^2(\Gamma_h(t))}}{||\phi_h||_{H^1(\Gamma_h(t))}}. \tag{21}$$

Proof. Our proof is almost the same as the one for Theorem 5.1 in [14].
We estimate the terms of the right-hand side of (19). At first we connect $||.||_{*,t}$ and $||.||_{H_h^{-1}(\Gamma_h(t))}$:

$$\begin{aligned}
||Mr||_* &= (r^T M(\mathbf{A}+M)^{-1}Mr)^{1/2} = ||(\mathbf{A}+M)^{-1/2}Mr||_2 \\
&= \sup_{0\neq w\in\mathbb{R}^N} \frac{r^T(\mathbf{A}+M)^{-1/2}w}{w^T w} = \sup_{0\neq z\in\mathbb{R}^N} \frac{r^T M z}{(z^T(\mathbf{A}+M)z)^{1/2}} \\
&= \sup_{0\neq\phi_h\in S_h} \frac{\langle R_h,\phi_h\rangle_{L^2(\Gamma_h)}}{||\phi_h||_{H^1\Gamma_h}} = ||R_h||_{H_h^{-1}\Gamma_h}.
\end{aligned}$$

By Taylor expansion, the definition of stage and classical order, and with the bounded Peano kernels K and K_i, the defects satisfy

$$\begin{aligned}
\delta_{n+1} &= \tau^{q+1}\int_{t_n}^{t_{n+1}} K\left(\frac{t-t_n}{\tau}\right)(M\tilde{\alpha})^{(q+2)}(t)\mathrm{d}t, \\
\Delta_{ni} &= \tau^{q}\int_{t_n}^{t_{n+1}} K_i\left(\frac{t-t_n}{\tau}\right)(M\tilde{\alpha})^{(q+1)}(t)\mathrm{d}t,
\end{aligned}$$

hence, by a simple but lengthy calculation (given in detail in [14]) the following bound is obtained:

$$\tau\sum_{k=1}^{n}|\frac{\delta_k}{\tau}|_{M_k}^2 + C\tau\sum_{k=0}^{n-1}\sum_{i=1}^{s}(|M_{ki}^{-1}\Delta_{ki}|_{M_{ki}}^2 + |M_{ki}^{-1}\Delta_{ki}|_{\mathbf{A}_{ki}}^2) \leq C\tilde{\beta}_{h,q}^2(\tau^{q+1})^2,$$

and therefore, by inserting everything into (19), the proof is completed. ∎

B. Backward Differentiation Formulae

We apply a k-step backward difference formula (BDF) for $k \leq 5$ as a discretization to the ODE system (8), coming from the ESFEM space discretization of the quasilinear parabolic evolving surface PDE. Both implicit and linearly implicit methods are discussed.

In the following, we extend the stability result for BDF methods of [7] (Lemma 4.1), to the case quasilinear problems. Apart from the properties of the ESFEM the proof is based on Dahlquist's G–stability theory [38] and on the multiplier technique of Nevanlinna and Odeh [39].

We recall the k-step BDF method for (8) with step size $\tau > 0$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A(\alpha_n)\alpha_n = 0, \qquad (n \geq k), \tag{22}$$

where the coefficients of the method are given by $\delta(\zeta) = \sum_{j=0}^{k}\delta_j\zeta^j = \sum_{\ell=1}^{k}\frac{1}{\ell}(1-\zeta)^\ell$, while the starting values are $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$. The method is known to be 0-stable for $k \leq 6$ and have order k (for more details, see [36; Chapter V]).

The linearly implicit modification is, using the polynomial $\gamma(\zeta) = \sum_{j=1}^{k}\gamma_j\zeta^j = \zeta^k - (\zeta-1)^{k-1}$:

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\alpha_{n-j} + A\left(\sum_{j=1}^{k}\gamma_j\alpha_{n-j}\right)\alpha_n = 0, \qquad (n \geq k). \tag{23}$$

For more details we refer to [5], in particular for existence and uniqueness of the BDF solution see Section III.A in [5].

Instead of (8) let us consider again the perturbed problem (16). By substituting the true solution $\tilde{\alpha}(t)$ of the perturbed problem into the BDF method (22), we obtain

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j M(t_{n-j})\tilde{\alpha}_{n-j} + A(\tilde{\alpha}_n)\tilde{\alpha}_n = -d_n, \qquad (n \geq k).$$

By introducing the error $e_n = \alpha_n - \tilde{\alpha}(t_n)$, multiplying by τ, and by subtraction we have the error equation

$$\sum_{j=0}^{k}\delta_j M_{n-j}e_{n-j} + \tau A(\alpha_n)e_n + \tau(A(\alpha_n) - A(\tilde{\alpha}_n))\tilde{\alpha}_n = \tau d_n, \quad (n \geq k). \tag{24}$$

In the linearly implicit case we obtain:

$$\begin{aligned}\sum_{j=0}^{k}\delta_j M_{n-j}e_{n-j} + \tau A\left(\sum_{j=1}^{k}\gamma_j\alpha_{n-j}\right)e_n + \tau\Bigg(A\left(\sum_{j=1}^{k}\gamma_j\alpha_{n-j}\right)\\ - A\left(\sum_{j=1}^{k}\gamma_j\tilde{\alpha}_{n-j}\right)\Bigg)\tilde{\alpha}_n = \tau\hat{d}_n, \quad (n \geq k),\end{aligned}$$

where $\hat{d}_n$ have similar properties as d_n, therefore it will be also denoted by d_n.

The stability results for BDF methods are the following.

Lemma 4.2. *For a k-step implicit or linearly implicit BDF method with $k \leq 5$ there exists a $\tau_0 > 0$, such that for $\tau \leq \tau_0$ and $t_n = n\tau \leq T$, that the error e_n is bounded by*

$$|e_n|_{M_n}^2 + \tau\sum_{j=k}^{n}|e_j|_{\mathbf{A}_j}^2 \leq C\tau\sum_{j=k}^{n}||d_j||_{*,t_j}^2 + C\max_{0\leq i\leq k-1}|e_i|_{M_i}^2$$

where $\|w\|_{*,t}^2 = w^T(\mathbf{A}(t) + M(t))^{-1}w$. *The constant* C *is independent of* h, τ, *and* n *(but depends on* $\mathbf{m}$, $\mathcal{M}$, L, μ, κ *and* T*)*

Proof. The proof follows the proof of Lemma 4.1 from [7], and [5] Section VI, using G-stability from [38] and multiplier techniques from [39]. Except in those terms where the nonlinearity appears, see Theorem 1 in [5].

(a) The starting point of the proof is the following reformulation of the error equation (24):

$$M_n \sum_{j=0}^{k} \delta_j e_{n-j} + \tau A(\alpha_n) e_n + \tau (A(\alpha_n) - A(\tilde{\alpha}_n))\tilde{\alpha}_n = \tau d_n + \sum_{j=1}^{k} \delta_j (M_n - M_{n-j}) e_{n-j}$$

and using a modified energy estimate. Following [39], we multiply both sides with the multiplier $e_n - \eta e_{n-1}$, where the smallest possible values of η is found to be $\eta = 0$, 0, 0.0836, 0.2878, 0.8160 for $k = 1, 2, \dots, 5$, respectively, cf. [39]. This gives us, for $n \geq k+1$:

$$I_n + II_n^1 + II_n^2 = III_n + IV_n,$$

where

$$\begin{aligned}
I_n &= \left\langle \sum_{j=0}^{k} \delta_j e_{n-j} | M_n | e_n - \eta e_{n-1} \right\rangle, \\
II_n^1 &= \tau \langle e_n | A(\alpha_n) | e_n - \eta e_{n-1} \rangle, \\
II_n^2 &= \tau \langle (A(\alpha_n) - A(\tilde{\alpha}_n))\tilde{\alpha}_n | e_n - \eta e_{n-1} \rangle, \\
III_n &= \tau \langle d_n | e_n - \eta e_{n-1} \rangle, \\
IV_n &= \sum_{j=1}^{k} \langle e_{n-j} | M_n - M_{n-j} | e_n - \eta e_{n-1} \rangle.
\end{aligned}$$

We only have to estimate these terms in a suitable way.

(b) We start by bounding the nonlinear terms. First, we will estimate II_n^1 from below using (2) and Lemma 3.1:

$$\begin{aligned}
\tau^{-1} II_n^1 &= \langle e_n | A(\alpha_n) | e_n \rangle - \eta |\langle e_n | A(\alpha_n) | e_{n-1} \rangle| \\
&\geq \mathbf{m} |e_n|_{\mathbf{A}_n}^2 - \mathcal{M}\, \eta |e_n|_{\mathbf{A}_n} |e_{n-1}|_{\mathbf{A}_n} \\
&\geq \left(\mathbf{m} - \frac{\mathbf{m}}{4}\eta\right) |e_n|_{\mathbf{A}_n}^2 - \frac{1}{\mathbf{m}} \mathcal{M}^2 \eta (1 + 2\kappa\tau) |e_{n-1}|_{\mathbf{A}_{n-1}}^2.
\end{aligned}$$

The other term is estimated using (4), Young's inequality, and again by Lemma 3.1:

$$\begin{aligned}
\tau^{-1} II_n^2 &\geq -|\langle (A(\alpha_n) - A(\tilde{\alpha}_n))\tilde{\alpha}_n | e_n - \eta e_{n-1} \rangle| \\
&\geq -L |e_n|_{M_n} (|e_n|_{\mathbf{A}_n} + \eta |e_{n-1}|_{\mathbf{A}_n}) \\
&\geq -\frac{\mathbf{m}}{4} |e_n|_{\mathbf{A}_n}^2 - \frac{1}{\mathbf{m}} L^2 |e_n|_{M_n}^2 - \frac{L}{2} \eta |e_n|_{M_n}^2 + \frac{L}{2} \eta (1 + 2\kappa\tau) |e_{n-1}|_{\mathbf{A}_{n-1}}^2.
\end{aligned}$$

Combined, and using that $0 \leq \eta < 1$, we have

$$II_n^1 + II_n^2 \geq \tau \frac{1}{2}\mathbf{m}|e_n|^2_{\mathbf{A}_n} - \tau\eta\left(\frac{1}{\mathbf{m}}L^2 + \frac{L}{2}\right)|e_n|^2_{M_n}$$
$$- \tau\eta\left(\frac{1}{\mathbf{m}}\mathcal{M}^2 + \frac{L}{2}\right)(1+2\kappa\tau)|e_{n-1}|^2_{\mathbf{A}_{n-1}}$$

The estimations of I_n, III_n and IV_n are the same as in the proof in [13], with G-stability of [38] as the main tool.

(c) Combining all estimates and summing up gives, for $\tau \leq \tau_0$ and for $n \geq k+1$:

$$|E_n|^2_{G,n} + \frac{\mathbf{m}}{4}\tau \sum_{j=k+1}^{n} |e_j|^2_{\mathbf{A}_j} \leq C\tau \sum_{j=k}^{n-1} |E_j|^2_{G,j} + C\tau \sum_{j=k+1}^{n} ||d_j||^2_{*,j} + C\eta\tau|e_k|^2_{\mathbf{A}_k},$$

where $E_n = (e_n, \ldots, e_{n-k+1})$, and the $|E_n|^2_{G,n} := \sum_{i,j=1}^k g_{ij}\langle e_{n-k+i}|M_n|e_{n-k+j}\rangle$. This is the same inequality as in [7], hence we can also proceed with the discrete Gronwall inequality.

(d) To achieve the stated result we have to estimate the extra term $C\,(|e_k|^2_{M_k} + \tau|e_k|^2_{\mathbf{A}_k})$. For that we take the inner product of the error equation for $n=k$ with e_k to obtain Similarly as for II_n^i, use the properties of operator A and Lemma 3.1, yields

$$|e_k|^2_{M_k} + \tau\rho|e_k|^2_{\mathbf{A}_k} \leq C\tau||d_k||^2_{*,k} + C\max_{0\leq i\leq k-1}|e_i|^2_{M_i}.$$

The insertion of this completes the proof.

The result follows from analogous arguments for linearly implicit methods, cf. [5, Section VI]. ■

Again, using the above stability results, the error bounds are following analogously as in [7] (Theorem 5.1) or [14] (Theorem 5.3).

Theorem 4.2. *Consider the quasilinear parabolic problem (1), having a solution in $\mathcal{S}(t)$ for $0 \leq t \leq T$. Couple the ESFEM as space discretization with time discretization by a k-step implicit or linearly implicit backward difference formula of order $k \leq 5$. Assume that the Ritz map of the solution has continuous discrete material derivatives up to order k + 1. Then there exists $\tau_0 > 0$, independent of h, such that for $\tau \leq \tau_0$, for the error $E_h^n = U_h^n - \mathcal{P}_h u(.,t_n)$ the following estimate holds for $t_n = n\tau \leq T$:*

$$||E_h^n||_{L^2(\Gamma_h(t_n))} + \left(\tau\sum_{j=1}^{n}||\nabla_{\Gamma_h(t_j)}E_h^j||^2_{L^2(\Gamma_h(t_j))}\right)^{\frac{1}{2}}$$
$$\leq C\tilde{\beta}_{h,k}\tau^k + \left(\tau\sum_{j=1}^{n}||R_h(.,t_j)||^2_{H_h^{-1}(\Gamma_h(t_j))}\right)^{\frac{1}{2}} + C\max_{0\leq i\leq k-1}||E_h^i||_{L^2(\Gamma_h(t_i))},$$

where the constant C is independent of h, n, and τ (but depends on $\mathbf{m}$, $\mathcal{M}$, L, μ, κ, and T). Furthermore

$$\tilde{\beta}_{h,k}^2 = \int_0^T \sum_{\ell=1}^{k+1} ||(\partial_h^\bullet)^{(\ell)}(\mathcal{P}_h u)(.,t)||_{L^2(\Gamma_h(t))}\mathrm{d}t.$$

Proof. The proof of this result is analogous to that of Theorem 4.1, it uses the norm identity, and bounded Peano kernels. For details see the above references. ■

V. ERROR BOUNDS FOR THE FULLY DISCRETE SOLUTIONS

We follow the approach of [7] (Section V) by defining the FEM residual $R_h(.,t) = \sum_{j=1}^N r_j(t)\chi_j(.,t) \in S_h(t)$ as

$$\int_{\Gamma_h} R_h\phi_h = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma_h} \tilde{\mathcal{P}}_h u\phi_h + \int_{\Gamma_h} \mathcal{A}(\tilde{\mathcal{P}}_h u)\nabla_\Gamma(\tilde{\mathcal{P}}_h u)\cdot\nabla_\Gamma\phi_h - \int_{\Gamma_h}(\tilde{\mathcal{P}}_h u)\partial_h^\bullet\phi_h, \tag{25}$$

where $\phi_h \in S_h(t)$, and the Ritz map of the true solution u is given as

$$\tilde{\mathcal{P}}_h u(.,t) = \sum_{j=1}^N \tilde{\alpha}_j(t)\chi_j(.,t).$$

The above problem is equivalent to the ODE system with the vector $r(t) = (r_j(t)) \in \mathbb{R}^N$:

$$\frac{\mathrm{d}}{\mathrm{d}t}(M(t)\tilde{\alpha}(t)) + A(\tilde{\alpha}(t))\tilde{\alpha}(t) = M(t)r(t),$$

which is the perturbed ODE system (16).

A. Bound of the Semidiscrete Residual

We now show the optimal second order estimate of the residual R_h.

Theorem 5.1. *Let u, the solution of the parabolic problem, be in $\mathcal{S}(t)$ for $0 \le t \le T$. Then there exists a constant $C>0$ and $h_0 > 0$, such that for all $h \le h_0$ and $t \in [0,T]$, the finite element residual R_h of the Ritz map is bounded as*

$$||R_h||_{H^{-1}(\Gamma_h(t))} \le ch^2.$$

Proof. (a) We start by applying the discrete transport property to the residual Eq. (24)

$$\begin{aligned} m_h(R_h,\phi_h) &= \frac{\mathrm{d}}{\mathrm{d}t}m_h(\tilde{\mathcal{P}}_h u,\phi_h) + a_h(\tilde{\mathcal{P}}_h u;\tilde{\mathcal{P}}_h u,\phi_h) - m_h(\tilde{\mathcal{P}}_h u,\partial_h^\bullet\phi_h) \\ &= m_h(\partial_h^\bullet\tilde{\mathcal{P}}_h u,\phi_h) + a_h(\tilde{\mathcal{P}}_h u;\tilde{\mathcal{P}}_h u,\phi_h) + g_h(V_h;\tilde{\mathcal{P}}_h u,\phi_h). \end{aligned}$$

(b) We continue by the transport property with discrete material derivatives from Lemma 3.4, but for the weak form, with $\varphi := \varphi_h = (\phi_h)^l$:

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}m(u,\varphi_h) + a(u;u,\varphi_h) - m(u,\partial^\bullet\varphi_h) \\ &= m(\partial_h^\bullet u,\varphi_h) + a(u;u,\varphi_h) + g(v_h;u,\varphi_h) + m(u,\partial_h^\bullet\varphi_h - \partial^\bullet\varphi_h). \end{aligned}$$

(c) Subtraction of the two equations, using the definition of the Ritz map with $\xi = u$ in (12), that is,

$$a_h^*(u^{-l}; \tilde{\mathcal{P}}_h u, \phi_h) = a^*(u; u, \varphi_h),$$

and using that

$$\partial_h^\bullet \varphi_h - \partial^\bullet \varphi_h = (v_h - v) \cdot \nabla_\Gamma \varphi_h$$

holds, we obtain

$$\begin{aligned} m_h(R_h, \phi_h) = {} & m_h(\partial_h^\bullet \tilde{\mathcal{P}}_h u, \phi_h) - m(\partial_h^\bullet u, \varphi_h) + g_h(V_h; \tilde{\mathcal{P}}_h u, \phi_h) - g(v_h; u, \varphi_h) \\ & + a_h^*(\tilde{\mathcal{P}}_h u; \tilde{\mathcal{P}}_h u, \phi_h) - a_h^*(u^{-l}; \tilde{\mathcal{P}}_h u, \phi_h) + m(u, \varphi_h) - m_h(\tilde{\mathcal{P}}_h u, \phi_h) \\ & + m(u, (v_h - v) \cdot \nabla_\Gamma \varphi_h). \end{aligned}$$

All the pairs can be easily estimated separately as $ch^2 ||\varphi_h||_{L^2(\Gamma(t))}$, by combining the estimates of Lemma 3.5, and Theorem 3.1 and 3.2, except the third, and the last term.

The term containing the velocity difference $(v_h - v)$ can be estimated, using $|v_h - v| + h|\nabla_\Gamma(v_h - v)| \leq ch^2$ from [10, Lemma 5.6], as $ch^2 ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))}$.

The nonlinear terms are rewritten as:

$$\begin{aligned} a_h^*(\tilde{\mathcal{P}}_h u; \tilde{\mathcal{P}}_h u, \phi_h) - a_h^*(u^{-l}; \tilde{\mathcal{P}}_h u, \phi_h) = {} & a_h^*(\tilde{\mathcal{P}}_h u; \tilde{\mathcal{P}}_h u, \phi_h) - a^*(\mathcal{P}_h u; \mathcal{P}_h u, \varphi_h) \\ & + a^*(\mathcal{P}_h u; \mathcal{P}_h u, \varphi_h) - a^*(u; \mathcal{P}_h u, \varphi_h) \\ & + a^*(u; \mathcal{P}_h u, \varphi_h) - a_h^*(u^{-l}; \tilde{\mathcal{P}}_h u, \phi_h) \end{aligned}$$

For the first and the third term Lemma 3.5 provides an upper bound $ch^2 ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))}$ (similarly like before).

Finally, using Lemma 3.9 we obtain, similarly to (4), that the second term can be bounded as

$$\begin{aligned} |a^*(\mathcal{P}_h u; \mathcal{P}_h u, \varphi_h) - a^*(u; \mathcal{P}_h u, \varphi_h)| &= \left| \int_{\Gamma(t)} (\mathcal{A}(\mathcal{P}_h u) - \mathcal{A}(u)) \nabla_\Gamma \mathcal{P}_h u \cdot \nabla_\Gamma \varphi_h \right| \\ &\leq c\ell ||\mathcal{P}_h u - u||_{L^2(\Gamma(t))} ||\nabla_\Gamma \mathcal{P}_h u||_{L^\infty(\Gamma(t))} ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))} \\ &\leq c\ell ||\mathcal{P}_h u - u||_{L^2(\Gamma(t))}\, c\, r\, ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))} \\ &\leq c\ell r h^2 ||\nabla_\Gamma \varphi_h||_{L^2(\Gamma(t))}. \end{aligned}$$

Therefore, by (21), and using the equivalence of norms [8] ($\phi_h^l = \varphi_h$), we have

$$||R_h(.,t)||_{H_h^{-1}(\Gamma_h(t))} = \sup_{0 \neq \phi_h \in S_h(t)} \frac{m_h(R_h(.,t), \phi_h)}{||\phi_h||_{H^1(\Gamma_h(t))}} \leq ch^2 \frac{||\varphi_h||_{H^1(\Gamma(t))}}{||\phi_h||_{H^1(\Gamma_h(t))}} \leq ch^2.$$

■

B. Error Estimates for the Full Discretizations

We compare the lifted fully discrete numerical solution $u_h^n := (U_h^n)^l$ with the exact solution $u(., t_n)$ of the evolving surface PDE (1), where $U_h^n = \sum_{j=1}^N \alpha_j^n \chi_j(., t)$, where the vectors α^n are generated by a R–K or a BDF method.

Theorem 5.2 (ESFEM and R–K). *Consider the ESFEM as space discretization of the quasilinear parabolic problem (1), with time discretization by an s-stage implicit R–K method satisfying Assumption 4.1. Let u be a sufficiently smooth solution of the problem, which satisfies $u(., t) \in \mathcal{S}(t)$ $(0 \leq t \leq T)$, and assume that the initial value is approximated as*

$$||u_h^0 - (\mathcal{P}_h u)(., 0)||_{L^2(\Gamma(0))} \leq C_0 h^2.$$

Then, there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leq h_0$ and $\tau \leq \tau_0$, the following error estimate holds for $t_n = n\tau \leq T$:

$$||u_h^n - u(., t_n)||_{L^2(\Gamma(t_n))} + h\left(\tau \sum_{j=1}^n ||\nabla_{\Gamma(t_j)}(u_h^j - u(., t_j))||^2_{L^2(\Gamma(t_j))}\right)^{\frac{1}{2}} \leq C(\tau^{q+1} + h^2).$$

The constant C is independent of h, τ, and n, but depends on $\mathbf{m}$, $\mathcal{M}$, *and L, from (2), (3) and (4), on μ, κ, from Lemma 3.1, and on T.*

Theorem 5.3 (ESFEM and BDF). *Consider the ESFEM as space discretization of the quasilinear parabolic problem (1), with time discretization by a k-step implicit or linearly implicit backward difference formula of order $k \leq 5$. Let u be a sufficiently smooth solution of the problem, which satisfies $u(., t) \in \mathcal{S}(t)$ $(0 \leq t \leq T)$, and assume that the starting values are satisfying*

$$\max_{0 \leq i \leq k-1} ||u_h^i - (\mathcal{P}_h u)(., t_i)||_{L^2(\Gamma(0))} \leq C_0 h^2.$$

Then, there exists $h_0 > 0$ and $\tau_0 > 0$, such that for $h \leq h_0$ and $\tau \leq \tau_0$, the following error estimate holds for $t_n = n\tau \leq T$:

$$||u_h^n - u(., t_n)||_{L^2(\Gamma(t_n))} + h\left(\tau \sum_{j=1}^n ||\nabla_{\Gamma(t_j)}(u_h^j - u(., t_j))||^2_{L^2(\Gamma(t_j))}\right)^{\frac{1}{2}} \leq C(\tau^k + h^2).$$

The constant C is independent of h, τ, and n, but depends on $\mathbf{m}$, $\mathcal{M}$, *and L, from (2), (3) and (4), on μ, κ, from Lemma 3.1, and on T.*

Proof of Theorem 5.2–5.3. The global error is decomposed into two parts:

$$u_h^n - u(., t_n) = (u_h^n - (\mathcal{P}_h u)(., t_n)) + ((\mathcal{P}_h u)(., t_n) - u(., t_n)),$$

and the terms are estimated by previous results.

The first one is estimated by our results for R–K or BDF methods: Theorem 4.1 or 4.2, respectively, together with the residual bound Theorem 5.1, and by the Ritz error estimates Theorem 3.1 and 3.2.

The second term is estimated by the error estimates for the Ritz map (Theorem 3.1 and 3.2). ■

VI. SEMILINEAR PROBLEMS EXTENSION

The presented results, in particular Theorem 5.2 and 5.3, can be generalized to semilinear problems. Convergence results for BDF method were already shown for semilinear problems in [5]. For the analogous results for R–K methods follow [4] (Remark 1.1). Problems fitting into this framework can be found in the references given in the introduction.

Following Remark 1.1 from [4], the inhomogeneity $f(t)$ in the evolving surface PDE (1) can be replaced by $f(t, u)$ satisfying a local Lipschitz condition [similar to (4)]: for every $\delta > 0$ there exists $L = L(\delta, r)$ such that

$$||f(t, w_1) - f(t, w_2)||_{V(t)'} \leq \delta ||w_1 - w_2||_{V(t)} + L||w_1 - w_2||_{H(t)} \quad (0 \leq t \leq T)$$

holds for arbitrary $w_1, w_2 \in V(t)$ with $||w_1||_{V(t)}, ||w_2||_{V(t)} \leq r$, uniformly in t. Such a condition can be satisfied using the same $\mathcal{S}$ set as for quasilinear problems.

To be precise: In this case the bilinear form $a(t; ., .)$ is not depending on ξ, it reduces to the case presented in [10]. Therefore, Section III here would reduce to recall results mainly from [9, 10]. There is no ξ dependency in the definition of the generalized Ritz map, hence, it is the one appeared in [13, 14] together with the error bounds presented there. The regularity result of the Ritz map is still needed from Section III.G.

The stability estimates for the R–K and BDF methods are needed to be revised in a straightforward way, cf. [4, 5], respectively. To give more insight we give some details in the case of BDF methods. R–K methods can be handled in a similar way.

The error equation for the semilinear problem reads as

$$\sum_{j=0}^{k} \delta_j M_{n-j} e_{n-j} + \tau A_n e_n = \tau (f(t_n, \alpha_n) - f(t_n, \tilde{\alpha}_n)) + \tau d_n, \quad (n \geq k).$$

After testing with the multiplier $e_n - \eta e_{n-1}$ we obtain

$$I_n + II_n = III_n + IV_n + V_n.$$

The new nonlinear term is now estimated as

$$\begin{aligned}
\tau^{-1}|V_n| &= |\langle f(t_n, \alpha_n) - f(t_n, \tilde{\alpha}_n)|e_n - \eta e_{n-1}\rangle| \\
&\leq ||f(t, \alpha_n) - f(t, \tilde{\alpha}_n)||_{H_h^{-1}(\Gamma_h(t))} ||e_n - \eta e_{n-1}||_{A_n} \\
&\leq (\delta ||e_n||_{A_n} + L||e_n||_{M_n})(||e_n||_{A_n} + \eta ||e_{n-1}||_{A_n}) \\
&\leq 2\delta ||e_n||_{A_n}^2 + C\eta ||e_{n-1}||_{A_n}^2 + C||e_n||_{M_n}^2.
\end{aligned}$$

The other terms are either estimated as before, or in a much simple way, for instance in the case of II_n which is now linear, cf. [7].

VII. NUMERICAL EXPERIMENTS

We present a numerical experiment for an evolving surface quasilinear parabolic problem discretized by evolving surface finite elements coupled with the backward Euler method as a time

TABLE I. Errors and EOCs in the $L^\infty(L^2)$ and $L^2(H^1)$ norms.

Level	Dof	$L^\infty(L^2)$	EOCs	$L^2(H^1)$	EOCs
1	126	0.07121892	–	0.1404349	–
2	516	0.02077452	1.78	0.0404614	1.80
3	2070	0.00540906	1.94	0.0111377	1.86
4	8208	0.00136755	1.98	0.0033538	1.73
5	32682	0.00034289	2.00	0.0011904	1.49

integrator. The fully discrete methods were implemented in DUNE-FEM [40], while the initial triangulations were generated using DistMesh [41].

The evolving surface is given by

$$\Gamma(t) = \left\{x \in \mathbb{R}^3 \mid a(t)^{-1}x_1^2 + x_2^2 + x_3^2 - 1 = 0\right\},$$

where $a(t) = 1 + 0.25\sin(2\pi t)$, see for example [9, 6, 14]. The problem is considered over the time interval $[0, 1]$. We consider the problem with the nonlinearity $\mathcal{A}(u) = 1 - \frac{1}{2}e^{-u^2/4}$. This satisfies the conditions in Assumption 2.1, as it has lower bound 1/2, upper bound 1, and its derivative $\mathcal{A}'(u) = \frac{u}{4}e^{-u^2/4}$ is also bounded, hence, $\mathcal{A}$ is Lipschitz continuous. The right-hand side f is computed as to have $u(x, t) = e^{-6t}x_1x_2$ as the true solution of the quasilinear problem

$$\begin{cases} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \nabla_{\Gamma(t)} \cdot (\mathcal{A}(u)\nabla_{\Gamma(t)}u) = f & \text{on } \Gamma(t), \\ \qquad\qquad\qquad\qquad u(.,0) = u_0 & \text{on } \Gamma(0). \end{cases}$$

The time integrations require the solution of a nonlinear system at every timestep. As it is usual for R–K methods, we used the simplified Newton iterations, cf. [36] (Section IV.H].

Let $(\mathcal{T}_k(t))_{k=1,2,\ldots,n}$ and $(\tau_k)_{k=1,2,\ldots,n}$ be a series of triangulations and timesteps, respectively, such that $2h_k \approx h_{k-1}$ and $4\tau_k = \tau_{k-1}$, with $\tau_1 = 0.1$. By e_k we denote the error corresponding to the mesh $\mathcal{T}_k(t)$ and step size τ_k. Then the experimental order of convergences (EOCs) are given as

$$EOC_k = \frac{\ln(e_k/e_{k-1})}{\ln(2)}, \qquad (k = 2, 3, \ldots, n).$$

In Table I, we report on the EOCs, for the ESFEM coupled with backward Euler method, corresponding to the norm and seminorm

$$L^\infty(L^2): \qquad \max_{1\le n\le N} ||u_h^n - u(., t_n)||_{L^2(\Gamma(t_n))},$$

$$L^2(H^1): \qquad \left(\tau \sum_{n=1}^{N} ||\nabla_{\Gamma(t_n)}(u_h^n - u(., t_n))||^2_{L^2(\Gamma(t_n))}\right)^{1/2}.$$

We computed the numerical solution using the backward Euler method coupled with ESFEM for four different meshes and a series of time steps, until the final time $T = 1$. Then we computed the errors in the discrete norm and seminorm, cf. (10), these error curves are displayed in Fig. 1. The convergence in time can be seen (note the reference line), while for sufficiently small τ the spatial error is dominating, in agreement with the theoretical results.

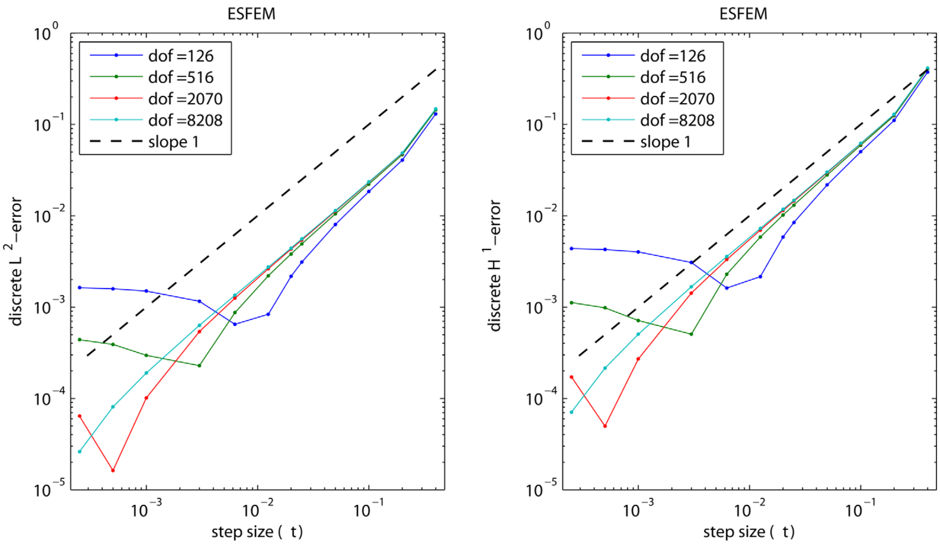

FIG. 1. Errors of the ESFEM and the backward Euler method at time $T = 1$.

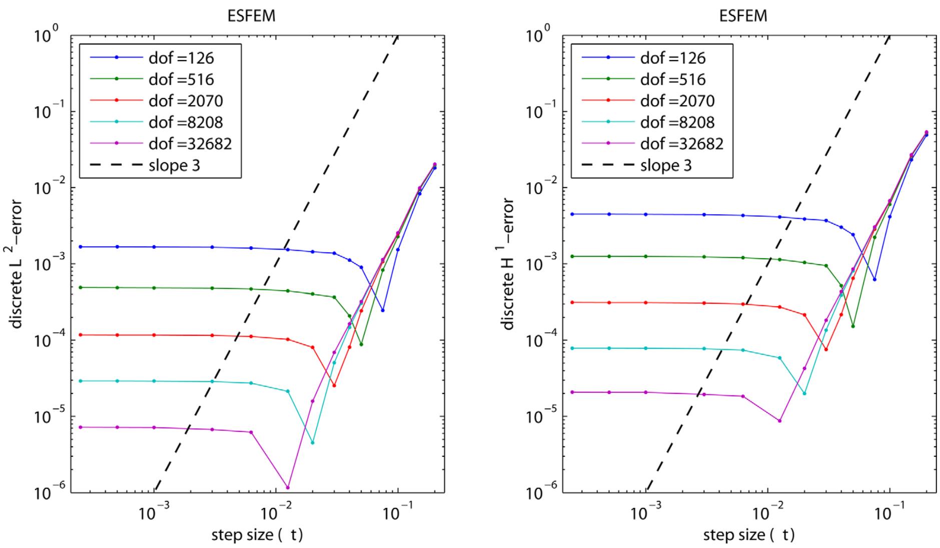

FIG. 2. Errors of the ESFEM and the three step linearly implicit BDF method at time $T = 1$.

Figure 2 shows a similar plot: the errors here were obtained by the three step linearly implicit BDF method coupled with ESFEM for five different meshes and a series of time steps. Again the results are matching with the theoretical ones.

We note that, for this example, no significant difference appeared between the fully implicit and linearly implicit BDF methods.

APPENDIX: A PRIORI ESTIMATES

The result presented here gives regularity result, with a t independent constant, for the elliptic problems appeared in the proofs of the errors in the Ritz map.

Theorem A.1 *(Elliptic regularity for evolving surfaces). i. Let $\Gamma(t)$ be an evolving surface, fix a $t \in [0, T]$ and a function $\xi : \Gamma(t) \to \mathbb{R}$.*

i. Let $f \in H^{-1}(\Gamma(t))$ and

$$L(u) := -\nabla_\Gamma \cdot (\mathcal{A}(\xi)\nabla_\Gamma u) + u. \tag{26}$$

Then, there exists a weak solution $u \in H^1(\Gamma(t))$ of the problem

$$L(u) = f \tag{27}$$

with the estimate

$$||u||_{H^1(\Gamma(t))} \leq c||f||_{H^{-1}(\Gamma(t))}, \tag{28}$$

where the constant above is independent of t.

ii. Let L(u) be (26), let $f \in L^2(\Gamma(t))$ and let $u \in H^1(\Gamma(t))$ be a weak solution of (27). Then u is a strong solution of (27), that is, u solves (27) almost everywhere and there exists a constant c > 0 independent of t and u such that

$$||u||_{H^2(\Gamma(t))} \leq c(||u||_{L^2(\Gamma(t))} + ||f||_{L^2(\Gamma(t))}).$$

Proof. For (i): The Lax–Milgram lemma shows the existence of the weak solution u. Because the coercivity and boundedness constants (2) and (3) are independent of t, the constant in (28) also not depends on t. For (ii): Basically we consider pullback of the operator L to $\Gamma(0)$, rewrite it in a local chart and then apply the corresponding results of [42].

By assumption there exists a diffeomorphic parametrization of our evolving surface $\Gamma(t)$, that is, we have a smooth map

$$\Phi : \Gamma(0) \times [0, T] \to \mathbb{R}^{m+1}$$

such that

$$\Phi_t : \Gamma(0) \to \mathbb{R}^{m+1}, \quad \Phi_t(x) := \Phi(x, t)$$

is an injective immersion which is a homeomorphism onto its image with $\Phi_t(\Gamma(0)) = \Gamma(t)$. Because $\Gamma(0)$ is compact, there exists a finite atlas

$$(\varphi_n(0) : U_n(0) \subset \Gamma(0) \to \mathbb{R}^m)_{n=1}^k$$

such that $\varphi_n(U_n(0)) \subset \mathbb{R}^m$ is bounded and a finite family of compact sets $(V_n(0))_{n=1}^k$ with $V_n(0) \subset U_n(0)$, and $\bigcup_{n=1}^k V_n(0) = \Gamma(0)$. Using the properties of the diffeomorphic parametrization the new collections,

$$V_n(t) := \Phi_t(V_n(0)), \quad U_n(t) := \Phi_t(U_n(0)), \quad \varphi_n(t) := \varphi_n(0)^\circ \Phi_t^{-1},$$

still have the same properties. Now consider the following standard formulae of Riemannian geometry [43]:

$$\nabla_\Gamma h(x,t) = \sum_{i,j=1}^{m} g_n^{ij}(x,t)\frac{\partial(h^\circ\varphi_n(t)^{-1})}{\partial x^i}\frac{\partial(\varphi_n(t)^{-1})}{\partial x^j},$$

where

$$g_{ij,n}(x,t) := \frac{\partial(\varphi_n(t)^{-1})}{\partial x^i}\cdot\frac{\partial(\varphi_n(t)^{-1})}{\partial x^j}\bigg|_x$$

is the first fundamental form and $g_n^{ij}(x,t)$ are entries of the inverse matrix of $g_n := (g_{ij,n})$, and

$$\nabla_\Gamma \cdot X = \sum_{i,j=1}^{m}\frac{1}{\sqrt{g_n}}\frac{\partial}{\partial x^i}(\sqrt{g_n}g_n^{ij}X_j)$$

where X is a smooth tangent vector field with $X_j = X\cdot\frac{\partial(\varphi(t)^{-1})}{\partial x^j}$ and $\sqrt{g_n} := \sqrt{\det(g_n)}$. It is straightforward to calculate that

$$(-\nabla_\Gamma\cdot\mathcal{A}\nabla_\Gamma u + u)^\circ\varphi_n(t)^{-1}(x) = \sum_{i,j=1}^{m} a_{ij,n}(x,t)\frac{\partial^2(u^\circ\varphi_n(t)^{-1})}{\partial x^i\partial x^j} + \sum_{i=1}^{m} b_{i,n}(x,t)\frac{\partial(u^\circ\varphi_n(t)^{-1})}{\partial x^i}$$
$$+\ c_n(x,t)u^\circ\varphi_n(t)^{-1}$$

for some appropriate functions $a_{ij,n}\in W^{1,\infty}(U_n(t))$, $b_{i,n}, c_n \in L^\infty(U_n(t))$ where $a_{ij,n}$ represents a uniform elliptic matrix. Observe that the assumptions (2), (3), and (4) implies that the function above can be bounded independently of t. Now [42] (Theorem 8.8) states that, if $u^\circ\varphi_n(t)^{-1}$ is the H^1-weak solution of (27), then it must be a strong solution as well.

For the estimate in (ii) observe that [42] (Theorem 9.11) gives us for $V_n(t)$ in particular the estimate

$$||u^\circ\varphi_n(t)^{-1}{}_{H^2(V_n'(t))} \le c(||u^\circ\varphi_n(t)^{-1}||_{L^2(U_n'(t))} + ||f^\circ\varphi_n(t)^{-1}||_{L^2(U_n'(t)))}, \tag{29}$$

where $V_n' := \varphi_n(t)(V_n(t))$ and $U_n' := \varphi_n(t)(U_n(t))$ are obviously independent of t. Thus the constant above is independent of t. Then Theorem 3.41 in [44] shows that

$$||u||_{H^2(V_n(t))} \le c(t)||u^\circ\varphi_n(t)^{-1}||_{H^2(V_n'(t))} \le c||u^\circ\varphi_n(t)^{-1}||_{H^2(V_n'(t))},$$

where the constant in the middle depends continuously on t, hence the last constant is independent of t. A similar estimate holds for the right-hand side of (29). An easy calculation finishes the proof for (ii). ■

The authors would like to thank Prof. Christian Lubich for the invaluable discussions on the topic and for his encouragement and help during the preparation of this paper. We would also like to thank Prof. Frank Loose and Christopher Nerz for our discussions.

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Appendix F. Convergence of finite elements on an evolving surface driven by diffusion on the surface

Numer. Math. (2017) 137:643–689
DOI 10.1007/s00211-017-0888-4

Numerische Mathematik

Convergence of finite elements on an evolving surface driven by diffusion on the surface

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Received: 22 July 2016 / Revised: 6 February 2017 / Published online: 27 April 2017

Abstract For a parabolic surface partial differential equation coupled to surface evolution, convergence of the spatial semidiscretization is studied in this paper. The velocity of the evolving surface is not given explicitly, but depends on the solution of the parabolic equation on the surface. Various velocity laws are considered: elliptic regularization of a direct pointwise coupling, a regularized mean curvature flow and a dynamic velocity law. A novel stability and convergence analysis for evolving surface finite elements for the coupled problem of surface diffusion and surface evolution is developed. The stability analysis works with the matrix–vector formulation of the method and does not use geometric arguments. The geometry enters only into the consistency estimates. Numerical experiments complement the theoretical results.

Mathematics Subject Classification 35R01 · 65M60 · 65M15 · 65M12

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1 Introduction

Starting from a paper by Dziuk and Elliott [10], much insight into the stability and convergence properties of finite elements on evolving surfaces has been obtained by studying a linear parabolic equation on a given moving closed surface $\Gamma(t)$. The strong formulation of this model problem is to find a solution $u(x,t)$ (for $x \in \Gamma(t)$ and $0 \le t \le T$) with given initial data $u(x,0) = u_0(x)$ to the linear partial differential equation

$$\partial^\bullet u(x,t) + u(x,t)\nabla_{\Gamma(t)} \cdot v(x,t) - \Delta_{\Gamma(t)} u(x,t) = 0, \quad x \in \Gamma(t), \quad 0 < t \le T,$$

where $\partial^\bullet$ denotes the material time derivative, $\Delta_{\Gamma(t)}$ is the Laplace–Beltrami operator on the surface, and $\nabla_{\Gamma(t)} \cdot v$ is the tangential divergence of the *given* velocity v of the surface. We refer to [12] for an excellent review article (up to 2012) on the numerical analysis of this and related problems. Optimal-order L^2 error bounds for piecewise linear finite elements are shown in [13] and maximum-norm error bounds in [23]. Stability and convergence of full discretizations obtained by combining the evolving surface finite element method (ESFEM) with various time discretizations are shown in [11,15,24]. Convergence of semi- and full discretizations using high-order evolving surface finite elements is studied in [20]. Arbitrary Euler–Lagrangian (ALE) variants of the ESFEM method for this equation are studied in [16,17,21]. Convergence properties of the ESFEM and of full discretizations for quasilinear parabolic equations on prescribed moving surfaces are studied in [22].

Beyond the above model problem, there is considerable interest in cases where the velocity of the evolving surface is *not given explicitly*, but depends on the solution u of the parabolic equation; see, e.g., [1,6,16,18] for physical and biological models where such situations arise. Contrary to the case of surfaces with prescribed motion, there exists so far no numerical analysis for solution-driven surfaces in $\mathbb{R}^3$, to the best of our knowledge.

For the case of evolving *curves* in $\mathbb{R}^2$, there are recent papers by Pozzi and Stinner [25] and Barrett et al. [2], who couple the curve-shortening flow with diffusion on the curve and study the convergence of finite element discretizations without and with a tangential part in the discrete velocity, respectively. The analogous problem for two- or higher-dimensional surfaces would be to couple mean curvature flow with diffusion on the surface. Studying the convergence of finite elements for these coupled problems, however, remains illusive as long as the convergence of ESFEM for mean curvature flow of closed surfaces is not understood. This has remained an open problem since Dziuk's formulation of such a numerical method for mean curvature flow in his 1990 paper [9].

In this paper we consider different velocity laws for coupling the surface motion with the diffusion on the surface. Conceivably the simplest velocity law would be to prescribe the normal velocity at any surface point as a function of the solution value and possibly its tangential gradient at this point: $v(x,t) = g(u(x,t), \nabla_{\Gamma(t)} u(x,t))\, \nu_{\Gamma(t)}(x)$ for $x \in \Gamma(t)$, where $\nu_{\Gamma(t)}(x)$ denotes the outer normal vector and g is a given smooth scalar-valued function. This does, however, not appear to lead to a well-posed problem, and in fact we found no mention of this seemingly obvious choice in the literature. Here we study instead a *regularized velocity law*:

$$v(x,t) - \alpha \Delta_{\Gamma(t)} v(x,t) = g\big(u(x,t), \nabla_{\Gamma(t)} u(x,t)\big)\, \nu_{\Gamma(t)}(x), \quad x \in \Gamma(t),$$

with a fixed regularization parameter $\alpha > 0$. This elliptic regularization will turn out to permit us to give a complete stability and convergence analysis of the ESFEM semidiscretization, for finite elements of polynomial degree at least two. The case of linear finite elements is left open in the theory of this paper, but will be considered in our numerical experiments. The stability and convergence results can be extended to full discretizations with linearly implicit backward difference time-stepping, as we plan to show in later work.

Our approach also applies to the ESFEM discretization of coupling a *regularized mean curvature flow* and diffusion on the surface:

$$v - \alpha \Delta_{\Gamma(t)} v = \Big(-H + g\big(u, \nabla_{\Gamma(t)} u\big)\Big) \nu_{\Gamma(t)},$$

where H denotes mean curvature on the surface $\Gamma(t)$.

The error analysis is further extended to a *dynamic velocity law*

$$\partial^\bullet v + v \nabla_{\Gamma(t)} \cdot v - \alpha \Delta_{\Gamma(t)} v = g\big(u, \nabla_{\Gamma(t)} u\big)\, \nu_{\Gamma(t)}.$$

A physically more relevant dynamic velocity law would be based on momentum and mass balance, such as incompressible Navier–Stokes motion of the surface coupled to diffusion on the surface. We expect that our analysis extends to such a system, but this is beyond the scope of this paper. Surface evolutions under Navier–Stokes equations and under Willmore flow have recently been considered in [3–5].

The paper is organized as follows.

In Sect. 2 we describe the considered problems and give the weak formulation. We recall the basics of the evolving surface finite element method and describe the semidiscrete problem. Its matrix–vector formulation is useful not only for the implementation, but will play a key role in the stability analysis of this paper.

In Sect. 3 we present the main result of the paper, which gives convergence estimates for the ESFEM semidiscretization with finite elements of polynomial degree at least 2. We further outline the main ideas and the organization of the proof.

In Sect. 4 we present auxiliary results that are used to relate different surfaces to one another. They are the key technical results used later on in the stability analysis. Section 5 contains the stability analysis for the regularized velocity law with a prescribed driving term. In Sect. 6 this is extended to the stability analysis for coupling surface PDEs and surface motion. The stability analysis works with the matrix–vector formulation of the ESFEM semidiscretization and does not use geometric arguments.

In Sect. 7 we briefly recall some geometric estimates used for estimating the consistency errors, which are the defects obtained on inserting the interpolated exact solution into the scheme. Section 8 deals with the defect estimates. Section 9 proves the main result by combining the results of the previous sections.

In Sect. 10 we give extensions to other velocity laws: the regularized mean curvature flow and the dynamic velocity law addressed above.

Section 11 presents numerical experiments that are complementary to our theoretical results in that they show the numerical behaviour of piecewise linear finite elements on some examples.

We use the notational convention to denote vectors in $\mathbb{R}^3$ by italic letters, but to denote finite element nodal vectors in $\mathbb{R}^N$ and $\mathbb{R}^{3N}$ by boldface lowercase letters and finite element mass and stiffness matrices by boldface capitals. All boldface symbols in this paper will thus be related to the matrix–vector formulation of the ESFEM.

2 Problem formulation and evolving surface finite element semidiscretization

2.1 Basic notions and notation

We consider the evolving two-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^3$ as the image

$$\Gamma(t) = \{X(p,t) : p \in \Gamma^0\}$$

of a sufficiently regular vector-valued function $X : \Gamma^0 \times [0,T] \to \mathbb{R}^3$, where Γ^0 is the smooth closed initial surface, and $X(p,0) = p$. In view of the subsequent numerical discretization, it is convenient to think of $X(p,t)$ as the position at time t of a moving particle with label p, and of $\Gamma(t)$ as a collection of such particles. To indicate the dependence of the surface on X, we will write

$$\Gamma(t) = \Gamma(X(\cdot,t)), \quad \text{or briefly} \quad \Gamma(X)$$

when the time t is clear from the context. The *velocity* $v(x,t) \in \mathbb{R}^3$ at a point $x = X(p,t) \in \Gamma(t)$ equals

$$\partial_t X(p,t) = v(X(p,t),t). \tag{2.1}$$

Note that for a known velocity field $v : \mathbb{R}^3 \times [0,T] \to \mathbb{R}^3$, the position $X(p,t)$ at time t of the particle with label p is obtained by solving the ordinary differential equation (2.1) from 0 to t for a fixed p.

For a function $u(x,t)$ ($x \in \Gamma(t)$, $0 \le t \le T$) we denote the *material derivative* as

$$\partial^\bullet u(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} u(X(p,t),t) \quad \text{for} \quad x = X(p,t).$$

At $x \in \Gamma(t)$ and $0 \le t \le T$, we denote by $\nu_{\Gamma(X)}(x,t)$ the outer normal, by $\nabla_{\Gamma(X)} u(x,t)$ the tangential gradient of u, by $\Delta_{\Gamma(X)} u(x,t)$ the Laplace–Beltrami operator applied to u, and by $\nabla_{\Gamma(X)} \cdot v(x,t)$ the tangential divergence of v; see, e.g., [12] for these notions.

2.2 Surface motion coupled to a surface PDE: strong and weak formulation

As outlined in the introduction, we consider a parabolic equation on an evolving surface that moves according to an elliptically regularized velocity law:

$$\begin{aligned} \partial^{\bullet} u + u \nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u &= f(u, \nabla_{\Gamma(X)} u), \\ v - \alpha \Delta_{\Gamma(X)} v &= g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)}. \end{aligned} \tag{2.2}$$

Here, $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are given continuously differentiable functions, and $\alpha > 0$ is a fixed parameter. This system is considered together with the collection of ordinary differential equations (2.1) for every label p. Initial values are specified for u and X.

On applying the Leibniz formula as in [10], the weak formulation reads as follows: Find $u(\cdot, t) \in W^{1,\infty}(\Gamma(X(\cdot, t)))$ and $v(\cdot, t) \in W^{1,\infty}(\Gamma(X(\cdot, t)))^3$ such that for all test functions $\varphi(\cdot, t) \in H^1(\Gamma(X(\cdot, t)))$ with $\partial^{\bullet}\varphi = 0$ and $\psi(\cdot, t) \in H^1(\Gamma(X(\cdot, t)))^3$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \int_{\Gamma(X)} u\varphi + \int_{\Gamma(X)} \nabla_{\Gamma(X)} u \cdot \nabla_{\Gamma(X)} \varphi &= \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u) \varphi, \\ \int_{\Gamma(X)} v \cdot \psi + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)} \psi &= \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u)\, \nu_{\Gamma(X)} \cdot \psi, \end{aligned} \tag{2.3}$$

alongside with the ordinary differential equations (2.1) for the positions X determining the surface $\Gamma(X)$.

We assume throughout this paper that the problem (2.2) or (2.3) admits a unique solution with sufficiently high Sobolev regularity on the time interval $[0, T]$ for the given initial data $u(\cdot, 0)$ and $X(\cdot, 0)$. We assume further that the flow map $X(\cdot, t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ is non-degenerate for $0 \le t \le T$, so that $\Gamma(t)$ is a regular surface.

2.3 Evolving surface finite elements

We describe the surface finite element discretization of our problem, following [7,8]. We use simplicial elements and continuous piecewise polynomial basis functions of degree k, as defined in [7, Section 2.5].

We triangulate the given smooth surface Γ^0 by an admissible family of triangulations $\mathcal{T}_h$ of decreasing maximal element diameter h; see [10] for the notion of an admissible triangulation, which includes quasi-uniformity and shape regularity. For a momentarily fixed h, we denote by $\mathbf{x}^0 = (x_1^0, \ldots, x_N^0)$ the vector in $\mathbb{R}^{3N}$ that collects all N nodes of the triangulation. By piecewise polynomial interpolation of degree k, the nodal vector defines an approximate surface Γ_h^0 that interpolates Γ^0 in the nodes x_j^0. We will evolve the jth node in time, denoted $x_j(t)$ with $x_j(0) = x_j^0$, and collect the nodes at time t in a vector

$$\mathbf{x}(t) = (x_1(t), \ldots, x_N(t)) \in \mathbb{R}^{3N}.$$

Provided that $x_j(t)$ is sufficiently close to the exact position $x_j^*(t) := X(p_j, t)$ (with $p_j = x_j^0$) on the exact surface $\Gamma(t) = \Gamma(X(\cdot, t))$, the nodal vector $\mathbf{x}(t)$ still corresponds to an admissible triangulation. In the following discussion we omit the omnipresent argument t and just write $\mathbf{x}$ for $\mathbf{x}(t)$ when the dependence on t is not important.

By piecewise polynomial interpolation on the plane reference triangle that corresponds to every curved triangle of the triangulation, the nodal vector $\mathbf{x}$ defines a closed surface denoted by $\Gamma_h(\mathbf{x})$. We can then define finite element *basis functions*

$$\phi_j[\mathbf{x}] : \Gamma_h(\mathbf{x}) \to \mathbb{R}, \quad j = 1, \dots, N,$$

which have the property that on every triangle their pullback to the reference triangle is polynomial of degree k, and which satisfy

$$\phi_j[\mathbf{x}](x_k) = \delta_{jk} \quad \text{for all} \quad j, k = 1, \dots, N.$$

These functions span the finite element space on $\Gamma_h(\mathbf{x})$,

$$S_h(\mathbf{x}) = \operatorname{span}\big\{\phi_1[\mathbf{x}], \phi_2[\mathbf{x}], \dots, \phi_N[\mathbf{x}]\big\}.$$

For a finite element function $u_h \in S_h(\mathbf{x})$ the tangential gradient $\nabla_{\Gamma_h(\mathbf{x})} u_h$ is defined piecewise.

We set

$$X_h(p_h, t) = \sum_{j=1}^{N} x_j(t)\, \phi_j[\mathbf{x}(0)](p_h), \quad p_h \in \Gamma_h^0,$$

which has the properties that $X_h(p_j, t) = x_j(t)$ for $j = 1, \dots, N$, that $X_h(p_h, 0) = p_h$ for all $p_h \in \Gamma_h^0$, and

$$\Gamma_h(\mathbf{x}(t)) = \Gamma(X_h(\cdot, t)).$$

The *discrete velocity* $v_h(x, t) \in \mathbb{R}^3$ at a point $x = X_h(p_h, t) \in \Gamma(X_h(\cdot, t))$ is given by

$$\partial_t X_h(p_h, t) = v_h(X_h(p_h, t), t).$$

A key property of the basis functions is the *transport property* [10]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_j[\mathbf{x}(t)](X_h(p_h, t))\Big) = 0,$$

which by integration from 0 to t yields

$$\phi_j[\mathbf{x}(t)](X_h(p_h, t)) = \phi_j[\mathbf{x}(0)](p_h).$$

This implies that the discrete velocity is simply

$$v_h(x,t) = \sum_{j=1}^{N} v_j(t)\,\phi_j[\mathbf{x}(t)](x) \quad \text{for } x \in \Gamma_h\big(\mathbf{x}(t)\big), \quad \text{with } v_j(t) = \dot{x}_j(t),$$

where the dot denotes the time derivative $\mathrm{d}/\mathrm{d}t$.

The *discrete material derivative* of a finite element function

$$u_h(x,t) = \sum_{j=1}^{N} u_j(t)\,\phi_j[\mathbf{x}(t)](x), \quad x \in \Gamma_h\big(\mathbf{x}(t)\big),$$

is defined as

$$\partial_h^{\bullet} u_h(x,t) = \frac{\mathrm{d}}{\mathrm{d}t} u_h(X_h(p_h,t),t) \quad \text{for } \; x = X_h(p_h,t).$$

By the transport property of the basis functions, this is just

$$\partial_h^{\bullet} u_h(x,t) = \sum_{j=1}^{N} \dot{u}_j(t)\,\phi_j[\mathbf{x}(t)](x), \quad x \in \Gamma_h\big(\mathbf{x}(t)\big).$$

2.4 Semidiscretization of the evolving surface problem

The finite element spatial semidiscretization of the problem (2.3) reads as follows: Find the unknown nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element functions $u_h(\cdot,t) \in S_h(\mathbf{x}(t))$ and $v_h(\cdot,t) \in S_h(\mathbf{x}(t))^3$ such that, for all $\varphi_h(\cdot,t) \in S_h(\mathbf{x}(t))$ with $\partial_h^{\bullet}\varphi_h = 0$ and all $\psi_h(\cdot,t) \in S_h(\mathbf{x}(t))^3$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Gamma_h(\mathbf{x})} u_h \varphi_h + \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} u_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \varphi_h &= \int_{\Gamma_h(\mathbf{x})} f(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h)\,\varphi_h, \\
\int_{\Gamma_h(\mathbf{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} v_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \psi_h &= \int_{\Gamma_h(\mathbf{x})} g(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h)\,\nu_{\Gamma_h(\mathbf{x})} \cdot \psi_h,
\end{aligned} \tag{2.4}$$

and

$$\partial_t X_h(p_h,t) = v_h(X_h(p_h,t),t), \quad p_h \in \Gamma_h^0. \tag{2.5}$$

The initial values for the nodal vector $\mathbf{u}$ corresponding to u_h and the nodal vector $\mathbf{x}$ of the initial positions are taken as the exact initial values at the nodes x_j^0 of the triangulation of the given initial surface Γ^0:

$$x_j(0) = x_j^0, \quad u_j(0) = u\left(x_j^0, 0\right), \quad (j = 1, \ldots, N).$$

2.5 Differential-algebraic equations of the matrix–vector formulation

We now show that the nodal vectors $\mathbf{u} \in \mathbb{R}^N$ and $\mathbf{v} \in \mathbb{R}^{3N}$ of the finite element functions u_h and v_h, respectively, together with the surface nodal vector $\mathbf{x} \in \mathbb{R}^{3N}$ satisfy a system of differential-algebraic equations (DAEs). Using the above finite element setting, we set (omitting the argument t)

$$\begin{aligned} u_h &= \sum_{j=1}^N u_j \phi_j[\mathbf{x}], \quad u_h(x_j) = u_j \in \mathbb{R}, \\ v_h &= \sum_{j=1}^N v_j \phi_j[\mathbf{x}], \quad v_h(x_j) = v_j \in \mathbb{R}^3, \end{aligned}$$

and collect the nodal values in column vectors $\mathbf{u} = (u_j) \in \mathbb{R}^N$ and $\mathbf{v} = (v_j) \in \mathbb{R}^{3N}$.

We define the surface-dependent mass matrix $\mathbf{M}(\mathbf{x})$ and stiffness matrix $\mathbf{A}(\mathbf{x})$ on the surface determined by the nodal vector $\mathbf{x}$:

$$\begin{aligned} \mathbf{M}(\mathbf{x})|_{jk} &= \int_{\Gamma_h(\mathbf{x})} \phi_j[\mathbf{x}]\phi_k[\mathbf{x}], \\ \mathbf{A}(\mathbf{x})|_{jk} &= \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h}\phi_j[\mathbf{x}] \cdot \nabla_{\Gamma_h}\phi_k[\mathbf{x}], \end{aligned} \qquad (j, k = 1, \ldots, N).$$

We further let (with the identity matrix $I_3 \in \mathbb{R}^{3\times 3}$)

$$\mathbf{K}(\mathbf{x}) = I_3 \otimes \Big(\mathbf{M}(\mathbf{x}) + \alpha \mathbf{A}(\mathbf{x})\Big). \tag{2.6}$$

The right-hand side vectors $\mathbf{f}(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^N$ and $\mathbf{g}(\mathbf{x}, \mathbf{u}) \in \mathbb{R}^{3N}$ are given by

$$\begin{aligned} \mathbf{f}(\mathbf{x}, \mathbf{u})|_j &= \int_{\Gamma_h(\mathbf{x})} f(u_h, \nabla_{\Gamma_h} u_h)\, \phi_j[\mathbf{x}], \\ \mathbf{g}(\mathbf{x}, \mathbf{u})|_{3(j-1)+\ell} &= \int_{\Gamma_h(\mathbf{x})} g(u_h, \nabla_{\Gamma_h} u_h) \left(\nu_{\Gamma_h(\mathbf{x})}\right)_\ell \phi_j[\mathbf{x}], \end{aligned}$$

for $j = 1, \ldots, N$, and $\ell = 1, 2, 3$.

We then obtain from (2.4)–(2.5) the following coupled DAE system for the nodal values $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{x}$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x})\mathbf{u}\Big) + \mathbf{A}(\mathbf{x})\mathbf{u} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{K}(\mathbf{x})\mathbf{v} &= \mathbf{g}(\mathbf{x}, \mathbf{u}), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{2.7}$$

With the auxiliary vector $\mathbf{w} = \mathbf{M}(\mathbf{x})\mathbf{u}$, this system becomes

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{v},\\ \dot{\mathbf{w}} &= -\mathbf{A}(\mathbf{x})\mathbf{u} + \mathbf{f}(\mathbf{x},\mathbf{u}),\\ \mathbf{0} &= -\mathbf{K}(\mathbf{x})\mathbf{v} + \mathbf{g}(\mathbf{x},\mathbf{u}),\\ \mathbf{0} &= -\mathbf{M}(\mathbf{x})\mathbf{u} + \mathbf{w}.\end{aligned}$$

This is of a form to which standard DAE time discretization can be applied; see, e.g., [19, Chap. VI].

As will be seen in later sections, the matrix–vector formulation is very useful in the stability analysis of the ESFEM, beyond its obvious role for practical computations.

2.6 Lifts

In the error analysis we need to compare functions on three different surfaces: the *exact surface* $\Gamma(t) = \Gamma(X(\cdot,t))$, the *discrete surface* $\Gamma_h(t) = \Gamma_h(\mathbf{x}(t))$, and the *interpolated surface* $\Gamma_h^*(t) = \Gamma_h(\mathbf{x}^*(t))$, where $\mathbf{x}^*(t)$ is the nodal vector collecting the grid points $x_j^*(t) = X(p_j,t)$ on the exact surface. In the following definitions we omit the argument t in the notation.

A finite element function $w_h : \Gamma_h \to \mathbb{R}^m$ ($m = 1$ or 3) on the discrete surface, with nodal values w_j, is related to the finite element function $\widehat{w}_h$ on the interpolated surface that has the same nodal values:

$$\widehat{w}_h = \sum_{j=1}^{N} w_j \phi_j[\mathbf{x}^*].$$

The transition between the interpolated surface and the exact surface is done by the *lift operator*, which was introduced for linear surface approximations in [8]; see also [10,13]. Higher-order generalizations have been studied in [7]. The lift operator l maps a function on the interpolated surface Γ_h^* to a function on the exact surface Γ, provided that Γ_h^* is sufficiently close to Γ.

The exact regular surface $\Gamma(X(\cdot,t))$ can be represented by a (sufficiently smooth) signed distance function $d : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$, cf. [10, Section 2.1], such that

$$\Gamma(X(\cdot,t)) = \left\{x \in \mathbb{R}^3 \mid d(x,t) = 0\right\} \subset \mathbb{R}^3. \tag{2.8}$$

Using this distance function, the lift of a continuous function $\eta_h : \Gamma_h^* \to \mathbb{R}$ is defined as

$$\eta_h^l(y) := \eta_h(x), \quad x \in \Gamma_h^*,$$

where for every $x \in \Gamma_h^*$ the point $y = y(x) \in \Gamma$ is uniquely defined via

$$y = x - \nu(y)d(x).$$

For functions taking values in $\mathbb{R}^3$ the lift is componentwise. By η^{-l} we denote the function on Γ_h^* whose lift is η.

We denote the composed lift L from finite element functions on Γ_h to functions on Γ via Γ_h^* by

$$w_h^L = (\widehat{w}_h)^l.$$

3 Statement of the main result: semidiscrete error bound

We are now in the position to formulate the main result of this paper, which yields optimal-order error bounds for the finite element semidiscretization of a surface PDE on a solution-driven surface as specified in (2.2), for finite elements of polynomial degree $k \geq 2$. We denote by $\Gamma(t) = \Gamma(X(\cdot,t))$ the exact surface and by $\Gamma_h(t) = \Gamma(X_h(\cdot,t)) = \Gamma_h(\mathbf{x}(t))$ the discrete surface at time t. We introduce the notation

$$x_h^L(x,t) = X_h^L(p,t) \in \Gamma_h(t) \quad \text{for} \quad x = X(p,t) \in \Gamma(t).$$

Theorem 3.1 *Consider the space discretization (2.4)–(2.5) of the coupled problem (2.1)–(2.2), using evolving surface finite elements of polynomial degree $k \geq 2$. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for u. Suppose that the problem admits an exact solution (u, v, X) that is sufficiently smooth (say, in the Sobolev class H^{k+1}) on the time interval $0 \leq t \leq T$, and that the flow map $X(\cdot,t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ is non-degenerate for $0 \leq t \leq T$, so that $\Gamma(t)$ is a regular surface.*

Then, there exists $h_0 > 0$ such that for all mesh widths $h \leq h_0$ the following error bounds hold over the exact surface $\Gamma(t) = \Gamma(X(\cdot,t))$ for $0 \leq t \leq T$:

$$\left(\left\| u_h^L(\cdot,t) - u(\cdot,t) \right\|_{L^2(\Gamma(t))}^2 + \int_0^t \left\| u_h^L(\cdot,s) - u(\cdot,s) \right\|_{H^1(\Gamma(s))}^2 \mathrm{d}s \right)^{\frac{1}{2}} \leq Ch^k$$

and

$$\left(\int_0^t \left\| v_h^L(\cdot,s) - v(\cdot,s) \right\|_{H^1(\Gamma(s))^3}^2 \mathrm{d}s \right)^{1/2} \leq Ch^k,$$
$$\left\| x_h^L(\cdot,t) - \mathrm{id}_{\Gamma(t)} \right\|_{H^1(\Gamma(t))^3} \leq Ch^k.$$

The constant C is independent of t and h, but depends on bounds of the H^{k+1} norms of the solution (u, v, X), on local Lipschitz constants of f and g, on the regularization parameter $\alpha > 0$ and on the length T of the time interval.

We note that the last error bound is equivalent to

$$\left\| X_h^L(\cdot,t) - X(\cdot,t) \right\|_{H^1(\Gamma^0)^3} \leq ch^k.$$

Moreover, in the case of a coupling function g in (2.2) that is independent of the solution gradient, so that $g = g(u)$, we obtain an error bound for the velocity that is pointwise in time: uniformly for $0 \le t \le T$,

$$\left\| v_h^L(\cdot, t) - v(\cdot, t) \right\|_{H^1(\Gamma(t))^3} \le Ch^k.$$

A key issue in the proof is to ensure that the $W^{1,\infty}$ norm of the position error of the curves remains small. The H^1 error bound and an inverse estimate yield an $O(h^{k-1})$ error bound in the $W^{1,\infty}$ norm. This is small only for $k \ge 2$, which is why we impose the condition $k \ge 2$ in the above result.

Since the exact flow map $X(\cdot, t) : \Gamma_0 \to \Gamma(t)$ is assumed to be smooth and non-degenerate, it is locally close to an invertible linear transformation, and (using compactness) it therefore preserves the admissibility of grids with sufficiently small mesh width $h \le h_0$. Our assumptions therefore guarantee that the triangulations formed by the nodes $x_j^*(t) = X(p_j, t)$ remain admissible uniformly for $t \in [0, T]$ for sufficiently small h (though the bounds in the admissibility inequalities and the largest possible mesh width may deteriorate with growing time). Since $k \ge 2$, the position error estimate implies that for sufficiently small h also the triangulations formed by the numerical nodes $x_j(t)$ remain admissible uniformly for $t \in [0, T]$. This cannot be concluded for $k = 1$.

The error bound will be proven by clearly separating the issues of consistency and stability. The consistency error is the defect on inserting a projection (interpolation or Ritz projection) of the exact solution into the discretized equation. The defect bounds involve *geometric estimates* that were obtained for the time dependent case and for higher order $k \ge 2$ in [20], by combining techniques of Dziuk and Elliott [10,13] and Demlow [7]. This is done with the ESFEM formulation of Sect. 2.4.

The main issue in the proof of Theorem 3.1 is to prove *stability* in the form of an h-independent bound of the error in terms of the defect. The stability analysis is done in the matrix–vector formulation of Sect. 2.5. It uses energy estimates and transport formulae that relate the mass and stiffness matrices and the coupling terms for different nodal vectors $\mathbf{x}$. *No geometric estimates* enter in the proof of stability.

In Sect. 4 we prove important auxiliary results for the stability analysis. The stability is first analysed for the discretized velocity law without coupling to the surface PDE in Sect. 5 and is then extended to the coupled problem in Sect. 6. The necessary geometric estimates for the consistency analysis are collected in Sect. 7, and the defects are then bounded in Sect. 8. The proof of Theorem 3.1 is then completed in Sect. 9 by putting together the results on stability, defect bounds and interpolation error bounds.

4 Auxiliary results for the stability analysis: relating different surfaces

The finite element matrices of Sect. 2.5 induce discrete versions of Sobolev norms. For any $\mathbf{w} = (w_j) \in \mathbb{R}^N$ with corresponding finite element function $w_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}] \in S_h(\mathbf{x})$ we note

$$\|\mathbf{w}\|^2_{\mathbf{M}(\mathbf{x})} := \mathbf{w}^T\mathbf{M}(\mathbf{x})\mathbf{w} = \|w_h\|^2_{L^2(\Gamma_h(\mathbf{x}))}, \tag{4.1}$$

$$\|\mathbf{w}\|^2_{\mathbf{A}(\mathbf{x})} := \mathbf{w}^T\mathbf{A}(\mathbf{x})\mathbf{w} = \left\|\nabla_{\Gamma_h(\mathbf{x})} w_h\right\|^2_{L^2(\Gamma_h(\mathbf{x}))}. \tag{4.2}$$

In our stability analysis we need to relate finite element matrices corresponding to different nodal vectors. We use the following setting. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3N}$ be two nodal vectors defining discrete surfaces $\Gamma_h(\mathbf{x})$ and $\Gamma_h(\mathbf{y})$, respectively. We let $\mathbf{e} = (e_j) = \mathbf{x} - \mathbf{y} \in \mathbb{R}^{3N}$. For the parameter $\theta \in [0, 1]$, we consider the intermediate surface $\Gamma_h^\theta = \Gamma_h(\mathbf{y} + \theta\mathbf{e})$ and the corresponding finite element functions given as

$$e_h^\theta = \sum_{j=1}^N e_j \phi_j[\mathbf{y} + \theta\mathbf{e}]$$

and, for any vectors $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$,

$$w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta\mathbf{e}] \quad \text{and} \quad z_h^\theta = \sum_{j=1}^N z_j \phi_j[\mathbf{y} + \theta\mathbf{e}].$$

Lemma 4.1 *In the above setting the following identities hold:*

$$\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{y}))\mathbf{z} = \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta \left(\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\right) z_h^\theta \,\mathrm{d}\theta,$$

$$\mathbf{w}^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{y}))\mathbf{z} = \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} w_h^\theta \cdot \left(D_{\Gamma_h^\theta} e_h^\theta\right) \nabla_{\Gamma_h^\theta} z_h^\theta \,\mathrm{d}\theta,$$

with $D_{\Gamma_h^\theta} e_h^\theta = \mathrm{trace}(E)I_3 - (E + E^T)$ *for* $E = \nabla_{\Gamma_h^\theta} e_h^\theta \in \mathbb{R}^{3\times 3}$.

Proof Using the fundamental theorem of calculus and the Leibniz formula we write

$$\begin{aligned}\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{y}))\mathbf{z} &= \int_{\Gamma_h(\mathbf{x})} w_h^1 z_h^1 - \int_{\Gamma_h(\mathbf{y})} w_h^0 z_h^0 = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta \mathrm{d}\theta \\ &= \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta \left(\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\right) z_h^\theta \,\mathrm{d}\theta.\end{aligned}$$

In the last formula we used that the material derivatives (with respect to θ) of w_h^θ and z_h^θ vanish, thanks to the transport property of the basis functions. The second identity is shown in the same way, using the formula for the derivative of the Dirichlet integral; see [10] and also [15, Lemma 3.1]. □

A direct consequence of Lemma 4.1 is the following conditional equivalence of norms:

Lemma 4.2 *If* $\|\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \le \mu$ *for* $0 \le \theta \le 1$, *then*

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{y}+\mathbf{e})} \le e^{\mu/2} \, \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y})}.$$

If $\|D_{\Gamma_h^\theta} e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \le \eta$ *for* $0 \le \theta \le 1$, *then*

$$\|\mathbf{w}\|_{\mathbf{A}(\mathbf{y}+\mathbf{e})} \le e^{\eta/2} \, \|\mathbf{w}\|_{\mathbf{A}(\mathbf{y})}.$$

Proof By Lemma 4.1 we have for $0 \le \tau \le 1$

$$\begin{aligned} \|\mathbf{w}\|^2_{\mathbf{M}(\mathbf{y}+\tau\mathbf{e})} - \|\mathbf{w}\|^2_{\mathbf{M}(\mathbf{y})} &= \mathbf{w}^T (\mathbf{M}(\mathbf{y}+\tau\mathbf{e}) - \mathbf{M}(\mathbf{y}))\mathbf{w} \\ = \int_0^\tau \int_{\Gamma_h^\theta} w_h^\theta \cdot \left(\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\right) w_h^\theta \mathrm{d}\theta &\le \mu \int_0^\tau \|w_h^\theta\|^2_{L^2(\Gamma_h^\theta)} \,\mathrm{d}\theta \\ = \mu \int_0^\tau \|\mathbf{w}\|^2_{\mathbf{M}(\mathbf{y}+\theta\mathbf{e})} \,\mathrm{d}\theta, \end{aligned}$$

and the first result follows from Gronwall's inequality. The second result is proved in the same way. □

The following result, when used with w_h^θ equal to components of e_h^θ, reduces the problem of checking the conditions of the previous lemma for $0 \le \theta \le 1$ to checking the condition just for the case $\theta = 0$.

Lemma 4.3 *In the above setting, assume that*

$$\left\|\nabla_{\Gamma_h[\mathbf{y}]} e_h^0\right\|_{L^\infty(\Gamma_h[\mathbf{y}])} \le \frac{1}{2}. \tag{4.3}$$

Then, for $0 \le \theta \le 1$ *the function* $w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y}+\theta\mathbf{e}]$ *on* $\Gamma_h^\theta = \Gamma[\mathbf{y}+\theta\mathbf{e}]$ *is bounded by*

$$\left\|\nabla_{\Gamma_h^\theta} w_h^\theta\right\|_{L^p(\Gamma_h^\theta)} \le c_p \left\|\nabla_{\Gamma_h^0} w_h^0\right\|_{L^p(\Gamma_h^0)} \quad \textit{for} \quad 1 \le p \le \infty,$$

where c_p *depends only on* p *(we have* $c_\infty = 2$*).*

Proof We describe the finite element parametrization of the discrete surfaces Γ_h^θ in the same way as in Sect. 2.3, with θ instead of t in the role of the time variable. We set

$$Y_h^\theta(q_h) = Y_h(q_h, \theta) = \sum_{j=1}^N (y_j + \theta e_j)\phi_j[\mathbf{y}](q_h), \quad q_h \in \Gamma_h[\mathbf{y}], \tag{4.4}$$

so that

$$\Gamma\left(Y_h^\theta\right) = \Gamma_h[\mathbf{y}+\theta\mathbf{e}] = \Gamma_h^\theta.$$

Since $Y_h^0(q_h) = q_h$ for all $q_h \in \Gamma_h^0 = \Gamma_h[\mathbf{y}]$, the above formula can be rewritten as

$$Y_h^\theta(q_h) = q_h + \theta e_h^0(q_h).$$

Tangent vectors to Γ_h^θ at $y_h^\theta = Y_h^\theta(q_h)$ are therefore of the form

$$\delta y_h^\theta = DY_h^\theta(q_h)\,\delta q_h = \delta q_h + \theta\big(\nabla_{\Gamma_h^0} e_h^0(q_h)\big)^T \delta q_h,$$

where δq_h is a tangent vector to Γ_h^0 at q_h, or written more concisely, $\delta q_h \in T_{q_h}\Gamma_h^0$.

Letting $|\cdot|$ denote the Euclidean norm of a vector in $\mathbb{R}^3$, we have at $y_h^\theta = Y_h^\theta(q_h)$

$$\begin{aligned}\left|\nabla_{\Gamma_h^\theta} w_h^\theta\left(y_h^\theta\right)\right| &= \sup_{\delta y_h^\theta \in T_{y_h^\theta}\Gamma_h^\theta} \frac{\big(\nabla_{\Gamma_h^\theta} w_h^\theta\left(y_h^\theta\right)\big)^T \delta y_h^\theta}{|\delta y_h^\theta|} = \sup_{\delta y_h^\theta \in T_{y_h^\theta}\Gamma_h^\theta} \frac{Dw_h^\theta\left(y_h^\theta\right)\delta y_h^\theta}{|\delta y_h^\theta|}\\ &= \sup_{\delta q_h \in T_{q_h}\Gamma_h^0} \frac{Dw_h^\theta\left(y_h^\theta\right) DY_h^\theta(q_h)\,\delta q_h}{\left|DY_h^\theta(q_h)\,\delta q_h\right|}.\end{aligned}$$

By construction of w_h^θ and the transport property of the basis functions, we have

$$w_h^\theta\left(Y_h^\theta(q_h)\right) = \sum_{j=1}^N w_j\phi_j[\mathbf{y}+\theta\mathbf{e}]\left(Y_h^\theta(q_h)\right) = \sum_{j=1}^N w_j\phi_j[\mathbf{y}](q_h) = w_h^0(q_h).$$

By the chain rule, this yields

$$Dw_h^\theta\left(y_h^\theta\right) DY_h^\theta(q_h) = Dw_h^0(q_h).$$

Under the imposed condition $\|\nabla_{\Gamma_h^0} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])} \le \frac{1}{2}$ we have for $0 \le \theta \le 1$

$$\left|DY_h^\theta(q_h)\,\delta q_h\right| \ge |\delta q_h| - \theta\left|\big(\nabla_{\Gamma_h^0} e_h^0(q_h)\big)^T \delta q_h\right| \ge \tfrac{1}{2}|\delta q_h|.$$

Hence we obtain

$$\begin{aligned}\left|\nabla_{\Gamma_h^\theta} w_h^\theta\left(y_h^\theta\right)\right| &= \sup_{\delta q_h \in T_{q_h}\Gamma_h^0} \frac{Dw_h^0(q_h)\,\delta q_h}{|DY_h^\theta(q_h)\,\delta q_h|}\\ &\le \sup_{\delta q_h \in T_{q_h}\Gamma_h^0} \frac{Dw_h^0(q_h)\,\delta q_h}{\frac{1}{2}|\delta q_h|} = 2\left|\nabla_{\Gamma_h^0} w_h^0(q_h)\right|.\end{aligned}$$

This yields the stated result for $p = \infty$. For $1 \le p < \infty$ we note in addition that in using the integral transformation formula we have a uniform bound between the surface elements, since DY_h^θ is close to the identity matrix by our smallness assumption on $\nabla_{\Gamma_h^0} e_h^0$. □

The arguments of the previous proof are also used in estimating the changes of the normal vectors on the various surfaces $\Gamma_h^\theta = \Gamma_h[\mathbf{y} + \theta \mathbf{e}]$.

Lemma 4.4 *Suppose that condition* (4.3) *is satisfied. Let* $y_h^\theta = Y_h^\theta(q_h) \in \Gamma_h^\theta$ *be related by the parametrization* (4.4) *of* Γ_h^θ *over* Γ_h^0, *for* $0 \le \theta \le 1$. *Then, the corresponding unit normal vectors differ by no more than*

$$\left| \nu_{\Gamma_h^\theta}\left(y_h^\theta\right) - \nu_{\Gamma_h^0}\left(y_h^0\right) \right| \le C\theta \left| \nabla_{\Gamma_h^0} e_h^0(y_h^0) \right|,$$

with some constant C.

Proof Let δq_h^1 and δq_h^2 be two linearly independent tangent vectors of Γ_h^0 at $q_h \in \Gamma_h^0$ (which may be chosen orthogonal to each other and of unit length with respect to the Euclidean norm). With $\delta y_h^{\theta,i} = DY_h^\theta(q_h)\, \delta q_h^i = \delta q_h^i + \theta \big(\nabla_{\Gamma_h^0} e_h(q_h)\big)^T \delta q_h^i$ for $i = 1, 2$ we then have, for $0 \le \theta \le 1$,

$$\nu_{\Gamma_h^\theta}\left(y_h^\theta\right) = \frac{\delta y_h^{\theta,1} \times \delta y_h^{\theta,2}}{|\delta y_h^{\theta,1} \times \delta y_h^{\theta,2}|}.$$

Since this expression is a locally Lipschitz continuous function of the two vectors, the result follows. (The imposed bound (4.3) is sufficient to ensure the linear independence of the vectors $\delta y_h^{\theta,i}$.) □

We denote by $\partial_\theta^\bullet f$ the material derivative of a function $f = f(y_h^\theta, \theta)$ depending on $\theta \in [0, 1]$ and $y_h^\theta \in \Gamma_h^\theta$:

$$\partial_\theta^\bullet f = \frac{\mathrm{d}}{\mathrm{d}\theta} f\left(y_h^\theta, \theta\right).$$

From Lemma 4.4 together with Lemma 4.3 we obtain the following bound:

Lemma 4.5 *If condition* (4.3) *is satisfied, then*

$$\left\| \partial_\theta^\bullet \nu_{\Gamma_h^\theta} \right\|_{L^p(\Gamma_h^\theta)} \le C \left\| \nabla_{\Gamma_h^0} e_h^0 \right\|_{L^p(\Gamma_h^0)},$$

where C *is independent of* $0 \le \theta \le 1$ *and* $1 \le p \le \infty$.

Proof By Lemma 4.4 with Γ_h^θ in the role of Γ_h^0, we obtain

$$\left| \partial_\theta^\bullet \nu_{\Gamma_h^\theta}\left(y_h^\theta\right) \right| = \big| \lim_{\tau \to 0} \big(\nu_{\Gamma_h^{\theta+\tau}}\left(y_h^{\theta+\tau}\right) - \nu_{\Gamma_h^\theta}\left(y_h^\theta\right)\big)/\tau \big| \le C \left| \nabla_{\Gamma_h^\theta} e_h^\theta\left(y_h^\theta\right) \right|,$$

which implies

$$\left\| \partial_\theta^\bullet \nu_{\Gamma_h^\theta} \right\|_{L^p(\Gamma_h^\theta)} \le C \left\| \nabla_{\Gamma_h^\theta} e_h^\theta \right\|_{L^p(\Gamma_h^\theta)},$$

and Lemma 4.3 completes the proof. □

We finally need a result that bounds the time derivatives of the mass and stiffness matrices corresponding to nodes on the exact smooth surface $\Gamma(t)$. The following result is a direct consequence of [15, Lemma 4.1].

Lemma 4.6 *Let $\Gamma(t) = \Gamma(X(\cdot,t))$, $t \in [0,T]$, be a smoothly evolving family of smooth closed surfaces, and let the vector $\mathbf{x}^*(t) \in \mathbb{R}^{3N}$ collect the nodes $x_j^*(t) = X(p_j,t)$. Then,*

$$\mathbf{w}^T \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}(\mathbf{x}^*(t))\mathbf{z} \le C\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}^*(t))}\|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}^*(t))},$$
$$\mathbf{w}^T \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{x}^*(t))\mathbf{z} \le C\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}^*(t))}\|\mathbf{z}\|_{\mathbf{A}(\mathbf{x}^*(t))},$$

for all $\mathbf{w},\mathbf{z} \in \mathbb{R}^N$. The constant C depends only on a bound of the $W^{1,\infty}$ norm of the surface velocity.

5 Stability of discretized surface motion under a prescribed driving-term

In this section we begin the stability analysis by first studying the stability of the spatially discretized velocity law with a given inhomogeneity instead of a coupling to the surface PDE. This allows us to present, in a technically simpler setting, some of the basic arguments that are used in our approach to stability estimates, which works with the matrix–vector formulation. The stability of the spatially discretized problem including coupling with the surface PDE is then studied in Sect. 6 by similar, but more elaborate arguments.

5.1 Uncoupled velocity law and its semidiscretization

In this section we consider the velocity law without coupling to a surface PDE:

$$v - \alpha\Delta_{\Gamma(X)}v = g\,\nu_{\Gamma(X)},$$

where $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given smooth function of (x,t), and $\alpha > 0$ is a fixed parameter. This problem is considered together with the ordinary differential equations (2.1) for the positions X determining the surface $\Gamma(X)$. Initial values are specified for X.

The weak formulation is given by the second formula of (2.3) with the function g considered here. This is considered together with the ordinary differential equations (2.1) for the positions X.

Then the finite element spatial semidiscretization of this problem reads as: Find the unknown nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element function $v_h(\cdot,t) \in S_h(\mathbf{x}(t))^3$ such that the following semidiscrete equation holds for every $\psi_h \in S_h(\mathbf{x}(t))^3$:

$$\int_{\Gamma_h(\mathbf{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} v_h \cdot \nabla_{\Gamma_h(\mathbf{x})}\psi_h = \int_{\Gamma_h(\mathbf{x})} g\,\nu_{\Gamma_h(\mathbf{x})} \cdot \psi_h, \qquad (5.1)$$

together with the ordinary differential equations (2.5). As before, the nodal vector of the initial positions $\mathbf{x}(0)$ is taken from the exact initial values at the nodes x_j^0 of the triangulation of the given initial surface Γ^0: $x_j(0) = x_j^0$ for $j = 1, \ldots, N$.

As in Sect. 2.5, the nodal vectors $\mathbf{v} \in \mathbb{R}^{3N}$ of the finite element function v_h together with the surface nodal vector $\mathbf{x} \in \mathbb{R}^{3N}$ satisfy a system of differential-algebraic equations (DAEs). We obtain from (5.1) and (2.5) the following coupled DAE system for the nodal values $\mathbf{v}$ and $\mathbf{x}$:

$$\begin{aligned} \mathbf{K}(\mathbf{x})\mathbf{v} &= \mathbf{g}(\mathbf{x}, t), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{5.2}$$

Here the matrix $\mathbf{K}(\mathbf{x}) = I_3 \otimes (\mathbf{M}(\mathbf{x}) + \alpha \mathbf{A}(\mathbf{x}))$ is from (2.6), and the driving term $\mathbf{g}(\mathbf{x}, t)$ is given by

$$\mathbf{g}(\mathbf{x}, t))|_{3(j-1)+\ell} = \int_{\Gamma_h(\mathbf{x})} g(\cdot, t) \left(\nu_{\Gamma_h(\mathbf{x})}\right)_\ell \phi_j[\mathbf{x}], \quad (j = 1, \ldots, N, \ell = 1, 2, 3).$$

5.2 Error equations

We denote by

$$\mathbf{x}^*(t) = \left(x_j^*(t)\right) \in \mathbb{R}^{3N} \quad \text{with} \quad x_j^*(t) = X(p_j, t), \quad (j = 1, \ldots, N)$$

the nodal vector of the *exact* positions on the surface $\Gamma(X(\cdot, t))$. This defines a discrete surface $\Gamma_h(\mathbf{x}^*(t))$ that interpolates the exact surface $\Gamma(X(\cdot, t))$.

We consider the interpolated exact velocity

$$v_h^*(\cdot, t) = \sum_{j=1}^N v_j^*(t)\phi_j[\mathbf{x}^*(t)] \quad \text{with} \quad v_j^*(t) = \dot{x}_j^*(t),$$

with the corresponding nodal vector

$$\mathbf{v}^*(t) = \left(v_j^*(t)\right) = \dot{\mathbf{x}}^*(t) \in \mathbb{R}^{3N}.$$

Inserting v_h^* and $\mathbf{x}^*$ in place of the numerical solution v_h and $\mathbf{x}$ into (5.1) yields a defect $d_h(\cdot, t) \in S_h(\mathbf{x}^*(t))^3$: for every $\psi_h \in S_h(\mathbf{x}^*(t))^3$,

$$\int_{\Gamma_h(\mathbf{x}^*)} v_h^* \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x}^*)} \nabla_{\Gamma_h(\mathbf{x}^*)} v_h^* \cdot \nabla_{\Gamma_h(\mathbf{x}^*)} \psi_h = \int_{\Gamma_h(\mathbf{x}^*)} g\, \nu_{\Gamma_h(\mathbf{x}^*)} \cdot \psi_h + \int_{\Gamma_h(\mathbf{x}^*)} d_h \cdot \psi_h.$$

With $d_h(\cdot, t) = \sum_{j=1}^N d_j(t)\phi_j[\mathbf{x}^*(t)]$ and the corresponding nodal vector $\mathbf{d}_\mathbf{v}(t) = \left(d_j(t)\right) \in \mathbb{R}^{3N}$ we then have $(I_3 \otimes \mathbf{M}(\mathbf{x}^*(t)))\mathbf{d}_\mathbf{v}(t)$ as the defect on inserting $\mathbf{x}^*$ and $\mathbf{v}^*$ in the first equation of (5.2). With $\mathbf{M}^{[3]}(\mathbf{x}^*) = I_3 \otimes \mathbf{M}(\mathbf{x}^*)$, we thus have

$$\begin{aligned}\mathbf{K}(\mathbf{x}^*)\mathbf{v}^* &= \mathbf{g}(\mathbf{x}^*) + \mathbf{M}^{[3]}(\mathbf{x}^*)\mathbf{d}_\mathbf{v},\\ \dot{\mathbf{x}}^* &= \mathbf{v}^*.\end{aligned} \tag{5.3}$$

We denote the errors in the surface nodes and in the velocity by $\mathbf{e}_\mathbf{x} = \mathbf{x} - \mathbf{x}^*$ and $\mathbf{e}_\mathbf{v} = \mathbf{v} - \mathbf{v}^*$, respectively. We rewrite the velocity law in (5.2) as

$$\mathbf{K}(\mathbf{x}^*)\mathbf{v} = -\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* - \big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e}_\mathbf{v} + \mathbf{g}(\mathbf{x}).$$

Then, by subtracting (5.3) from the above version of (5.2), we obtain the following error equations for the uncoupled problem:

$$\begin{aligned}\mathbf{K}(\mathbf{x}^*)\mathbf{e}_\mathbf{v} &= -\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* - \big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e}_\mathbf{v}\\ &\quad + \big(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\big) - \mathbf{M}^{[3]}(\mathbf{x}^*)\mathbf{d}_\mathbf{v},\\ \dot{\mathbf{e}}_\mathbf{x} &= \mathbf{e}_\mathbf{v}.\end{aligned} \tag{5.4}$$

When no confusion can arise, we write in the following $\mathbf{M}(\mathbf{x}^*)$ for $\mathbf{M}^{[3]}(\mathbf{x}^*)$ and $\|\cdot\|_{H^1(\Gamma)}$ for $\|\cdot\|_{H^1(\Gamma)^3}$, etc.

5.3 Norms

We recall that $\mathbf{K}(\mathbf{x}^*) = I_3 \otimes (\mathbf{M}(\mathbf{x}^*) + \alpha\mathbf{A}(\mathbf{x}^*))$ and, for $\mathbf{w} \in \mathbb{R}^{3N}$ and the corresponding finite element function $w_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}^*] \in S_h(\mathbf{x}^*)^3$, we consider the norm

$$\begin{aligned}\|\mathbf{w}\|^2_{\mathbf{K}(\mathbf{x}^*)} &:= \mathbf{w}^T\mathbf{K}(\mathbf{x}^*)\mathbf{w}\\ &= \|w_h\|^2_{L^2(\Gamma_h(\mathbf{x}^*))} + \alpha\|\nabla_{\Gamma_h(\mathbf{x}^*)} w_h\|^2_{L^2(\Gamma_h(\mathbf{x}^*))} \sim \|w_h\|^2_{H^1(\Gamma_h(\mathbf{x}^*))}.\end{aligned}$$

For convenience, we will take $\alpha = 1$ in the remainder of this section, so that the last norm equivalence becomes an equality. For the defect $d_h \in S_h(\mathbf{x}^*)^3$ we use the dual norm (cf. [24, Proof of Theorem 5.1])

$$\begin{aligned}\|d_h\|_{H_h^{-1}(\Gamma_h(\mathbf{x}^*))} &:= \sup_{0\neq\psi_h\in S_h(\mathbf{x}^*)^3} \frac{\int_{\Gamma_h(\mathbf{x}^*)} d_h\cdot\psi_h}{\|\psi_h\|_{H^1(\Gamma_h(\mathbf{x}^*))^3}}\\ &= \sup_{0\neq\mathbf{z}\in\mathbb{R}^{3N}} \frac{\mathbf{d}_\mathbf{v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{z}}{(\mathbf{z}^T\mathbf{K}(\mathbf{x}^*)\mathbf{z})^{\frac12}} = \sup_{0\neq\mathbf{w}\in\mathbb{R}^{3N}} \frac{\mathbf{d}_\mathbf{v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{K}(\mathbf{x}^*)^{-\frac12}\mathbf{w}}{(\mathbf{w}^T\mathbf{w})^{\frac12}}\\ &= \|\mathbf{K}(\mathbf{x}^*)^{-\frac12}\mathbf{M}(\mathbf{x}^*)\mathbf{d}_\mathbf{v}\|_2 = \Big(\mathbf{d}_\mathbf{v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{K}(\mathbf{x}^*)^{-1}\mathbf{M}(\mathbf{x}^*)\mathbf{d}_\mathbf{v}\Big)^{\frac12}.\end{aligned} \tag{5.5}$$

We denote

$$\|\mathbf{d}_\mathbf{v}\|^2_{\star,\mathbf{x}^*} := \mathbf{d}_\mathbf{v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{K}(\mathbf{x}^*)^{-1}\mathbf{M}(\mathbf{x}^*)\mathbf{d}_\mathbf{v},$$

so that

$$\|\mathbf{d_v}\|_{\star,\mathbf{x}^*} = \|d_h\|_{H_h^{-1}(\Gamma_h(\mathbf{x}^*))}.$$

5.4 Stability estimate

The following stability result holds for the errors $\mathbf{e_v}$ and $\mathbf{e_x}$, under an assumption of small defects. It will be shown in Sect. 8 that this assumption is satisfied if the exact solution is sufficiently smooth.

Proposition 5.1 *Suppose that the defect is bounded as follows, with $\kappa > 1$:*

$$\|\mathbf{d_v}(t)\|_{\star,\mathbf{x}^*(t)} \le ch^\kappa, \quad t \in [0, T].$$

Then there exists $h_0 > 0$ such that the following error bounds hold for $h \le h_0$ and $0 \le t \le T$:

$$\|\mathbf{e_x}(t)\|^2_{\mathbf{K}(\mathbf{x}^*(t))} \le C \int_0^t \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*} \mathrm{d}s, \tag{5.6}$$

$$\|\mathbf{e_v}(t)\|^2_{\mathbf{K}(\mathbf{x}^*(t))} \le C\|\mathbf{d_v}(t)\|^2_{\star,\mathbf{x}^*} + C \int_0^t \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*} \mathrm{d}s. \tag{5.7}$$

The constant C is independent of t and h, but depends on the final time T and on the regularization parameter α.

We note that the error functions $e_v(\cdot,t), e_x(\cdot,t) \in S_h(\mathbf{x}^*(t))^3$ with nodal vectors $\mathbf{e_v}(t)$ and $\mathbf{e_x}(t)$, respectively, are then bounded by

$$\|e_v(\cdot,t)\|_{H^1(\Gamma_h(\mathbf{x}^*(t)))} \le Ch^\kappa \quad \text{and} \quad \|e_x(\cdot,t)\|_{H^1(\Gamma_h(\mathbf{x}^*(t)))} \le Ch^\kappa, \quad t \in [0, T].$$

Proof The proof uses energy estimates for the error equations (5.4) in the matrix–vector formulation, and it relies on the results of Sect. 4. In the course of this proof c and C will be generic constants that take on different values on different occurrences.

In view of condition (4.3) for $\mathbf{y} = \mathbf{x}^*(t)$, we will need to control the $W^{1,\infty}$ norm of the position error $e_x(\cdot,t)$. Let $0 < t^* \le T$ be the maximal time such that

$$\|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot,t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} \le h^{(\kappa-1)/2} \quad \text{for} \quad t \in [0, t^*]. \tag{5.8}$$

At $t = t^*$ either this inequality becomes an equality, or else we have $t^* = T$.

We will first prove the stated error bounds for $0 \le t \le t^*$. Then the proof will be finished by showing that in fact t^* coincides with T.

By testing the first equation in (5.4) with $\mathbf{e_v}$, and dropping the omnipresent argument $t \in [0, t^*]$, we obtain:

$$\begin{aligned}\|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)} = \mathbf{e_v}^T\mathbf{K}(\mathbf{x}^*)\mathbf{e_v} = &-\mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* \\ &-\mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e_v} \\ &+\mathbf{e_v}^T\big(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\big) - \mathbf{e_v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{d_v}.\end{aligned}$$

We separately estimate the four terms on the right-hand side in an appropriate way, with Lemmas 4.1–4.4 as our main tools.

(i) We denote, for $0 \le \theta \le 1$, by e_v^θ and $v_h^{*,\theta}$ the finite element functions in $S_h(\Gamma_h^\theta)^3$ for $\Gamma_h^\theta = \Gamma_h(\mathbf{x}^* + \theta \mathbf{e_x})$ with nodal vectors $\mathbf{e_v}$ and $\mathbf{v}^*$, respectively. Lemma 4.1 then gives us

$$\begin{aligned} &\mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* \\ &\quad = \int_0^1 \int_{\Gamma_h^\theta} e_v^\theta \cdot \big(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\big) v_h^{*,\theta} \, \mathrm{d}\theta + \alpha \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_v^\theta \cdot \big(D_{\Gamma_h^\theta} e_x^\theta\big) \nabla_{\Gamma_h^\theta} v_h^{*,\theta} \, \mathrm{d}\theta. \end{aligned}$$

Using the Cauchy–Schwarz inequality, we estimate the integral with the product of the $L^2 - L^2 - L^\infty$ norms of the three factors. We thus have

$$\begin{aligned} \mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* &\le \int_0^1 \|e_v^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\ &\quad + \alpha \int_0^1 \|\nabla_{\Gamma_h^\theta} e_v^\theta\|_{L^2(\Gamma_h^\theta)} \|D_{\Gamma_h^\theta} e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\ &\le c \int_0^1 \|e_v^\theta\|_{H^1(\Gamma_h^\theta)} \|e_x^\theta\|_{H^1(\Gamma_h^\theta)} \|v_h^{*,\theta}\|_{W^{1,\infty}(\Gamma_h^\theta)} \, \mathrm{d}\theta. \end{aligned}$$

By (5.8) and Lemma 4.3, this is bounded by

$$\mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* \le c \|e_v\|_{H^1(\Gamma_h(\mathbf{x}^*))} \|e_x\|_{H^1(\Gamma_h(\mathbf{x}^*))} \|v_h^*\|_{W^{1,\infty}(\Gamma_h(\mathbf{x}^*))},$$

where the last factor is bounded independently of h. By the Young inequality, we thus obtain

$$\begin{aligned} \mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* &\le \tfrac{1}{6} \|e_v\|^2_{H^1(\Gamma_h(\mathbf{x}^*))} + C \|e_x\|^2_{H^1(\Gamma_h(\mathbf{x}^*))} \\ &= \tfrac{1}{6} \|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)} + C \|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)}. \end{aligned}$$

(ii) Similarly, estimating the three factors in the integrals by $L^2 - L^\infty - L^2$, we obtain

$$\begin{aligned} \mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e_v} &\le c \|e_\mathbf{v}\|^2_{L^2(\Gamma_h(\mathbf{x}^*))} \|\nabla_{\Gamma_h} \cdot e_\mathbf{x}\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\ &\quad + c\alpha \|\nabla_{\Gamma_h} e_\mathbf{v}\|^2_{L^2(\Gamma_h(\mathbf{x}^*))} \|D_{\Gamma_h} e_\mathbf{x}\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\ &\le c h^{(\kappa-1)/2} \|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)}, \end{aligned}$$

where in the last inequality we used the bound (5.8).

(iii) In the following bound we use Lemma 4.5. Again with the finite element function $e_v^\theta = \sum_{j=1}^N (\mathbf{e_v})_j \phi_j[\mathbf{x}^* + \theta \mathbf{e_x}]$ on the surface $\Gamma_h^\theta = \Gamma_h(\mathbf{x}^* + \theta \mathbf{e_x})$, for $0 \le \theta \le 1$, we write

$$\mathbf{e}_\mathbf{v}^T\big(\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{x}^*)\big)=\int_{\Gamma_h^1} g\nu_{\Gamma_h^1}\cdot e_v^1-\int_{\Gamma_h^0} g\nu_{\Gamma_h^0}\cdot e_v^0=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\Gamma_h^\theta} g\nu_{\Gamma_h^\theta}\cdot e_v^\theta \mathrm{d}\theta.$$

Using the Leibniz formula, this becomes

$$\mathbf{e}_\mathbf{v}^T\big(\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{x}^*)\big)=\int_0^1\int_{\Gamma_h^\theta}\Big(\partial_\theta^\bullet\big(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta\big)+\Big(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta\Big)\Big(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\Big)\Big)\,\mathrm{d}\theta.$$

Here we have, noting that $\partial_\theta^\bullet e_v^\theta=0$,

$$\partial_\theta^\bullet\big(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta\big)=g'e_x^\theta\,\nu_{\Gamma_h^\theta}\cdot e_v^\theta+g\,\partial_\theta^\bullet\nu_{\Gamma_h^\theta}\cdot e_v^\theta.$$

With Lemmas 4.3 and 4.5 we therefore obtain via the Cauchy–Schwarz inequality

$$\begin{aligned}\int_{\Gamma_h^\theta}\partial_\theta^\bullet\big(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta\big)&\le c_2^2\,\|g'\|_{L^\infty}\,\|e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))}\|e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))}\\&\quad+c_2^2\,\|g\|_{L^\infty}\,\|\nabla_{\Gamma_h(\mathbf{x}^*)}e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))}\,\|e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))},\end{aligned}$$

and again with Lemma 4.3,

$$\int_{\Gamma_h^\theta}\Big(g\nu_{\Gamma_h^\theta}\cdot e_v^\theta\Big)\Big(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\Big)\le c_2^2\,\|g\|_{L^\infty}\,\|e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))}\,\|\nabla_{\Gamma_h(\mathbf{x}^*)}\cdot e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))}.$$

In total, we obtain a bound of the same type as for the terms in (i) and (ii):

$$\begin{aligned}\mathbf{e}_\mathbf{v}^T\big(\mathbf{g}(\mathbf{x})-\mathbf{g}(\mathbf{x}^*)\big)&\le c\|e_x\|_{H^1(\Gamma_h(\mathbf{x}^*))}\,\|e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))}\\&=c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}\,\|\mathbf{e}_\mathbf{v}\|_{\mathbf{M}(\mathbf{x}^*)}\le\tfrac{1}{6}\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2+C\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2.\end{aligned}$$

The combination of the estimates of the three terms (i)–(iii) with absorptions (for sufficiently small $h\le h_0$), and a simple dual norm estimate, based on (5.5), for the defect term, yield the bound

$$\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2\le c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2+c\|\mathbf{d}_\mathbf{v}\|_{\star,\mathbf{x}^*}^2. \tag{5.9}$$

Using this estimate, together with taking the $\|\cdot\|_{\mathbf{K}(\mathbf{x}^*)}$ norm of both sides of the second equation in (5.4), we obtain

$$\|\dot{\mathbf{e}}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2=\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2\le c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2+c\|\mathbf{d}_\mathbf{v}\|_{\star,\mathbf{x}^*}^2. \tag{5.10}$$

In order to apply Gronwall's inequality, we connect $\dfrac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2$ and $\|\dot{\mathbf{e}}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2$ as follows:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2&=\mathbf{e}_\mathbf{x}^T\mathbf{K}(\mathbf{x}^*)\dot{\mathbf{e}}_\mathbf{x}+\frac{1}{2}\mathbf{e}_\mathbf{x}^T\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}(\mathbf{x}^*)\Big)\mathbf{e}_\mathbf{x}\\&\le\|\dot{\mathbf{e}}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2+c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2,\end{aligned}$$

where we use the Cauchy–Schwarz inequality and Lemma 4.6 in the estimate. Inserting (5.10), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} \le c\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} + c\|\mathbf{d_v}\|^2_{\star,\mathbf{x}^*}.$$

A Gronwall inequality then yields (5.6), using $e_j(0) = x_j(0) - x_j^0 = 0$ for $j = 1, \dots, N$. Inserting this estimate in (5.9), we can bound $\mathbf{e_v}(t)$ for $0 \le t \le t^*$ by (5.7).

Now it only remains to show that $t^* = T$ for h sufficiently small. For $0 \le t \le t^*$ we use an inverse inequality and (5.6) to bound the left-hand side in (5.8):

$$\begin{aligned}\|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot,t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} &\le ch^{-1}\|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot,t)\|_{L^2(\Gamma_h(\mathbf{x}^*(t)))}\\ &\le ch^{-1}\|\mathbf{e_x}(t)\|_{\mathbf{K}(\mathbf{x}^*(t))} \le cCh^{\kappa-1} \le \tfrac{1}{2}h^{(\kappa-1)/2}\end{aligned}$$

for sufficiently small h. Hence, we can extend the bound (5.8) beyond t^*, which contradicts the maximality of t^* unless we have already $t^* = T$. □

6 Stability of coupling surface PDEs to surface motion

Now we turn to the stability bounds of the original problem (2.4)–(2.5), or in DAE form (2.7), which is the formulation we will actually use for the stability analysis.

6.1 Error equations

Similarly as before, in order to derive stability estimates we consider the DAE system when we insert the nodal values $\mathbf{u}^*(t) \in \mathbb{R}^N$ of the exact solution $u(\cdot,t)$, the nodal values $\mathbf{x}^*(t) \in \mathbb{R}^{3N}$ of the exact positions $X(\cdot,t)$, and the nodal values $\mathbf{v}^*(t) \in \mathbb{R}^{3N}$ of the exact velocity $v(\cdot,t)$. Inserting them into (2.7) yields defects $\mathbf{d_u}(t) \in \mathbb{R}^N$ and $\mathbf{d_v}(t) \in \mathbb{R}^{3N}$: omitting the argument t in the notation, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{u}^*\Big) + \mathbf{A}(\mathbf{x}^*)\mathbf{u}^* &= \mathbf{f}(\mathbf{x}^*,\mathbf{u}^*) + \mathbf{M}(\mathbf{x}^*)\mathbf{d_u},\\ \mathbf{K}(\mathbf{x}^*)\mathbf{v}^* &= \mathbf{g}(\mathbf{x}^*,\mathbf{u}^*) + \mathbf{M}^{[3]}(\mathbf{x}^*)\mathbf{d_v},\\ \dot{\mathbf{x}}^* &= \mathbf{v}^*,\end{aligned} \tag{6.1}$$

where again $\mathbf{M}^{[3]}(\mathbf{x}^*) = I_3 \otimes \mathbf{M}(\mathbf{x}^*)$. As no confusion can arise, we write again $\mathbf{M}(\mathbf{x}^*)$ for $\mathbf{M}^{[3]}(\mathbf{x}^*)$.

We denote the PDE error by $\mathbf{e_u} = \mathbf{u} - \mathbf{u}^*$, and as in the previous section, $\mathbf{e_v} = \mathbf{v} - \mathbf{v}^*$ and $\mathbf{e_x} = \mathbf{x} - \mathbf{x}^*$ denote the velocity error and surface error, respectively. Subtracting (6.1) from (2.7), we obtain the following error equation:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{e_u}\Big) + \mathbf{A}(\mathbf{x}^*)\mathbf{e_u} = &- \frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{u}^*\Big)\\ &- \frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big)\end{aligned}$$

$$
\begin{aligned}
&-\big(\mathbf{A}(\mathbf{x})-\mathbf{A}(\mathbf{x}^*)\big)\mathbf{u}^*\\
&-\big(\mathbf{A}(\mathbf{x})-\mathbf{A}(\mathbf{x}^*)\big)\mathbf{e}_\mathbf{u}\\
&+\big(\mathbf{f}(\mathbf{x},\mathbf{u})-\mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)\big)-\mathbf{M}(\mathbf{x}^*)\mathbf{d}_\mathbf{u},\\
\mathbf{K}(\mathbf{x}^*)\mathbf{e}_\mathbf{v} &= -\big(\mathbf{K}(\mathbf{x})-\mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^*-\big(\mathbf{K}(\mathbf{x})-\mathbf{K}(\mathbf{x}^*)\big)\mathbf{e}_\mathbf{v}\\
&+\big(\mathbf{g}(\mathbf{x},\mathbf{u})-\mathbf{g}(\mathbf{x}^*,\mathbf{u}^*)\big)-\mathbf{M}(\mathbf{x}^*)\mathbf{d}_\mathbf{v},\\
\dot{\mathbf{e}}_\mathbf{x} &= \mathbf{e}_\mathbf{v}. \qquad (6.2)
\end{aligned}
$$

6.2 Stability estimate

We now formulate the stability result for the errors $\mathbf{e_u}$, $\mathbf{e_v}$ and $\mathbf{e_x}$ of the surface motion coupled to the surface PDE. Here, we use the norms (4.1)–(4.2) and those of Sect. 5.3.

Proposition 6.1 *Assume that the following bounds hold for the defects, for some $\kappa > 1$:*

$$\|\mathbf{d_u}(t)\|_{\star,\mathbf{x}^*(t)} \le ch^\kappa, \quad \|\mathbf{d_v}(t)\|_{\star,\mathbf{x}^*(t)} \le ch^\kappa, \quad \textit{for } t \in [0,T].$$

Then there exists $h_0 > 0$ such that the following stability estimate holds for all $h \le h_0$ and $0 \le t \le T$:

$$
\begin{aligned}
&\|\mathbf{e_u}(t)\|^2_{\mathbf{M}(\mathbf{x}^*)} + \int_0^t \|\mathbf{e_u}(s)\|^2_{\mathbf{A}(\mathbf{x}^*)}\mathrm{d}s + \|\mathbf{e_x}(t)\|^2_{\mathbf{K}(\mathbf{x}^*)} + \int_0^t \|\mathbf{e_v}(s)\|^2_{\mathbf{K}(\mathbf{x}^*)}\mathrm{d}s\\
&\le C\int_0^t \Big(\|\mathbf{d_u}(s)\|^2_{\star,\mathbf{x}^*} + \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\Big)\mathrm{d}s. \qquad (6.3)
\end{aligned}
$$

The constant C is independent of t and h, but depends on the final time T and on the regularization parameter α.

We note that the error functions $e_u(\cdot,t) \in S_h(\mathbf{x}^*(t))$ and $e_v(\cdot,t), e_x(\cdot,t) \in S_h(\mathbf{x}^*(t))^3$ with nodal vectors $\mathbf{e_u}(t)$ and $\mathbf{e_v}(t)$, $\mathbf{e_x}(t)$, respectively, are then bounded by

$$
\begin{aligned}
&\|e_u(\cdot,t)\|_{L^2(\Gamma_h(\mathbf{x}^*(t)))} + \left(\int_0^t \|e_u(\cdot,t)\|^2_{H^1(\Gamma_h(\mathbf{x}^*(t)))}\,\mathrm{d}s\right)^{1/2} \le Ch^\kappa,\\
&\left(\int_0^t \|e_v(\cdot,t)\|^2_{H^1(\Gamma_h(\mathbf{x}^*(t)))^3}\mathrm{d}s\right)^{1/2} \le Ch^\kappa, \qquad (6.4)\\
&\|e_x(\cdot,t)\|_{H^1(\Gamma_h(\mathbf{x}^*(t)))^3} \le Ch^\kappa, \quad t\in[0,T].
\end{aligned}
$$

Proof The proof is an extension of the proof of Proposition 5.1, again based on the matrix–vector formulation and the auxiliary results of Sect. 4. We handle the surface PDE and the surface equations separately: we first estimate the errors of the PDE, while those for the surface equation are based on Sect. 5. Finally we will combine the results to obtain the stability estimates for the coupled problem.

In the course of this proof c and C will be generic constants that take on different values on different occurrences.

Let $0 < t^* \le T$ be the maximal time such that the following inequalities hold:

$$\begin{aligned} \|\nabla_{\Gamma_h(\mathbf{x}^*(t))} e_x(\cdot,t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} &\le h^{(\kappa-1)/2}, \\ \|e_u(\cdot,t)\|_{L^\infty(\Gamma_h(\mathbf{x}^*(t)))} &\le 1, \end{aligned} \qquad \text{for} \quad t \in [0,t^*]. \tag{6.5}$$

Note that $t^* > 0$ since initially both $e_x(\cdot,0) = 0$ and $e_u(\cdot,0) = 0$.

We first prove the stated error bounds for $0 \le t \le t^*$. At the end, the proof will be finished by showing that in fact t^* coincides with T.

Testing the first two equations of (6.2) with $\mathbf{e_u}$ and $\mathbf{e_v}$, and dropping the omnipresent argument $t \in [0,t^*]$, we obtain:

$$\begin{aligned} \mathbf{e_u}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{e_u}\Big) + \mathbf{e_u}^T\mathbf{A}(\mathbf{x}^*)\mathbf{e_u} &= -\mathbf{e_u}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{u}^*\Big) \\ &\quad -\mathbf{e_u}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big) \\ &\quad -\mathbf{e_u}^T\big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{u}^* \\ &\quad -\mathbf{e_u}^T\big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{e_u} \\ &\quad +\mathbf{e_u}^T\big(\mathbf{f}(\mathbf{x},\mathbf{u}) - \mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)\big) - \mathbf{e_u}^T\mathbf{M}(\mathbf{x}^*)\mathbf{d_u}, \\ \|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)} &= -\mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* - \mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e_v} \\ &\quad +\mathbf{e_v}^T\big(\mathbf{g}(\mathbf{x},\mathbf{u}) - \mathbf{g}(\mathbf{x}^*,\mathbf{u}^*)\big) - \mathbf{e_v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{d_v}, \\ \dot{\mathbf{e}}_\mathbf{x} &= \mathbf{e_v}. \end{aligned}$$

(A) *Estimates for the surface PDE:* We estimate the terms separately, with Lemmas 4.1–4.3 as our main tools.

(i) The symmetry of $\mathbf{M}(\mathbf{x}^*)$ and a simple calculation yield

$$\begin{aligned} \mathbf{e_u}^T \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{e_u}\Big) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{e_u}^T\mathbf{M}(\mathbf{x}^*)\mathbf{e_u}\Big) + \frac{1}{2}\mathbf{e_u}^T\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}(\mathbf{x}^*)\Big)\mathbf{e_u} \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + \frac{1}{2}\mathbf{e_u}^T\Big(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}(\mathbf{x}^*)\Big)\mathbf{e_u}, \end{aligned}$$

where the last term is bounded by Lemma 4.6 as

$$\Big|\mathbf{e_u}^T \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}(\mathbf{x}^*)\mathbf{e_u}\Big| \le c\,\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)}.$$

(ii) By the definition of the $\mathbf{A}$-norm we have

$$\mathbf{e_u}^T\mathbf{A}(\mathbf{x}^*)\mathbf{e_u} = \|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)}.$$

(iii) With the product rule we write

$$\begin{aligned}
&\mathbf{e}_{\mathbf{u}}^{T}\,\frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\mathbf{u}^{*}\Big)\\
&\quad=\mathbf{e}_{\mathbf{u}}^{T}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\dot{\mathbf{u}}^{*}+\mathbf{e}_{\mathbf{u}}^{T}\Big(\frac{\mathrm{d}}{\mathrm{dt}}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\Big)\mathbf{u}^{*}. \qquad (6.6)
\end{aligned}$$

With $\Gamma_h^\theta(t) = \Gamma_h[\mathbf{x}^*(t)+\theta\mathbf{e}_\mathbf{x}(t)]$ and with the finite element functions $e_u^\theta(\cdot,t)$, $u_h^{*,\theta}(\cdot,t) \in S_h(\mathbf{x}^*(t)+\theta\mathbf{e}_\mathbf{x}(t))$ with nodal vectors $\mathbf{e}_\mathbf{u}(t)$, $\mathbf{u}^*(t)$, resp., Lemma 4.1 (with $\mathbf{x}^*(t)$ in the role of $\mathbf{y}$) yields for the first term, omitting again the argument t,

$$\mathbf{e}_{\mathbf{u}}^{T}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\dot{\mathbf{u}}^{*}=\int_0^1\int_{\Gamma_h^\theta} e_u^\theta\,\big(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\big)\,\partial_h^\bullet u_h^{*,\theta}\,\mathrm{d}\theta.$$

Using the Cauchy–Schwarz inequality we obtain

$$\Big|\mathbf{e}_{\mathbf{u}}^{T}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\dot{\mathbf{u}}^{*}\Big|\le\int_0^1 \|e_u^\theta\|_{L^2(\Gamma_h^\theta)}\,\|\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)}\,\|\partial_h^\bullet u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)}\,\mathrm{d}\theta.$$

Under condition (6.5) we obtain from Lemmas 4.2 and 4.3 that for $0\le t\le t^*$,

$$\Big|\mathbf{e}_{\mathbf{u}}^{T}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\dot{\mathbf{u}}^{*}\Big|\le c\,\|e_u^0\|_{L^2(\Gamma_h^0)}\,\|e_x^0\|_{H^1(\Gamma_h^0)}\,\|\partial_h^\bullet u_h^{*,0}\|_{L^\infty(\Gamma_h^0)}.$$

Now, the last factor is bounded by

$$\Big\|\partial_h^\bullet u_h^{*,0}\Big\|_{L^\infty(\Gamma_h^0)}\le c\,\big\|\dot{\mathbf{u}}^*\big\|_\infty\le C$$

because of the assumed smoothness of the exact solution u and hence of its material derivative $\partial^\bullet u(\cdot,t)$, whose values at the nodes are the entries of the vector $\dot{\mathbf{u}}^*(t)$. Hence we obtain, on recalling the definitions of the discrete norms,

$$-\mathbf{e}_{\mathbf{u}}^{T}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\dot{\mathbf{u}}^{*}\le C\|\mathbf{e}_\mathbf{u}\|_{\mathbf{M}(\mathbf{x}^*)}\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}.$$

Using Lemma 4.1 together with the Leibniz formula, the last term in (6.6) becomes

$$\begin{aligned}
\mathbf{e}_{\mathbf{u}}^{T}\Big(\frac{\mathrm{d}}{\mathrm{dt}}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^{*})\big)\Big)\mathbf{u}^{*}&=\int_0^1\int_{\Gamma_h^\theta} e_u^\theta\,\partial_h^\bullet\big(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\big)\,u_h^{*,\theta}\,\mathrm{d}\theta\\
&\quad+\int_0^1\int_{\Gamma_h^\theta} e_u^\theta\,\big(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\big)\,u_h^{*,\theta}\,\big(\nabla_{\Gamma_h^\theta}\cdot v_h^\theta\big)\,\mathrm{d}\theta,
\end{aligned}$$

where v_h^θ is the velocity of Γ_h^θ (as a function of t), which is the finite element function in $S_h(\mathbf{x}^*+\theta\mathbf{e}_\mathbf{x})$ with nodal vector $\dot{\mathbf{x}}^*+\theta\dot{\mathbf{e}}_\mathbf{x}=\mathbf{v}^*+\theta\mathbf{e}_\mathbf{v}$. Thus,

$$v_h^\theta = v_h^{*,\theta}+\theta e_v^\theta, \qquad (6.7)$$

where $v_h^{*,\theta}$ and e_v^θ are the finite element functions on Γ_h^θ with nodal vectors $\mathbf{v}^*$ and $\mathbf{e_v}$, respectively. In the first integral we further use, cf. [14, Lemma 2.6],

$$\partial_h^\bullet \left(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\right) = \nabla_{\Gamma_h^\theta} \cdot \partial_h^\bullet e_x^\theta - \left(\left(I_3 - \nu_h^\theta \left(\nu_h^\theta\right)^T\right) \nabla_{\Gamma_h^\theta} v_h^\theta\right) : \nabla_{\Gamma_h^\theta} e_x^\theta,$$

where : symbolizes the Euclidean inner product of the vectorization of two matrices. Here we note that $\partial_h^\bullet e_x^\theta$ is the finite element function on Γ_h^θ with nodal vector $\dot{\mathbf{e}}_\mathbf{x} = \mathbf{e_v}$, so that $\partial_h^\bullet e_x^\theta = e_v^\theta$.

We then estimate, using the Cauchy–Schwarz inequality in the first step, Lemmas 4.2 and 4.3 in the second step (using (6.5) to ensure the smallness condition in these lemmas), the definition of the discrete norms in the third step, and using the first bound of (6.5) and the boundedness of the discrete gradient of the interpolated exact velocity $\nabla_{\Gamma_h(\mathbf{x}^*)} v_h^*$ and of the interpolated exact solution u_h^* in the fourth step,

$$\begin{aligned}
&\left| \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \, \partial_h^\bullet \left(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\right) u_h^{*,\theta} \, \mathrm{d}\theta \right| \\
&\quad \le \int_0^1 \int_{\Gamma_h^\theta} \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \Big(\|\nabla_{\Gamma_h^\theta} \cdot e_v^\theta\|_{L^2(\Gamma_h^\theta)} \\
&\qquad + \|\nabla_{\Gamma_h^\theta} v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \cdot \|\nabla_{\Gamma_h^\theta} e_x^\theta\|_{L^2(\Gamma_h^\theta)} \\
&\qquad + \|\nabla_{\Gamma_h^\theta} e_v^\theta\|_{L^2(\Gamma_h^\theta)} \cdot \|\nabla_{\Gamma_h^\theta} e_x^\theta\|_{L^\infty(\Gamma_h^\theta)} \Big) \|u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\
&\quad \le c \, \|e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))} \Big(\|\nabla_{\Gamma_h(\mathbf{x}^*)} e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \\
&\qquad + \|\nabla_{\Gamma_h(\mathbf{x}^*)} v_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \cdot \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))} \\
&\qquad + \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_v\|_{L^2(\Gamma_h(\mathbf{x}^*))} \cdot \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \Big) \|u_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\
&\quad \le c \, \|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)} \Big(\|\mathbf{e_v}\|_{\mathbf{A}(\mathbf{x}^*)} + \|\nabla_{\Gamma_h(\mathbf{x}^*)} v_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \, \|\mathbf{e_x}\|_{\mathbf{A}(\mathbf{x}^*)} \\
&\qquad + \|\mathbf{e_v}\|_{\mathbf{A}(\mathbf{x}^*)} \|\nabla_{\Gamma_h(\mathbf{x}^*)} e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \Big) \|\mathbf{u}^*\|_\infty \\
&\quad \le c \|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)} \Big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)} + C \|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)} h^{(\kappa-1)/2} \Big) C \\
&\quad \le C' \|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)} \Big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)} \Big).
\end{aligned}$$

With the same arguments we estimate, on inserting (6.7),

$$\begin{aligned}
&\left| \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \left(\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\right) u_h^{*,\theta} \left(\nabla_{\Gamma_h^\theta} \cdot v_h^\theta\right) \mathrm{d}\theta \right| \\
&\quad \le \int_0^1 \int_{\Gamma_h^\theta} \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^2(\Gamma_h^\theta)} \|u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot v_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \, \mathrm{d}\theta \\
&\qquad + \int_0^1 \int_{\Gamma_h^\theta} \|e_u^\theta\|_{L^2(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_x^\theta\|_{L^\infty(\Gamma_h^\theta)} \|u_h^{*,\theta}\|_{L^\infty(\Gamma_h^\theta)} \|\nabla_{\Gamma_h^\theta} \cdot e_v^\theta\|_{L^2(\Gamma_h^\theta)} \, \mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&\le c\,\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\|\mathbf{u}^*\|_\infty\|\nabla_{\Gamma_h(\mathbf{x}^*)}\cdot v_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\\
&\quad + c\,\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\,\|\nabla_{\Gamma_h(\mathbf{x}^*)}e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\,\|\mathbf{u}^*\|_\infty\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)}\\
&\le C\,\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)}+\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\Big).
\end{aligned}
$$

Altogether we obtain the bound

$$
-\mathbf{e_u}^T\frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^*)\big)\mathbf{u}^*\Big)\le C\,\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)}+\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\Big).
$$

(iv) We obtain similarly

$$
\begin{aligned}
&-\mathbf{e_u}^T\frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big)\\
&\quad=-\frac{1}{2}\mathbf{e_u}^T\Big(\frac{\mathrm{d}}{\mathrm{dt}}\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^*)\big)\Big)\mathbf{e_u}-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{e_u}^T\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big)\\
&\quad\le c\,\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)}+\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\Big)\|e_u\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\\
&\qquad-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{e_u}^T\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big)\\
&\quad\le C\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)}+\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\big)-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{e_u}^T\big(\mathbf{M}(\mathbf{x})-\mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big),
\end{aligned}
$$

where we used the second bound of (6.5) in the last inequality.

(v) Lemma 4.1, the Cauchy–Schwarz inequality and Lemma 4.3 yield

$$
\begin{aligned}
-\mathbf{e_u}^T\big(\mathbf{A}(\mathbf{x})-\mathbf{A}(\mathbf{x}^*)\big)\mathbf{u}^* &= -\int_0^1\int_{\Gamma_h^\theta}\nabla_{\Gamma_h^\theta}e_u^\theta\cdot\big(D_{\Gamma_h^\theta}e_x^\theta\big)\nabla_{\Gamma_h^\theta}u_h^{*,\theta}\,\mathrm{d}\theta\\
&\le c\|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)}\,\|\mathbf{e_x}\|_{\mathbf{A}(\mathbf{x}^*)}\,\|\nabla_{\Gamma_h(\mathbf{x}^*)}u_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\\
&\le C\|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)}\,\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}.
\end{aligned}
$$

(vi) Similarly we estimate

$$
\begin{aligned}
-\mathbf{e_u}^T\big(\mathbf{A}(\mathbf{x})-\mathbf{A}(\mathbf{x}^*)\big)\mathbf{e_u} &\le c\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)}\|D_{\Gamma_h(\mathbf{x}^*)}e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\\
&\le Ch^{(\kappa-1)/2}\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)},
\end{aligned}
$$

where we used the first bound of (6.5).

(vii) The coupling term is estimated similarly to (iii) in the proof of Proposition 5.1:

$$
\mathbf{e_u}^T\big(\mathbf{f}(\mathbf{x},\mathbf{u})-\mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)\big)=\int_{\Gamma_h^1}f\left(u_h,\nabla_{\Gamma_h^1}u_h\right)e_u^1-\int_{\Gamma_h^0}f\left(u_h^*,\nabla_{\Gamma_h^0}u_h^*\right)e_u^0.
$$

With

$$
u_h^\theta=\sum_{j=1}^N(u_j^*+\theta(\mathbf{e_u})_j)\,\phi_j[\mathbf{x}^*+\theta\mathbf{e_x}]=u_h^{*,\theta}+\theta e_u^\theta \tag{6.8}
$$

we therefore have

$$\mathbf{e}_{\mathbf{u}}^T\big(\mathbf{f}(\mathbf{x},\mathbf{u})-\mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)\big)=\int_0^1\frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\Gamma_h^\theta} f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)\,e_u^\theta\,\mathrm{d}\theta$$

and with the Leibniz formula (noting that e_x^θ is the velocity of the surface Γ_h^θ considered as a function of θ), we rewrite this as

$$\begin{aligned}&\mathbf{e}_{\mathbf{u}}^T\big(\mathbf{f}(\mathbf{x},\mathbf{u})-\mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)\big)\\&=\int_0^1\int_{\Gamma_h^\theta}\Big(\partial_\theta^\bullet f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)\,e_u^\theta+f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)\,e_u^\theta\,\left(\nabla_{\Gamma_h^\theta}\cdot e_x^\theta\right)\Big)\mathrm{d}\theta.\end{aligned}$$

Here we use the chain rule

$$\partial_\theta^\bullet f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)=\partial_1 f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)\,\partial_\theta^\bullet u_h^\theta+\partial_2 f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)\,\partial_\theta^\bullet\nabla_{\Gamma_h^\theta}u_h^\theta$$

and observe the following: by the assumed smoothness of f and the exact solution u, and by the bound (6.5) for e_u (and hence for e_u^θ by Lemmas 4.2 and 4.3), we have on recalling (6.8)

$$\left\|\partial_i f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)\right\|_{L^\infty(\Gamma_h^\theta)}\le C,\quad i=1,2.$$

We note

$$\partial_\theta^\bullet u_h^\theta=e_u^\theta$$

and the relation, see [14, Lemma 2.6],

$$\partial_\theta^\bullet\nabla_{\Gamma_h^\theta}u_h^\theta=\nabla_{\Gamma_h^\theta}\partial_\theta^\bullet u_h^\theta-\nabla_{\Gamma_h^\theta}e_x^\theta\,\nabla_{\Gamma_h^\theta}u_h^\theta+\nu_h^\theta\,(\nu_h^\theta)^T\left(\nabla_{\Gamma_h^\theta}e_x^\theta\right)^T\,\nabla_{\Gamma_h^\theta}u_h^\theta.$$

We then have, on inserting (6.8) and using once again Lemmas 4.2 and 4.3 and the bound (6.5),

$$\begin{aligned}\mathbf{e}_{\mathbf{u}}^T\big(\mathbf{f}(\mathbf{x},\mathbf{u})-\mathbf{f}(\mathbf{x}^*,\mathbf{u}^*)\big)&=\int_0^1\int_{\Gamma_h^\theta}e_u^\theta\Big(\partial_1 f\left(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta\right)e_u^\theta\\&\quad+\partial_2 f(u_h^\theta,\nabla_{\Gamma_h^\theta}u_h^\theta)\big(\nabla_{\Gamma_h^\theta}e_u^\theta-\nabla_{\Gamma_h^\theta}e_x^\theta\,\nabla_{\Gamma_h^\theta}u_h^\theta+\nu_h^\theta\,(\nu_h^\theta)^T\left(\nabla_{\Gamma_h^\theta}e_x^\theta\right)^T\,\nabla_{\Gamma_h^\theta}u_h^\theta\big)\Big)\mathrm{d}\theta\\&\le c\|e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))}\Big(\|e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))}\\&\quad+\|\nabla_{\Gamma_h(\mathbf{x}^*)}e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))}+\|\nabla_{\Gamma_h(\mathbf{x}^*)}e_x\|_{L^2(\Gamma_h(\mathbf{x}^*))}\,\|\nabla_{\Gamma_h(\mathbf{x}^*)}u_h^*\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\\&\quad+\|\nabla_{\Gamma_h(\mathbf{x}^*)}e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))}\,\|\nabla_{\Gamma_h(\mathbf{x}^*)}e_u\|_{L^2(\Gamma_h(\mathbf{x}^*))}\Big)\\&\le C\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)}+\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{A}(\mathbf{x}^*)}+\|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{A}(\mathbf{x}^*)}+\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{A}(\mathbf{x}^*)}\Big)\\&\le C\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{M}(\mathbf{x}^*)}+\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{A}(\mathbf{x}^*)}+\|\mathbf{e}_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^*)}\Big).\end{aligned}$$

Combined, the above estimates yield the following inequality:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)} \leq{} & C\,\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + C\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\Big)\\ & + C\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)} + c\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\\ & + C\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\|\mathbf{e_v}\|_{\mathbf{K}(\mathbf{x}^*)} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{e_u}^T\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big)\\ & + C\|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)}\|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)} + Ch^{(\kappa-1)/2}\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)}\\ & + C\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}\Big(\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\Big)\\ & + C\|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)}\|\mathbf{d_u}\|_{\star,\mathbf{x}^*}.\end{aligned}$$

Estimating further, using Young's inequality and absorptions into $\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)}$ (using $h \leq h_0$ for a sufficiently small h_0), we obtain the following estimate, where we can choose $\rho > 0$ small at the expense of enlarging the constant in front of $\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)}$:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + \frac{1}{2}\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)} \leq{} & c\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + c\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} + \rho\|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)}\\ & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{e_u}^T\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big) + c\|\mathbf{d_u}\|^2_{\star,\mathbf{x}^*}.\end{aligned} \tag{6.9}$$

(B) *Estimates in the surface equation:* Based on Sect. 5, we obtain

$$\|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)} \leq c\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} + |\mathbf{e_v}^T\big(\mathbf{g}(\mathbf{x},\mathbf{u}) - \mathbf{g}(\mathbf{x}^*,\mathbf{u}^*)\big)| + c\|\mathbf{d_v}\|^2_{\star,\mathbf{x}^*},$$

where the coupling term can be estimated based on (iii) in the proof of Proposition 5.1 and (vii) above:

$$|\mathbf{e_v}^T\big(\mathbf{g}(\mathbf{x},\mathbf{u}) - \mathbf{g}(\mathbf{x}^*,\mathbf{u}^*)\big)| \leq \|\mathbf{e_v}\|_{\mathbf{M}(\mathbf{x}^*)}\big(\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{e_x}\|_{\mathbf{K}(\mathbf{x}^*)}\big).$$

We then obtain

$$\|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)} \leq C\big(\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{d_v}\|^2_{\star,\mathbf{x}^*}\big). \tag{6.10}$$

As in Sect. 5, this provides the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} \leq C\big(\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} + \|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)} + \|\mathbf{d_v}\|^2_{\star,\mathbf{x}^*}\big). \tag{6.11}$$

(C) *Combination:* We first insert (6.10) into (6.9), where we can choose $\rho > 0$ so small that $C\rho \leq 1/2$ for the constant C in (6.10). Then we take a linear combination of (6.9) and (6.11) to obtain, for a sufficiently small $\sigma > 0$,

$$\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + \frac{1}{2}\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)} + \sigma\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)}$$

$$
\begin{aligned}
&\leq c\|\mathbf{e_u}\|^2_{\mathbf{M}(\mathbf{x}^*)} + c\|\mathbf{e_x}\|^2_{\mathbf{K}(\mathbf{x}^*)} + \frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{e_u}^T\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}\Big) \\
&\quad + c\|\mathbf{d_u}\|^2_{\star,\mathbf{x}^*} + c\|\mathbf{d_v}\|^2_{\star,\mathbf{x}^*}.
\end{aligned}
$$

We integrate both sides over $[0, t]$, for $0 \leq t \leq t^*$, to get

$$
\begin{aligned}
&\|\mathbf{e_u}(t)\|^2_{\mathbf{M}(\mathbf{x}^*)} + \frac{1}{2}\int_0^t \|\mathbf{e_u}(s)\|^2_{\mathbf{A}(\mathbf{x}^*)}\mathrm{d}s + \sigma\|\mathbf{e_x}(t)\|^2_{\mathbf{K}(\mathbf{x}^*)} \\
&\leq \|\mathbf{e_u}(0)\|^2_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_x}(0)\|^2_{\mathbf{K}(\mathbf{x}^*)} + c\int_0^t \Big(\|\mathbf{e_u}(s)\|^2_{\mathbf{M}(\mathbf{x}^*)} + \|\mathbf{e_x}(s)\|^2_{\mathbf{K}(\mathbf{x}^*)}\Big)\mathrm{d}s \\
&\quad - \mathbf{e_u}(t)^T\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}(t) + c\int_0^t \Big(\|\mathbf{d_u}(s)\|^2_{\star,\mathbf{x}^*} + \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\Big)\mathrm{d}s.
\end{aligned}
$$

The middle term can be further bounded using Lemmas 4.1–4.3 and an $L^2 - L^\infty - L^2$ estimate, as

$$
\begin{aligned}
\mathbf{e_u}(t)^T\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_u}(t) &= \int_0^1 \int_{\Gamma_h^\theta} e_u^\theta \cdot (\nabla_{\Gamma_h^\theta} \cdot e_x^\theta) e_u^\theta \,\mathrm{d}\theta \\
&\leq c\|\mathbf{e_u}(t)\|^2_{\mathbf{M}(\mathbf{x}^*)}\|\nabla_{\Gamma_h(\mathbf{x}^*)} \cdot e_x\|_{L^\infty(\Gamma_h(\mathbf{x}^*))} \\
&\leq Ch^{(\kappa-1)/2}\|\mathbf{e_u}(t)\|^2_{\mathbf{M}(\mathbf{x}^*)},
\end{aligned}
$$

where we used the first bound from (6.5) in the last inequality.

Absorbing this to the left-hand side and using Gronwall's inequality yields the stability estimate

$$
\begin{aligned}
&\|\mathbf{e_u}(t)\|^2_{\mathbf{M}(\mathbf{x}^*)} + \int_0^t \|\mathbf{e_u}(s)\|^2_{\mathbf{A}(\mathbf{x}^*)}\mathrm{d}s + \|\mathbf{e_x}(t)\|^2_{\mathbf{K}(\mathbf{x}^*)} \\
&\leq c\int_0^t \Big(\|\mathbf{d_u}(s)\|^2_{\star,\mathbf{x}^*} + \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\Big)\mathrm{d}s. \qquad (6.12)
\end{aligned}
$$

Inserting this bound in (6.10), squaring and integrating from 0 to t yields

$$
\int_0^t \|\mathbf{e_v}(s)\|^2_{\mathbf{K}(\mathbf{x}^*)}\,\mathrm{d}s \leq c\int_0^t \Big(\|\mathbf{d_u}(s)\|^2_{\star,\mathbf{x}^*} + \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\Big)\mathrm{d}s.
$$

With the assumed bounds of the defects, we obtain $O(h^k)$ error estimates for $0 \leq t \leq t^*$. Finally, to show that $t^* = T$, we use the same argument as at the end of the proof of Proposition 5.1. □

Remark 6.1 If the coupling function $g = g(u)$ in (2.2) does not depend on the tangential gradient of u, then the term $\|\mathbf{e_u}\|^2_{\mathbf{A}(\mathbf{x}^*)}$ does not appear in the bound (6.10). Therefore, inserting the estimate (6.12) into (6.10) then yields a pointwise stability estimate for $\mathbf{e_v}$: uniformly for $0 \le t \le T$,

$$\|\mathbf{e_v}(t)\|^2_{\mathbf{K}(\mathbf{x}^*)} \le C\|\mathbf{d_v}(t)\|^2_{\star,\mathbf{x}^*} + C\int_0^t \Big(\|\mathbf{d_u}(s)\|^2_{\star,\mathbf{x}^*} + \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\Big)\mathrm{d}s.$$

7 Geometric estimates

In this section we give further notations and some technical lemmas from [20] that will be used later on. Most of the results are high-order and time-dependent extensions of geometric approximation estimates shown in [7,8,10,13].

7.1 The interpolating surface

We return to the setting of Sect. 2, where $X(\cdot,t)$ defines a smooth surface $\Gamma(t) = \Gamma(X(\cdot,t))$. For an admissible triangulation of $\Gamma(t)$ with nodes $x_j^*(t) = X(p_j,t)$ and the corresponding nodal vector $\mathbf{x}^*(t) = (x_j^*(t))$, we define the interpolating surface by

$$X_h^*(p_h,t) = \sum_{j=1}^N x_j^*(t)\,\phi_j[\mathbf{x}(0)](p_h), \quad p_h \in \Gamma_h^0,$$

which has the properties that $X_h^*(p_j,t) = x_j^*(t) = X(p_j,t)$ for $j = 1,\dots,N$, and

$$\Gamma_h^*(t) := \Gamma_h(\mathbf{x}^*(t)) = \Gamma(X_h^*(\cdot,t)).$$

In the following we drop the argument t when it is not essential. The velocity of the interpolating surface Γ_h^*, defined as in Sect. 2.3, is denoted by v_h^*.

7.2 Approximation results

The lift of a function $\eta_h : \Gamma_h^*(t) \to \mathbb{R}$ is again denoted by $\eta_h^l : \Gamma(t) \to \mathbb{R}$, defined via the oriented distance function d between $\Gamma_h^*(t)$ and $\Gamma(t)$ provided that the surfaces are sufficiently close (which is the case if h is sufficiently small).

Lemma 7.1 (Equivalence of norms [8, Lemma 3], [7]) *Let $\eta_h : \Gamma_h^*(t) \to \mathbb{R}$ with lift $\eta_h^l : \Gamma(t) \to \mathbb{R}$. Then the L^p and $W^{1,p}$ norms on the discrete and continuous surfaces are equivalent for $1 \le p \le \infty$, uniformly in the mesh size $h \le h_0$ (with sufficiently small $h_0 > 0$) and in $t \in [0,T]$.*

In particular, there is a constant c such that for $h \le h_0$ and $0 \le t \le T$,

$$c^{-1}\|\eta_h\|_{L^2(\Gamma_h^*(t))} \le \|\eta_h^l\|_{L^2(\Gamma(t))} \le c\|\eta_h\|_{L^2(\Gamma_h^*(t))},$$
$$c^{-1}\|\eta_h\|_{H^1(\Gamma_h^*(t))} \le \|\eta_h^l\|_{H^1(\Gamma(t))} \le c\|\eta_h\|_{H^1(\Gamma_h^*(t))}.$$

Later on the following estimates will be used. They have been shown in [20], based on [7,13].

Lemma 7.2 *Let $\Gamma(t)$ and $\Gamma_h^*(t)$ be as above in Sect. 7.1. Then, for $h \le h_0$ with a sufficiently small $h_0 > 0$, we have the following estimates for the distance function d from (2.8), and for the error in the normal vector:*

$$\|d\|_{L^\infty(\Gamma_h^*(t))} \le ch^{k+1}, \quad \|\nu_{\Gamma(t)} - \nu^l_{\Gamma_h^*(t)}\|_{L^\infty(\Gamma(t))} \le ch^k,$$

with constants independent of $h \le h_0$ and $t \in [0, T]$.

7.3 Bilinear forms and their estimates

We use surface-dependent bilinear forms defined similarly as in [13]: Let X be a given surface with velocity v, with interpolation surface X_h^* with velocity v_h^*. For arbitrary $z, \varphi \in H^1(\Gamma(X))$ and for their discrete analogs $Z_h, \phi_h \in S_h(\mathbf{x}^*)$:

$$\begin{aligned} m(X; z, \varphi) &= \int_{\Gamma(X)} z\varphi, & m\left(X_h^*; Z_h, \phi_h\right) &= \int_{\Gamma(X_h^*)} Z_h\phi_h, \\ a(X; z, \varphi) &= \int_{\Gamma(X)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a\left(X_h^*; Z_h, \phi_h\right) &= \int_{\Gamma(X_h^*)} \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\ q(X; v; z, \varphi) &= \int_{\Gamma(X)} (\nabla_\Gamma \cdot v) z\varphi, & q\left(X_h^*; v_h^*; Z_h, \phi_h\right) &= \int_{\Gamma(X_h^*)} \left(\nabla_{\Gamma_h} \cdot v_h^*\right) Z_h\phi_h, \end{aligned}$$

where the discrete tangential gradients are understood in a piecewise sense. For more details see [13, Lemma 2.1] (and the references in the proof), or [12, Lemma 5.2].

We start by defining a discrete velocity on the smooth surface, denoted by $\hat{v}_h$. We follow Section 5.3 of [20], where the high-order ESFEM generalization of the discrete velocity on $\Gamma(X)$ from Sections 4.3 and 5.3 of [13] is discussed. Using the lifted elements, $\Gamma(X)$ is decomposed into curved elements whose Lagrange points move with the velocity $\hat{v}_h$ defined by

$$\hat{v}_h\big(\left(X_h^*\right)^l(\cdot, t), t\big) = \frac{\mathrm{d}}{\mathrm{dt}} \left(X_h^*\right)^l(\cdot, t).$$

Discrete material derivatives on $\Gamma(X_h^*)$ and $\Gamma(X)$ are given by

$$\begin{aligned} \partial^\bullet_{v_h^*}\varphi_h &= \partial_t\varphi_h + v_h^* \cdot \nabla\varphi_h, \\ \partial^\bullet_{\hat{v}_h}\varphi_h^l &= \partial_t\varphi_h^l + \hat{v}_h \cdot \nabla\varphi_h^l, \end{aligned} \quad (\varphi_h \in S_h(\mathbf{x}^*)).$$

In [13, Lemma 4.1] it was shown that the transport property of the basis functions carries over to the lifted basis functions $\phi_j[\mathbf{x}^*]$:

$$\partial^\bullet_{\hat{v}_h}\phi_j[\mathbf{x}^*]^l = \left(\partial^\bullet_{v_h^*}\phi_j[\mathbf{x}^*]\right)^l = 0, \quad (j = 1, \ldots, N).$$

Therefore, the above discrete material derivatives and the lift operator satisfy, for $\varphi_h \in S_h(X_h^*)$,

$$\partial^\bullet_{\hat{v}_h} \varphi_h^l = \left(\partial^\bullet_{v_h^*} \varphi_h\right)^l. \tag{7.1}$$

Lemma 7.3 (Transport properties [13, Lemma 4.2]) *For any $z(t), \varphi(t) \in H^1(\Gamma(X(\cdot, t)))$,*

$$\frac{\mathrm{d}}{\mathrm{dt}} m(X; z, \varphi) = m\left(X; \partial^\bullet z, \varphi\right) + m\left(X; z, \partial^\bullet \varphi\right) + q(X; v; z, \varphi).$$

The same formulas hold when $\Gamma(X)$ is considered as the lift of the discrete surface $\Gamma(X_h^*)$ (i.e. $\Gamma(X)$ can be decomposed into curved elements which are lifts of the elements of $\Gamma(X_h^*)$), moving with the velocity $\hat{v}_h$:

$$\frac{\mathrm{d}}{\mathrm{dt}} m(X; z, \varphi) = m(X; \partial^\bullet_{\hat{v}_h} z, \varphi) + m(X; z, \partial^\bullet_{\hat{v}_h} \varphi) + q(X; \hat{v}_h; z, \varphi).$$

Similarly, in the discrete case, for arbitrary $z_h(t)$, $\varphi_h(t)$, $\partial^\bullet_{v_h^*} z_h(t)$, $\partial^\bullet_{v_h^*} \varphi_h(t) \in S_h(\mathbf{x}^*(t))$ we have:

$$\frac{\mathrm{d}}{\mathrm{dt}} m(X_h^*; z_h, \varphi_h) = m\left(X_h^*; \partial^\bullet_{v_h^*} z_h, \varphi_h\right) + m\left(X_h^*; z_h, \partial^\bullet_{v_h^*} \varphi_h\right) + q\left(X_h^*; v_h^*; z_h, \varphi_h\right),$$

where v_h^* is the velocity of the surface $\Gamma(X_h^*)$.

The following estimates, proved in Lemma 5.6 of [20], will play a crucial role in the defect bounds later on.

Lemma 7.4 (Geometric perturbation errors) *For any $Z_h, \psi_h \in S_h(\mathbf{x}^*)$ where $\Gamma(X_h^*)$ is the interpolation surface of piecewise polynomial degree k, we have the following bounds, for $h \leq h_0$ with a sufficiently small $h_0 > 0$,*

$$\begin{aligned}
\left| m\left(X; Z_h^l, \varphi_h^l\right) - m\left(X_h^*; Z_h, \varphi_h\right) \right| &\leq c h^{k+1} \|Z_h^l\|_{L^2(\Gamma(X))} \|\varphi_h^l\|_{L^2(\Gamma(X))}, \\
\left| a\left(X; Z_h^l, \varphi_h^l\right) - a\left(X_h^*; Z_h, \varphi_h\right) \right| &\leq c h^{k+1} \|\nabla_\Gamma Z_h^l\|_{L^2(\Gamma(X))} \|\nabla_\Gamma \varphi_h^l\|_{L^2(\Gamma(X))}, \\
\left| q\left(X; \hat{v}_h; Z_h^l, \varphi_h^l\right) - q\left(X_h^*; v_h^*; Z_h, \varphi_h\right) \right| &\leq c h^{k+1} \|Z_h^l\|_{L^2(\Gamma(X))} \|\varphi_h^l\|_{L^2(\Gamma(X))}.
\end{aligned}$$

The constant c is independent of h and $t \in [0, T]$.

7.4 Interpolation error estimates for evolving surface finite element functions

For any $u \in H^{k+1}(\Gamma(X))$, there is a unique piecewise polynomial surface finite element interpolation of degree k in the nodes x_j^*, denoted by $\widetilde{I}_h u \in S_h(\mathbf{x}^*)$. We set $I_h u := (\widetilde{I}_h u)^l : \Gamma(X) \to \mathbb{R}$. Error estimates for this interpolation are obtained from

[7, Proposition 2.7] by carefully studying the time dependence of the constants, cf. [20].

Lemma 7.5 *There exists a constant $c > 0$ independent of $h \le h_0$, with a sufficiently small $h_0 > 0$, and t such that for $u(\cdot, t) \in H^{k+1}(\Gamma(t))$, for $0 \le t \le T$,*

$$\|u - I_h u\|_{L^2(\Gamma(X))} + h\|\nabla_\Gamma (u - I_h u)\|_{L^2(\Gamma(X))} \le ch^{k+1}\|u\|_{H^{k+1}(\Gamma(X))}.$$

The same result holds for vector valued functions. As it will always be clear from the context we do not distinguish between interpolations for scalar and vector valued functions.

8 Defect bounds

In this section we show that the assumed defect estimates of Propositions 5.1 and 6.1 are indeed fulfilled.

The interpolations satisfy the discrete problem (2.4)–(2.5) only up to some defects. These defects are denoted by $d_u \in S_h(\mathbf{x}^*)$, $d_v \in S_h(\mathbf{x}^*)^3$, with $\mathbf{x}^*(t)$ the vector of exact nodal values $x_j^*(t) = X(p_j, t) \in \Gamma(t)$, and are given as follows: for all $\varphi_h \in S_h(\mathbf{x}^*)$ with $\partial^\bullet_{v_h^*}\varphi_h = 0$ and $\psi_h \in S_h(\mathbf{x}^*)^3$,

$$\begin{aligned}
\int_{\Gamma_h(\mathbf{x}^*)} d_u \varphi_h &= \frac{\mathrm{d}}{\mathrm{dt}} \int_{\Gamma_h(\mathbf{x}^*)} \widetilde{I}_h u\, \varphi_h + \int_{\Gamma_h(\mathbf{x}^*)} \nabla_{\Gamma_h(\mathbf{x}^*)} \widetilde{I}_h u \cdot \nabla_{\Gamma_h(\mathbf{x}^*)} \varphi_h \\
&\quad - \int_{\Gamma_h(\mathbf{x}^*)} f\left(\widetilde{I}_h u, \nabla_{\Gamma_h(\mathbf{x}^*)} \widetilde{I}_h u\right) \varphi_h, \\
\int_{\Gamma_h(\mathbf{x}^*)} d_v \cdot \psi_h &= \int_{\Gamma_h(\mathbf{x}^*)} \widetilde{I}_h v \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x}^*)} \nabla_{\Gamma_h(\mathbf{x}^*)} \widetilde{I}_h v \cdot \nabla_{\Gamma_h(\mathbf{x}^*)} \psi_h \\
&\quad - \int_{\Gamma_h(\mathbf{x}^*)} g\left(\widetilde{I}_h u, \nabla_{\Gamma_h(\mathbf{x}^*)} \widetilde{I}_h u\right) \nu_{\Gamma_h(\mathbf{x}^*)} \cdot \psi_h.
\end{aligned}$$

Later on the vectors of nodal values of the defects d_u and d_v are denoted by $\mathbf{d_u} \in \mathbb{R}^N$ and $\mathbf{d_v} \in \mathbb{R}^{3N}$, respectively. These vectors satisfy (6.1).

Lemma 8.1 *Let the solution u, the surface X and its velocity v be all sufficiently smooth. Then there exists a constant $c > 0$ such that for all $h \le h_0$, with a sufficiently small $h_0 > 0$, and for all $t \in [0, T]$, the defects d_u and d_v of the kth-degree finite element interpolation are bounded as*

$$\begin{aligned}
\|\mathbf{d_u}\|_{\star,\mathbf{x}^*} &= \|d_u\|_{H_h^{-1}(\Gamma(X_h^*))} \le ch^k, \\
\|\mathbf{d_v}\|_{\star,\mathbf{x}^*} &= \|d_v\|_{H_h^{-1}(\Gamma(X_h^*))} \le ch^k,
\end{aligned}$$

where the H_h^{-1}-norm is defined in (5.5). The constant c is independent of h and $t \in [0, T]$.

Proof (i) We start from an identity for the dual norm as in (5.5), (omitting the argument $\mathbf{x}^*$ of the matrices):

$$\|\mathbf{d_u}\|_{\star,\mathbf{x}^*} = \left(\mathbf{d_u}^T\mathbf{M}(\mathbf{M}+\mathbf{A})^{-1}\mathbf{M}\mathbf{d_u}\right)^{\frac{1}{2}} = \|d_u\|_{H_h^{-1}(\Gamma(X_h^*))}.$$

In order to estimate the defect in u, we subtract (2.3) from the above equation, and perform almost the same proof as in [13, Section 7]. We use the bilinear forms and the discrete versions of the transport properties from Lemma 7.3. We obtain, for any $\varphi_h \in S_h(\mathbf{x}^*)$ with $\partial^\bullet_{v_h^*}\varphi_h = 0$,

$$\begin{aligned}
m\left(X_h^*; d_u, \varphi_h\right) &= \frac{\mathrm{d}}{\mathrm{dt}} m\left(X_h^*; \widetilde{I}_h u, \varphi_h\right) + a\left(X_h^*; \widetilde{I}_h u, \varphi_h\right) \\
&\quad - m\left(X_h^*; f\left(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u\right), \varphi_h\right) \\
&= m\left(X_h^*; \partial^\bullet_{v_h^*}\widetilde{I}_h u, \varphi_h\right) + q\left(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h\right) + a\left(X_h^*; \widetilde{I}_h u, \varphi_h\right) \\
&\quad - m\left(X_h^*; f\left(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u\right), \varphi_h\right),
\end{aligned}$$

and

$$\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{dt}} m\left(X; u, \varphi_h^l\right) + a\left(X; u, \varphi_h^l\right) - m\left(X; f\left(u, \nabla_{\Gamma(X)}u\right), \varphi_h^l\right) \\
&= m\left(X; \partial^\bullet_{\hat{v}_h} u, \varphi_h^l\right) + q\left(X; \hat{v}_h; u, \varphi_h^l\right) + a\left(X; u, \varphi_h^l\right) - m\left(X; f\left(u, \nabla_{\Gamma(X)}u\right), \varphi_h^l\right).
\end{aligned}$$

Subtracting the two equation yields

$$\begin{aligned}
m\left(X_h^*; d_u, \varphi_h\right) &= m\left(X_h^*; \partial^\bullet_{v_h^*}\widetilde{I}_h u, \varphi_h\right) - m\left(X; \partial^\bullet_{\hat{v}_h} u, \varphi_h^l\right) \\
&\quad + q\left(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h\right) - q\left(X; \hat{v}_h; u, \varphi_h^l\right) \\
&\quad + a\left(X_h^*; \widetilde{I}_h u, \varphi_h\right) - a\left(X; u, \varphi_h^l\right) \\
&\quad - \left(m\left(X_h^*; f\left(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u\right), \varphi_h\right) - m\left(X; f\left(u, \nabla_\Gamma u\right), \varphi_h^l\right)\right).
\end{aligned}$$

We bound all the terms pairwise, by using the interpolation estimates of Lemma 7.5 and the estimates for the geometric perturbation errors of the bilinear forms of Lemma 7.4. For the first pair, using that $(\partial^\bullet_{v_h^*}\widetilde{I}_h u)^l = \partial^\bullet_{\hat{v}_h} I_h u$, we obtain

$$\begin{aligned}
\left|m\left(X_h^*; \partial^\bullet_{v_h^*}\widetilde{I}_h u, \varphi_h\right) - m\left(X; \partial^\bullet_{\hat{v}_h} u, \varphi_h^l\right)\right| &\leq \left|m\left(X_h^*; \partial^\bullet_{v_h^*}\widetilde{I}_h u, \varphi_h\right) - m\left(X; \partial^\bullet_{\hat{v}_h} I_h u, \varphi_h^l\right)\right| \\
&\quad + \left|m\left(X; I_h\partial^\bullet_{\hat{v}_h} u - \partial^\bullet_{\hat{v}_h} u, \varphi_h^l\right)\right| \\
&\leq ch^{k+1}\|\varphi_h^l\|_{L^2(\Gamma(X))}.
\end{aligned}$$

For the second pair we obtain

$$\begin{aligned}\left|q\left(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h\right) - q\left(X; \hat{v}_h; u, \varphi_h^l\right)\right| &\leq \left|q\left(X_h^*; v_h^*; \widetilde{I}_h u, \varphi_h\right) - q\left(X; \hat{v}_h; I_h u, \varphi_h^l\right)\right| \\ &\quad + \left|q\left(X; v_h^*; I_h u - u, \varphi_h\right)\right| \\ &\leq c h^{k+1} \|\varphi_h^l\|_{L^2(\Gamma(X))}.\end{aligned}$$

The third pair is estimated by

$$\begin{aligned}\left|a\left(X_h^*; \widetilde{I}_h u, \varphi_h\right) - a\left(X; u, \varphi_h^l\right)\right| &\leq \left|a\left(X_h^*; \widetilde{I}_h u, \varphi_h\right) - a\left(X; I_h u, \varphi_h^l\right)\right| \\ &\quad + \left|a\left(X; I_h u - u, \varphi_h^l\right)\right| \\ &\leq c h^k \|\nabla_\Gamma \varphi_h^l\|_{L^2(\Gamma(X))}.\end{aligned}$$

For the last pair we use the fact that $(f(u, \nabla_\Gamma u))^{-l} = f(u^{-l}, (\nabla_\Gamma u)^{-l})$ and the local Lipschitz continuity of the function f, to obtain

$$\begin{aligned}&\left|m\left(X_h^*; f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h\right) - m\left(X; f\left(u, \nabla_\Gamma u\right), \varphi_h^l\right)\right| \\ &\quad \leq \left|m\left(X_h^*; f\left(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u\right) - f\left(u^{-l}, (\nabla_\Gamma u)^{-l}\right), \varphi_h\right)\right| \\ &\qquad + \left|m\left(X_h^*; f\left(u, \nabla_\Gamma u\right)^{-l}, \varphi_h\right) - m\left(X; f\left(u, \nabla_\Gamma u\right), \varphi_h^l\right)\right| \\ &\quad \leq c \|f\left(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u\right) - f\left(u^{-l}, (\nabla_\Gamma u)^{-l}\right)\|_{L^2(\Gamma(X_h^*))} \|\varphi_h^l\|_{L^2(\Gamma(X))} \\ &\qquad + c h^{k+1} \|\varphi_h^l\|_{L^2(\Gamma(X))}.\end{aligned}$$

The first term is estimated, using the local Lipschitz continuity of f and equivalence of norms, by

$$\begin{aligned}&\|f\left(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u\right) - f(u^{-l}, (\nabla_\Gamma u)^{-l})\|_{L^2(\Gamma(X_h^*))} \\ &\quad \leq \|f\|_{W^{1,\infty}} \Big(c\|I_h u - u\|_{L^2(\Gamma(X))} + c\|\nabla_\Gamma (I_h u - u)\|_{L^2(\Gamma(X))} \\ &\qquad + c\|(\nabla_{\Gamma_h} u^{-l})^l - \nabla_\Gamma u\|_{L^2(\Gamma(X))}\Big),\end{aligned}$$

where the first two terms are bounded by $O(h^k)$ using interpolation estimates, while the third term is bounded, using Remark 4.1 in [13] and Lemma 7.2, as

$$\left\|\left(\nabla_{\Gamma_h} u^{-l}\right)^l - \nabla_\Gamma u\right\|_{L^2(\Gamma(X))} \leq c h^k.$$

Thus for the fourth pair we obtained

$$\left|m\left(X_h^*; f\left(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u\right), \varphi_h\right) - m\left(X; f\left(u, \nabla_\Gamma u\right), \varphi_h^l\right)\right| \leq c h^k \|\varphi_h^l\|_{L^2(\Gamma(X))}.$$

Altogether, we have

$$m\left(X_h^*; d_u, \varphi_h\right) \le ch^k \|\varphi_h^l\|_{H^1(\Gamma(X))},$$

which, by the equivalence of norms given by Lemma 7.1, shows the first bound of the stated lemma.

(ii) In order to estimate the defect in v, similarly as previously we subtract (2.3) from the above equation and use the bilinear forms to obtain

$$\begin{aligned} m\left(X_h^*; d_{\mathbf{v}}, \psi_h\right) = {} & m\left(X_h^*; \widetilde{I}_h v, \psi_h\right) - m\left(X; v, \psi_h^l\right) \\ & + \alpha\Big(a\left(X_h^*; \widetilde{I}_h v, \psi_h\right) - a\left(X; v, \psi_h^l\right)\Big) \\ & + m\left(X_h^*; g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u)\nu_{\Gamma(X_h^*)}, \psi_h\right) - m\left(X; g(u, \nabla_\Gamma u)\nu_{\Gamma(X)}, \psi_h^l\right). \end{aligned}$$

Similarly as in the previous part, these three pairs are bounded pairwise. For the first pair we have

$$\begin{aligned} \left| m\left(X_h^*; \widetilde{I}_h v, \psi_h\right) - m\left(X; v, \psi_h^l\right)\right| \le {} & \left| m\left(X_h^*; \widetilde{I}_h v, \psi_h\right) - m\left(X; I_h v, \psi_h^l\right)\right| \\ & + \left| m\left(X; I_h v - v, \psi_h^l\right)\right| \\ \le {} & ch^{k+1} \|\psi_h^l\|_{L^2(\Gamma(X))}. \end{aligned}$$

For the second pair we use the interpolation estimate to bound

$$\begin{aligned} \left| a\left(X_h^*; \widetilde{I}_h v, \psi_h\right) - a\left(X; v, \psi_h^l\right)\right| \le {} & \left| a\left(X_h^*; \widetilde{I}_h v, \psi_h\right) - a\left(X; I_h v, \psi_h^l\right)\right| \\ & + \left| a\left(X; I_h v - v, \psi_h^l\right)\right| \\ \le {} & ch^k \left\|\nabla_\Gamma \psi_h^l\right\|_{L^2(\Gamma(X))}. \end{aligned}$$

The third pair we estimate, similarly to the nonlinear pair above, by

$$\begin{aligned} & \left| m\left(X_h^*; g\left(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u\right) \nu_{\Gamma(X_h^*)}, \psi_h\right) - m\left(X; g\left(u, \nabla_\Gamma u\right) \nu_{\Gamma(X)}, \psi_h^l\right)\right| \\ & \quad \le \left| m\left(X_h^*; \left(g\left(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u\right) - g\left(u, \nabla_\Gamma u\right)^{-l}\right) \nu_{\Gamma(X_h^*)}, \psi_h\right)\right| \\ & \qquad + \left| m\left(X_h^*; g\left(u, \nabla_\Gamma u\right)^{-l} \left(\nu_{\Gamma(X_h^*)} - \nu_{\Gamma(X)}^{-l}\right), \psi_h\right)\right| \\ & \qquad + \left| m\left(X_h^*; g\left(u, \nabla_\Gamma u\right)^{-l} \nu_{\Gamma(X)}^{-l}, \psi_h\right) - m\left(X; g\left(u, \nabla_\Gamma u\right) \nu_{\Gamma(X)}, \psi_h^l\right)\right| \\ & \quad \le ch^k \|g\|_{W^{1,\infty}} \|\psi_h^l\|_{L^2(\Gamma(X))} + c\|\nabla_\Gamma (X - X_h^*)\|_{L^2(\Gamma(X))} \|\psi_h^l\|_{L^2(\Gamma(X))} \\ & \qquad + ch^{k+1} \|g\|_{L^2} \|\psi_h^l\|_{L^2(\Gamma(X))} \\ & \quad \le ch^k \|g\|_{W^{1,\infty}} \|\psi_h^l\|_{L^2(\Gamma(X))} + ch^k \|\psi_h^l\|_{L^2(\Gamma(X))} \\ & \quad \le ch^k \|\psi_h^l\|_{L^2(\Gamma(X))}, \end{aligned}$$

where we have used the local Lipschitz boundedness of the function g, the interpolation estimate, Lemmas 7.2 and 7.4, through a similar argument as above for the semilinear term with f.

Finally, the combination of these bounds yields

$$m\left(X_h^*; d_v, \psi_h\right) \leq ch^k \|\psi_h^l\|_{H^1(\Gamma(X))},$$

providing the asserted bound on $\mathbf{d_v}$. □

9 Proof of Theorem 3.1

The errors are decomposed using interpolations and the definition of lifts from Sect. 2.6: omitting the argument t,

$$\begin{aligned} u_h^L - u &= \left(\widehat{u}_h - \widetilde{I}_h u\right)^l + \left(I_h u - u\right), \\ v_h^L - v &= \left(\widehat{v}_h - \widetilde{I}_h v\right)^l + \left(I_h v - v\right), \\ X_h^L - X &= \left(\widehat{X}_h - \widetilde{I}_h X\right)^l + \left(I_h X - X\right). \end{aligned}$$

The last terms in these formulas can be bounded in the $H^1(\Gamma)$ norm by Ch^k, using the interpolation bounds of Lemma 7.5.

To bound the first terms on the right-hand sides, we first use the defect bounds of Lemma 8.1, which then together with the stability estimate of Proposition 6.1 proves the result, since by the norm equivalences of Lemma 7.1 and Eqs. (4.1)–(4.2) we have (again omitting the argument t)

$$\begin{aligned} |\left(\widehat{u}_h - \widetilde{I}_h u\right)^l\|_{L^2(\Gamma)} &\leq c\|\widehat{u}_h - \widetilde{I}_h u\|_{L^2(\Gamma_h^*)} = c\|\mathbf{e_u}\|_{\mathbf{M}(\mathbf{x}^*)}, \\ |\nabla_\Gamma\left(\widehat{u}_h - \widetilde{I}_h u\right)^l\|_{L^2(\Gamma)} &\leq c\|\nabla_{\Gamma_h^*}\left(\widehat{u}_h - \widetilde{I}_h u\right)\|_{L^2(\Gamma_h^*)} = c\|\mathbf{e_u}\|_{\mathbf{A}(\mathbf{x}^*)}, \end{aligned}$$

and similarly for $\widehat{v}_h - \widetilde{I}_h v$ and $\widehat{X}_h - \widetilde{I}_h X$.

10 Extension to other velocity laws

In this section we consider the extension of our results to different velocity laws: adding a mean curvature term to the regularized velocity law considered so far, and a dynamic velocity law. We concentrate on the velocity laws without coupling to the surface PDE, since the coupling can be dealt with in the same way as previously. We only consider the stability of the evolving surface finite element discretization, since bounds for the consistency error are obtained by the same arguments as before.

10.1 Regularized mean curvature flow

We next extend our results to the case where the velocity law contains a mean curvature term:

$$v - \alpha \Delta_{\Gamma(X)} v - \beta \Delta_{\Gamma(X)} X = g(\cdot, t) \nu_{\Gamma(X)}, \tag{10.1}$$

where $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given Lipschitz continuous function of (x, t), and $\alpha > 0$ and $\beta > 0$ are fixed parameters. Here $\Delta_{\Gamma(X)} X$ is a suggestive notation for $-H\nu$, where H denotes the mean curvature of the surface $\Gamma(X)$. (More precisely, $\Delta_{\Gamma(X)}\mathrm{id} = -H\nu_{\Gamma(X)}$).

The corresponding differential-algebraic system reads

$$\mathbf{K}(\mathbf{x})\mathbf{v} + \mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{g}(\mathbf{x}), \tag{10.2}$$

where $\mathbf{K}(\mathbf{x})$ is again defined by (2.6) and where we now write $\mathbf{A}(\mathbf{x})$ for the matrix $\beta I_3 \otimes \mathbf{A}(\mathbf{x})$ with $\mathbf{A}(\mathbf{x})$ of Sect. 2.5.

Similarly as before the corresponding error equation is given as

$$\begin{aligned}
\mathbf{K}(\mathbf{x}^*)\mathbf{e_v} + \mathbf{A}(\mathbf{x}^*)\mathbf{e_x} = &-\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* - \big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e_v} \\
&- \big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{x}^* - \big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{e_x} \\
&+ \big(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\big) - \mathbf{M}(\mathbf{x}^*)\mathbf{d_v}
\end{aligned}$$

together with $\dot{\mathbf{e}}_\mathbf{x} = \mathbf{e_v}$.

Proposition 10.1 *Under the assumptions of Proposition 5.1, there exists $h_0 > 0$ such that the following stability estimate holds for all $h \le h_0$, for $0 \le t \le T$:*

$$\begin{aligned}
\|\mathbf{e_x}(t)\|^2_{\mathbf{K}(\mathbf{x}^*(t))} &\le C \int_0^t \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\, \mathrm{d}s, \\
\|\mathbf{e_v}(t)\|^2_{\mathbf{K}(\mathbf{x}^*(t))} &\le C \|\mathbf{d_v}(t)\|^2_{\star,\mathbf{x}^*} + C \int_0^t \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\, \mathrm{d}s.
\end{aligned}$$

The constant C is independent of t and h, but depends on the final time T, and on the parameters α and β.

Proof We detail only those parts of the proof of Proposition 5.1 where the mean curvature term introduces differences, otherwise exactly the same proof applies.

In order to prove the stability estimate we again test with $\mathbf{e_v}$, and obtain

$$\begin{aligned}
\|\mathbf{e_v}\|^2_{\mathbf{K}(\mathbf{x}^*)} = &-\mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{v}^* - \mathbf{e_v}^T\big(\mathbf{K}(\mathbf{x}) - \mathbf{K}(\mathbf{x}^*)\big)\mathbf{e_v} \\
&- \mathbf{e_v}^T\big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{x}^* - \mathbf{e_v}^T\big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{e_x} - \mathbf{e_v}^T\mathbf{A}(\mathbf{x}^*)\mathbf{e_x} \\
&+ \mathbf{e_v}^T\big(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\big) - \mathbf{e_v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{d_v}.
\end{aligned}$$

Every term is estimated exactly as previously in the proof of Proposition 5.1, except the terms corresponding to the mean curvature term, involving the stiffness matrix $\mathbf{A}$. They are estimated by the same techniques as previously:

$$\mathbf{e}_\mathbf{v}^T\big(\mathbf{A}(\mathbf{x})-\mathbf{A}(\mathbf{x}^*)\big)\mathbf{x}^* + \mathbf{e}_\mathbf{v}^T\big(\mathbf{A}(\mathbf{x})-\mathbf{A}(\mathbf{x}^*)\big)\mathbf{e}_\mathbf{x} \le \frac{1}{6}\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2,$$
$$\mathbf{e}_\mathbf{v}^T\mathbf{A}(\mathbf{x}^*)\mathbf{e}_\mathbf{x} \le \frac{1}{6}\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2.$$

Altogether we obtain the error bound

$$\|\mathbf{e}_\mathbf{v}\|_{\mathbf{K}(\mathbf{x}^*)}^2 \le c\|\mathbf{e}_\mathbf{x}\|_{\mathbf{K}(\mathbf{x}^*)}^2 + c\|\mathbf{d}_\mathbf{v}\|_{\star,\mathbf{x}^*}^2,$$

which is exactly (5.9). The proof is then completed as before. □

With Proposition 10.1 and the appropriate defect bounds, Theorem 3.1 extends directly to the system with mean curvature term in the regularized velocity law.

10.2 A dynamic velocity law

Let us consider the dynamic velocity law, again without coupling to a surface PDE:

$$\partial^\bullet v + v\nabla_{\Gamma(X)}\cdot v - \alpha\Delta_{\Gamma(X)}v = g(\cdot,t)\,\nu_{\Gamma(X)},$$

where again $g:\mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$ is a given Lipschitz continuous function of (x,t), and $\alpha>0$ is a fixed parameter. This problem is considered together with the ordinary differential equations (2.1) for the positions X determining the surface $\Gamma(X)$. Initial values are specified for X and v.

The weak formulation and the semidiscrete problem can be obtained by a similar argument as for the PDE on the surface in Sect, 6. Therefore we immediately present the ODE formulation of the semidiscretization. As in Sect. 2.5, the nodal vectors $\mathbf{v}\in\mathbb{R}^{3N}$ of the finite element function v_h, together with the surface nodal vector $\mathbf{x}\in\mathbb{R}^{3N}$ satisfy a system of ODEs with matrices and driving term as in Sect. 5:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x})\mathbf{v}\Big) + \mathbf{A}(\mathbf{x})\mathbf{v} &= \mathbf{g}(\mathbf{x},t),\\ \dot{\mathbf{x}} &= \mathbf{v}.\end{aligned}\tag{10.3}$$

By using the same notations for the exact positions $\mathbf{x}^*(t)\in\mathbb{R}^{3N}$, for the interpolated exact velocity $\mathbf{v}^*(t)\in\mathbb{R}^{3N}$, and for the defect $\mathbf{d}_\mathbf{v}(t)$, we obtain that they fulfill the following equation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{v}^*\Big) + \mathbf{A}(\mathbf{x}^*)\mathbf{v}^* &= \mathbf{g}(\mathbf{x}^*,t) + \mathbf{M}(\mathbf{x}^*)\mathbf{d}_\mathbf{v},\\ \dot{\mathbf{x}}^* &= \mathbf{v}^*.\end{aligned}$$

By subtracting this from (10.3), and using similar arguments as before, we obtain the error equations for the surface nodes and velocity:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{e_v}\Big) + \mathbf{A}(\mathbf{x}^*)\mathbf{e_v} &= -\frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{v}^*\Big) \\
&\quad - \frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_v}\Big) \\
&\quad - \big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{v}^* \\
&\quad - \big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{e_v} \\
&\quad + \big(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\big) - \mathbf{M}(\mathbf{x}^*)\mathbf{d_v} \\
\dot{\mathbf{e}}_\mathbf{x} &= \mathbf{e_v}.
\end{aligned}
$$

We then have the following stability result.

Proposition 10.2 *Under the assumptions of Proposition 5.1, there exists $h_0 > 0$ such that the following error estimate holds for all $h \le h_0$, uniformly for $0 \le t \le T$:*

$$
\|\mathbf{e_x}(t)\|^2_{\mathbf{K}(\mathbf{x}^*(t))} + \|\mathbf{e_v}(t)\|^2_{\mathbf{M}(\mathbf{x}^*(t))} + \int_0^t \|\mathbf{e_v}(s)\|^2_{\mathbf{A}(\mathbf{x}^*(s))}\,\mathrm{d}s \le C\int_0^t \|\mathbf{d_v}(s)\|^2_{\star,\mathbf{x}^*}\,\mathrm{d}s.
$$

The constant $C > 0$ is independent of t and h, but depends on the final time T and the parameter α.

Proof By testing the error equation with $\mathbf{e_v}$ we obtain

$$
\begin{aligned}
\mathbf{e_v}^T\frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x}^*)\mathbf{e_v}\Big) + \mathbf{e_v}^T\mathbf{A}(\mathbf{x}^*)\mathbf{e_v} &= -\mathbf{e_v}^T\frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{v}^*\Big) \\
&\quad - \mathbf{e_v}^T\frac{\mathrm{d}}{\mathrm{dt}}\Big(\big(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{x}^*)\big)\mathbf{e_v}\Big) \\
&\quad - \mathbf{e_v}^T\big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{v}^* \\
&\quad - \mathbf{e_v}^T\big(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{x}^*)\big)\mathbf{e_v} \\
&\quad + \mathbf{e_v}^T\big(\mathbf{g}(\mathbf{x}) - \mathbf{g}(\mathbf{x}^*)\big) - \mathbf{e_v}^T\mathbf{M}(\mathbf{x}^*)\mathbf{d_v}.
\end{aligned}
$$

The terms are bounded in the same way as the corresponding terms in the proofs of Propositions 5.1 and 6.1. With these estimates, a Gronwall inequality yields the result. □

With Proposition 10.2 and the appropriate defect bounds, Theorem 3.1 extends directly to the parabolic surface PDE coupled with the dynamic velocity law.

11 Numerical results

In this section we complement Theorem 3.1 by showing the numerical behaviour of piecewise linear finite elements, which are not covered by Theorem 3.1, but nevertheless perform remarkably well. Moreover, we compare our regularized velocity law with regularization by mean curvature flow.

11.1 A coupled problem

Our test problem is a combination of (2.2) with a mean curvature term as in (10.1):

$$\begin{aligned} \partial^\bullet u + u\nabla_\Gamma \cdot v - \Delta_\Gamma u &= f(t,x), \\ v - \alpha\Delta_\Gamma v - \beta\Delta_\Gamma X &= \delta u \nu_\Gamma + g(t,x)\nu_\Gamma, \end{aligned} \tag{11.1}$$

for non-negative parameters α, β, δ. The velocity law here is a special case of (2.2) for $\beta = 0$, and reduces to (10.1) for $\delta = 0$. The matrix–vector form reads

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{M}\big(\mathbf{x}(t)\big)\mathbf{u}(t)\Big) + \mathbf{A}\big(\mathbf{x}(t)\big)\mathbf{u}(t) &= \mathbf{f}\big(t,\mathbf{x}(t)\big), \quad t \in [0,T], \\ \mathbf{K}\big(\mathbf{x}(t)\big)\dot{\mathbf{x}}(t) + \beta\mathbf{A}\big(\mathbf{x}(t)\big)\mathbf{x}(t) &= \delta\mathbf{N}\big(\mathbf{x}(t)\big)\mathbf{u}(t) + \mathbf{g}\big(t,\mathbf{x}(t)\big), \quad t \in [0,T], \end{aligned}$$

for given $\mathbf{x}(0)$ and $\mathbf{u}(0)$, where

$$\mathbf{N}\big(\mathbf{x}\big)\mathbf{u}|_{3(j-1)+\ell} = \int_{\Gamma_h(\mathbf{x})} \big(\nu_{\Gamma_h}\big)_\ell \mathbf{u}_j \phi_j[\mathbf{x}],$$

for $j = 1,\dots,N$ and $\ell = 1,2,3$.

In our numerical experiments we used a linearly implicit Euler discretization of this system with step sizes chosen so small that the error is dominated by the spatial discretization error.

Example 11.1 We consider (11.1) and choose f and g such that $X(p,t) = r(t)p$ with

$$r(t) = \frac{r_0 r_K}{r_K e^{-kt} + r_0(1-e^{-kt})}$$

and $u(X,t) = X_1X_2e^{-6t}$ are the exact solution of the problem. The parameters are set to be $T = 1$, $\alpha = 1$, $\beta = 0$, $\delta = 0.4$, $r_0 = 1$, $r_K = 2$ and $k = 0.5$.

We choose $(\mathcal{T}_k)$ as a series of meshes such that $2h_k \approx h_{k-1}$. In Table 1 we report on the errors and the corresponding experimental orders of convergence (EOC). Using the notation of Sect. 2.6, the following norms are used:

$$\begin{aligned} \|\mathrm{err}_u\|_{L^\infty(L^2)} &= \sup_{[0,T]} \|\widehat{u}_h(\,\cdot\,,t) - \widetilde{I}_h u(\,\cdot\,,t)\|_{L^2(\Gamma_h^*(t))}, \\ \|\mathrm{err}_u\|_{L^2(H^1)} &= \left(\int_0^T \left\|\widehat{u}_h(\,\cdot\,,s) - \widetilde{I}_h u(\,\cdot\,,s)\right\|^2_{H^1(\Gamma_h^*(s))} \mathrm{d}s\right)^{\frac12}, \\ \|\mathrm{err}_v\|_{L^\infty(H^1)} &= \sup_{[0,T]} \|\widehat{v}_h(\,\cdot\,,t) - \widetilde{I}_h v(\,\cdot\,,t)\|_{H^1(\Gamma_h^*(t))}, \\ \|\mathrm{err}_x\|_{L^\infty(H^1)} &= \sup_{[0,T]} \|\widehat{x}_h(\,\cdot\,,t) - \mathrm{id}_{\Gamma_h^*(t)}\|_{H^1(\Gamma_h^*(t))}. \end{aligned}$$

Table 1 Errors and EOCs for Example 11.1

Level	DOF	$h(T)$	$\|\mathrm{err}_u\|_{L^\infty(L^2)}$	EOC	$\|\mathrm{err}_u\|_{L^2(H^1)}$	EOC
(a) Errors for u						
1	126	0.6664	0.1519165	–	0.2727214	–
2	516	0.4088	0.0896624	1.08	0.1498895	1.22
3	2070	0.1799	0.0222349	1.70	0.0344362	1.79
4	8208	0.0988	0.0070552	1.91	0.0109074	1.92
5	32,682	0.0499	0.0018319	1.98	0.0029375	1.92

Level	DOF	$h(T)$	$\|\mathrm{err}_v\|_{L^\infty(H^1)}$	EOC	$\|\mathrm{err}_x\|_{L^\infty(H^1)}$	EOC
(b) Surface and velocity errors						
1	126	0.6664	0.2260428	–	0.1473157	–
2	516	0.4088	0.0595755	2.73	0.0298673	3.27
3	2070	0.1799	0.0158342	1.61	0.0106836	1.25
4	8208	0.0988	0.0053584	1.81	0.0042312	1.54
5	32,682	0.0499	0.0019341	1.50	0.0017838	1.27

The EOCs for the errors $E(h_{k-1})$ and $E(h_k)$ with mesh sizes h_{k-1}, h_k are given via

$$EOC(h_{k-1}, h_k) = \frac{\log\left(\frac{E(h_{k-1})}{E(h_k)}\right)}{\log\left(\frac{h_{k-1}}{h_k}\right)}, \quad (k = 2, \ldots, n).$$

The degree of freedoms (DOF) and maximum mesh size at time T are also reported in the tables.

In Table 1 we report on the errors and EOCs observed using Example 11.1. The EOCs in the PDE are expected to be 2 for the $L^\infty(L^2)$ norm and 1 for the $L^2(H^1)$ norm, while the errors in the surface and in the surface velocity are expected to be 1 in the $L^\infty(H^1)$ norm.

Example 11.2 Again we consider (11.1), but this time we quantitatively compare the two different regularized velocity laws. Hence, we let δ vanish. We use a g like in Example 11.1 and run two tests with the common parameters $T = 2$, $r_0 = 1$, $r_K = 2$ and $k = 0.5$, and use the same mesh and time step levels as before. The first test uses $\alpha = 0$ and $\beta = 1$ and the second test uses $\alpha = 1$ and $\beta = 0$. The results are captured in Table 2. Our regularized velocity law provides smaller errors as regularizing with mean curvature flow. The EOCs in the errors in the surface and in the errors for the surface velocity are expected to be 1 in $L^\infty(H^1)_v$ and $L^\infty(H^1)_x$ norm, see Table 2b. While it can be observed that for this particular example the convergence rates for $\alpha \neq 0$ are higher then for $\beta \neq 0$.

Table 2 Errors and EOCs for Example 11.2

Level	DOF	$h(T)$	$L^\infty(L^2)_v$	EOC	$L^\infty(H^1)_v$	EOC	$L^\infty(H^1)_x$	EOC
(a) Surface and velocity errors with parameters $\alpha = 0$ and $\beta = 1$								
1	126	0.6664	0.756045	–	1.31532	–	1.601255	–
2	516	0.4088	0.393067	1.34	0.78538	1.06	0.522342	2.29
3	2070	0.1799	0.095914	1.72	0.96206	-0.25	0.137396	1.63
4	8208	0.0988	0.035166	1.67	1.48784	-0.73	0.044666	1.87
5	32,682	0.0499	0.019755	0.85	2.73584	-0.89	0.013507	1.75
Level	DOF	$h(T)$	$L^\infty(L^2)_v$	EOC	$L^\infty(H^1)_v$	EOC	$L^\infty(H^1)_x$	EOC
(b) Surface and velocity errors with parameters $\alpha = 1$ and $\beta = 0$								
1	126	0.6664	0.149836	–	0.225114	–	0.143419	–
2	516	0.4088	0.036118	2.91	0.058147	2.77	0.024087	3.65
3	2070	0.1799	0.009286	1.65	0.015843	1.58	0.009702	1.11
4	8208	0.0988	0.002705	2.06	0.005361	1.81	0.003990	1.48
5	32,682	0.0499	0.000686	2.01	0.001935	1.49	0.001746	1.21

11.2 A model for tumor growth

Our next test problem is the coupled system of equations

$$\begin{aligned} \partial^\bullet u + u\nabla_\Gamma \cdot v - \Delta_\Gamma u &= f_1(u,w), \\ \partial^\bullet w + w\nabla_\Gamma \cdot v - D_c\Delta_\Gamma w &= f_2(u,w), \\ v - \alpha\Delta_\Gamma v - \beta\Delta_\Gamma X &= \delta u \nu_\Gamma, \end{aligned} \tag{11.2}$$

where

$$f_1(u,w) = \gamma(a - u + u^2 w), \quad f_2(u,w) = \gamma(b - u^2 w),$$

with non-negative parameters $D_c, \gamma, a, b, \alpha, \beta$.

For $\alpha = 0$ this system has been used as a simplified model for tumor growth; see Barreira et al. [1] and [6,16]. These authors used the mean curvature term with a small parameter $\beta > 0$ to regularize their velocity law.

We used piecewise linear finite elements and the same time discretization scheme as in [1,16].

Example 11.3 We consider (11.2) and want to compare qualitatively the two different regularized velocity laws $\alpha \neq 0$ and $\beta \neq 0$. As common parameters we use $D_c = 10$, $\gamma = 100$, $a = 0.1$, $b = 0.9$ and $T = 5$. The initial surface is a sphere and the initial values u_0 and w_0 are calculated by solving an auxiliary surface PDE as follows. We take small perturbations around the steady state

$$\begin{pmatrix} \widetilde{u}_0 \\ \widetilde{w}_0 \end{pmatrix} = \begin{pmatrix} a + b + \varepsilon_1(x) \\ \frac{b}{(a+b)^2} + \varepsilon_2(x) \end{pmatrix},$$

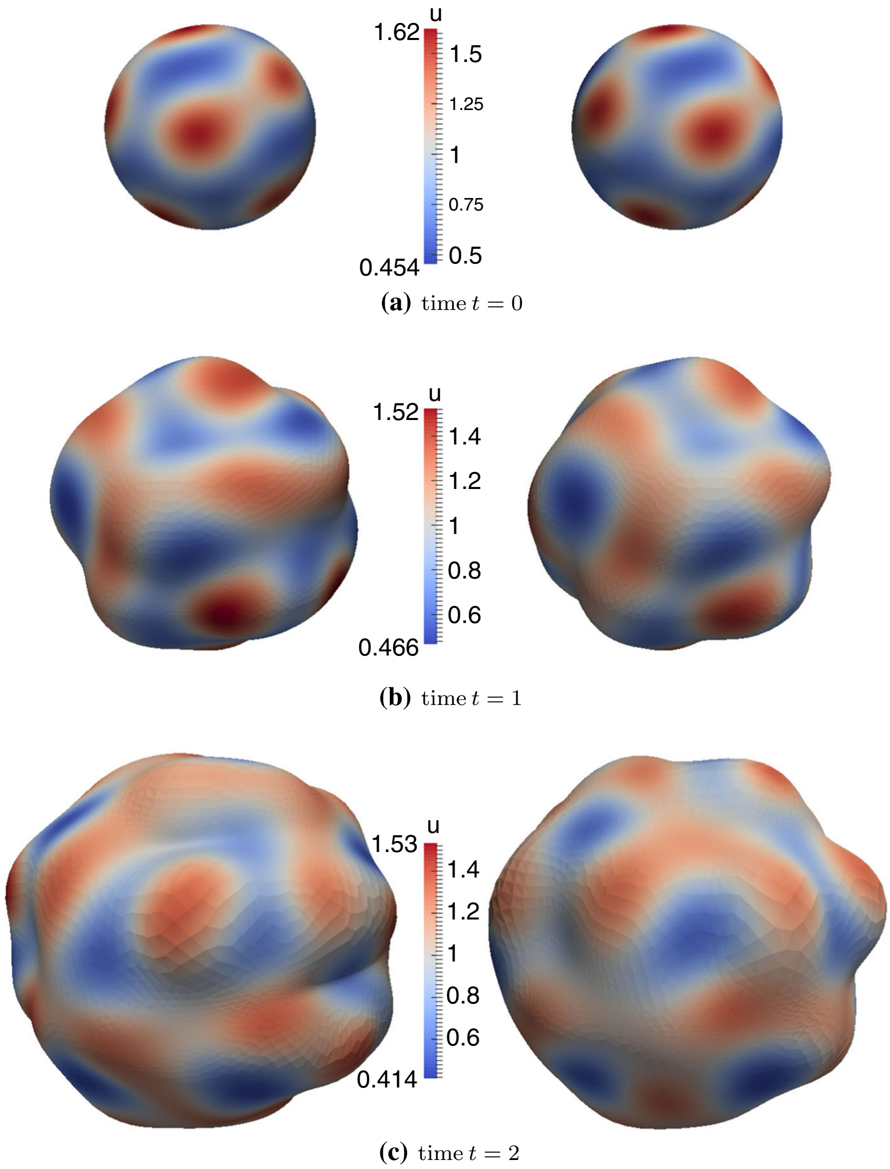

Fig. 1 Simulation for Example 11.3. The first column corresponds to $(\alpha, \beta) = (0, 0.01)$ and the second column to $(\alpha, \beta) = (0.01, 0)$. **a** Time $t = 0$, **b** time $t = 1$] and **c** time $t = 2$

where $\varepsilon_1(x), \varepsilon_2(x) \in [0, 0.01]$ take random values. We solve the auxiliary coupled diffusion equations with the stationary initial surface until time $\widetilde{T} = 5$. We set $u_0 = \widetilde{u}(\widetilde{T})$ and $w_0 = \widetilde{w}(\widetilde{T})$, which we used as initial values for (11.2).

We perform two experiments with $(\alpha, \beta) = (0, 0.01)$ and $(\alpha, \beta) = (0.01, 0)$. We present snapshots in Fig. 1. We observe that both velocity laws display the same qualitative behavior, also agreeing with [16].

Acknowledgements The research stay of Buyang Li at the University of Tübingen during the preparation of this paper was funded by the Alexander von Humboldt Foundation. The work of Buyang Li was partially supported by a grant from the Germany/Hong Kong Joint Research Scheme sponsored by the Research Grants Council of Hong Kong and the German Academic Exchange Service of Germany (Ref. No. G-PolyU502/16). The work of Balázs Kovács is funded by Deutsche Forschungsgemeinschaft, SFB 1173. We are grateful to Raquel Barreira and Anotida Madzvamuse for helpful comments on the tumor growth example of Sect. 11.2. We thank an anonymous referee for constructive comments on a previous version.

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Appendix G. Linearly implicit full discretisation of surface evolution

Version of March 19, 2018

Linearly implicit full discretization of surface evolution

Balázs Kovács · Christian Lubich

Abstract Stability and convergence of full discretizations of various surface evolution equations are studied in this paper. The proposed discretization combines a higher-order evolving-surface finite element method (ESFEM) for space discretization with higher-order linearly implicit backward difference formulae (BDF) for time discretization. The stability of the full discretization is studied in the matrix–vector formulation of the numerical method. The geometry of the problem enters into the bounds of the consistency errors, but does not enter into the proof of stability. Numerical examples illustrate the convergence behaviour of the full discretization.

Keywords Surface evolution · velocity law · evolving surface finite element method · time discretization · linearly implicit backward difference formulae · stability · convergence analysis

Mathematics Subject Classification (2000) 35R01 · 65M60 · 65M15 · 65M12

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1 Introduction

In this paper we study full discretizations of geometric evolution equations using the evolving surface finite element method (ESFEM) for space discretization and linearly implicit backward differentiation formulae (BDF) for time discretization. We consider the situation where the velocity $v(x,t)$ of a point x on an evolving two-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^3$ at time t is determined by one of the following velocity laws, for which finite element semi-discretization in space was studied in [KLLP17]:

(i) *Regularized mean curvature flow:* for $x \in \Gamma(t)$,

$$v(x,t) - \alpha \Delta_{\Gamma(t)} v(x,t) = -\beta H_{\Gamma(t)}(x)\, \nu_{\Gamma(t)}(x) + g\big(x,t\big)\, \nu_{\Gamma(t)}(x), \tag{1.1}$$

where $\Delta_{\Gamma(t)}$ is the Laplace–Beltrami operator on the surface $\Gamma(t)$, $H_{\Gamma(t)}$ is mean curvature, $\nu_{\Gamma(t)}$ is the outer normal, g is a smooth real-valued function, and $\alpha > 0$ and $\beta \geq 0$ are fixed parameters. This velocity law can be viewed as an elliptically regularized mean curvature flow with an additional driving term in the direction of the normal vector. In [KLLP17] this elliptic regularization allowed us to give a complete stability and convergence analysis of the ESFEM semi-discretization, for finite elements of polynomial degree at least two. In contrast, for pure mean curvature flow (that is, $\alpha = 0$), no convergence results appear to be known for ESFEM on two-dimensional closed surfaces.

(ii) *A dynamic velocity law:* for $x \in \Gamma(t)$,

$$\partial^{\bullet} v(x,t) + v(x,t)\, \nabla_{\Gamma(t)} \cdot v(x,t) - \alpha \Delta_{\Gamma(t)} v(x,t) = g(x,t)\, \nu_{\Gamma(t)}(x), \tag{1.2}$$

where $\partial^{\bullet} v$ denotes the material time derivative of v and $\nabla_{\Gamma} \cdot v$ denotes the surface divergence of v;

(iii) The case where the velocity law (i) or (ii) is coupled to diffusion on the evolving surface, as in [KLLP17].

We note that in all these cases, the considered velocity v is in general not normal to the surface, but contains tangential components.

The rigorous study of the stability and convergence properties of full discretizations obtained by combining the ESFEM with various time discretizations for problems on evolving surfaces was begun in the papers [DE12] (implicit Euler method), [DLM12] (implicit Runge–Kutta methods) and [LMV13] (BDF methods). These papers studied a linear parabolic equation on a *given* moving closed surface $\Gamma(t)$. Convergence of full discretizations of that problem using higher-order evolving surface finite elements is studied in [Kov17]. Convergence properties of full discretizations for quasi- and semilinear parabolic equations on prescribed moving surfaces are studied in [KP16]. For curves instead of two-dimensional surfaces, convergence of full discretizations of curve-shortening flow coupled to diffusion is studied by Barrett, Deckelnick & Styles [BDS17].

The main difficulty in proving the convergence of the full discretization of the surface-evolution equation in (i)–(iii) is the proof of stability in the sense of bounding errors in terms of defects in the discrete equations. The proof requires some auxiliary results from [KLLP17], which relate different finite element surfaces. For (1.1), the stability proof just uses the zero-stability of the BDF methods up to order 6. For (1.2), it is based on energy estimates that become available for BDF methods up to order 5 by the multiplier technique of Nevanlinna and Odeh [NO81], which in turn is based on the G-stability theory of Dahlquist [Dah78]. These techniques were originally developed for stiff ordinary differential equations and have recently been used for linear parabolic equations on given moving surfaces in [LMV13] and for various quasilinear parabolic problems in [AL15, ALL17, KP16].

The paper is organized as follows.

In Section 2 we describe the problem and the numerical methods. We recall the basics of the evolving surface finite element method and give its matrix–vector formulation, and we formulate the linearly implicit BDF time discretization.

In Section 3 we present the main result for (1.1), which gives optimal-order convergence estimates for the full discretization by ESFEM of polynomial degree at least 2 and linearly implicit BDF methods up to order 6. This result is proven in Sections 4 to 7.

Section 4 contains auxiliary results for the stability analysis of the discretized velocity law (1.1). We collect results from [KLLP17] that relate different finite element surfaces to one another. We also include a new auxiliary result for the linearly implicit BDF time discretization.

Section 5 contains the stability analysis, which works with the matrix–vector formulation of the discrete equations. Like the proof of stability of the ESFEM spatial semi-discretization in [KLLP17], it does not use geometric arguments.

Section 6 gives estimates for the consistency errors, that is, for the defects on inserting the interpolated exact solution into the discrete equations.

Section 7 proves the convergence result for the full discretization of (1.1) by combining the results of the previous sections.

In Section 8 we extend the convergence analysis to the full discretization of the dynamic velocity law (1.2). This is done for BDF methods up to order 5 using energy estimates based on the Nevanlinna–Odeh multiplier technique.

In Section 9 we extend the convergence result for the full discretization to the case where the velocity law (1.1) or (1.2) is coupled to diffusion on the evolving surface, as studied in [KLLP17] for the semi-discretization. The result is obtained by combining the techniques of [KLLP17] and [LMV13] with those of Sections 4 to 7 of the present paper.

Section 10 presents numerical experiments using quadratic ESFEM that illustrate the numerical behaviour of the proposed full discretization.

We use the notational convention to denote vectors in $\mathbb{R}^3$ by italic letters, but to denote finite element nodal vectors in $\mathbb{R}^{3N}$ by boldface lowercase letters and finite element mass and stiffness matrices by boldface capitals. All boldface symbols in this paper will thus be related to the matrix–vector formulation of the ESFEM.

2 Problem formulation and ESFEM / BDF full discretization

We use the same setting as in our previous work [KLLP17]. We recall basic notions, but refer to Section 2 of [KLLP17] for a more detailed description.

2.1 Basic notions and notation

We consider the evolving two-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^3$ as the image

$$\Gamma(t) = \{X(q,t) \,:\, q \in \Gamma^0\}$$

of a regular vector-valued function $X : \Gamma^0 \times [0,T] \to \mathbb{R}^3$, where Γ^0 is the smooth closed initial surface, and $X(q,0) = q$. To indicate the dependence of the surface on X, we write

$$\Gamma(t) = \Gamma(X(\cdot,t)), \quad \text{ or briefly } \quad \Gamma(X)$$

when the time t is clear from the context. The position $X(q,\cdot)$ is related to the *velocity* $v(x,t) \in \mathbb{R}^3$ at the point $x = X(q,t) \in \Gamma(t)$ via the ordinary differential equation

$$\partial_t X(q,t) = v(X(q,t),t). \tag{2.1}$$

For $x \in \Gamma(t)$ and $0 \leq t \leq T$, we denote by $\nu_{\Gamma(X)}(x)$ the outer normal, by $\nabla_{\Gamma(X)} u(x,t)$ the tangential gradient of a real-valued function u on $\Gamma(t)$, and by $\Delta_{\Gamma(X)} u(x,t)$ the Laplace–Beltrami operator applied to u.

2.2 Weak formulation of the surface-evolution equation

The space discretization is based on the weak formulation of the surface-evolution equation (1.1), which reads as follows: Find $v(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))^3$ such that for all test functions $\psi(\cdot,t) \in H^1(\Gamma(X(\cdot,t)))^3$,

$$\begin{aligned} \int_{\Gamma(X)} v \cdot \psi \ + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)} \psi & \\ + \beta \int_{\Gamma(X)} \nabla_{\Gamma(X)} X \cdot \nabla_{\Gamma(X)} \psi &= \int_{\Gamma(X)} g \, \nu_{\Gamma(X)} \cdot \psi, \end{aligned} \tag{2.2}$$

alongside with the ordinary differential equation (2.1) for the positions X determining the surface $\Gamma(X)$. (More precisely, the term $\nabla_{\Gamma(X)} X$ should read $\nabla_{\Gamma(X)} \mathrm{id}_{\Gamma(X)}$.)

We assume throughout this paper that the problem (1.1) or (2.2) admits a unique solution with sufficiently high Sobolev regularity on the time interval $[0,T]$ for the given initial data $X(\cdot,0)$. We assume further that the flow map $X(\cdot,t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ is non-degenerate for $0 \leq t \leq T$, so that $\Gamma(t)$ is a regular surface.

2.3 Evolving surface finite elements

From Section 2.3 of [KLLP17] we recall the description of the surface finite element discretization of our problem, which is based on [Dzi88] and [Dem09]. We use simplicial elements and continuous piecewise polynomial basis functions of degree k, as defined in [Dem09, Section 2.5].

We triangulate the given smooth surface Γ^0 by an admissible family of triangulations $\mathcal{T}_h$ of decreasing maximal element diameter h; see [DE07] for the notion of an admissible triangulation, which includes quasi-uniformity and shape regularity. For a momentarily fixed h, we denote by $\mathbf{x}^0 = (x_1^0, \dots, x_N^0)$ the vector in $\mathbb{R}^{3N}$ that collects all N nodes of the triangulation. By piecewise polynomial interpolation of degree k, the nodal vector defines an approximate surface Γ_h^0 that interpolates Γ^0 in the nodes x_j^0. We will evolve the jth node in time, denoted $x_j(t)$ with $x_j(0) = x_j^0$, and collect the nodes at time t in a column vector in $\mathbb{R}^{3N}$,

$$\mathbf{x}(t) \in \mathbb{R}^{3N}.$$

We just write $\mathbf{x}$ for $\mathbf{x}(t)$ when the dependence on t is not important.

By piecewise polynomial interpolation on the plane reference triangle that corresponds to every curved triangle of the triangulation, the nodal vector $\mathbf{x}$ defines a closed surface denoted by $\Gamma_h[\mathbf{x}]$. We can then define finite element *basis functions*

$$\phi_j[\mathbf{x}] : \Gamma_h[\mathbf{x}] \to \mathbb{R}, \qquad j = 1, \dots, N,$$

which have the property that on every triangle their pullback to the reference triangle is polynomial of degree k, and which satisfy

$$\phi_j[\mathbf{x}](x_k) = \delta_{jk} \quad \text{for all } j, k = 1, \dots, N.$$

These functions span the finite element space on $\Gamma_h[\mathbf{x}]$,

$$S_h[\mathbf{x}] = S_h(\Gamma_h[\mathbf{x}]) = \mathrm{span}\big\{\phi_1[\mathbf{x}], \phi_2[\mathbf{x}], \dots, \phi_N[\mathbf{x}]\big\}.$$

For a finite element function $u_h \in S_h[\mathbf{x}]$ the tangential gradient $\nabla_{\Gamma_h[\mathbf{x}]} u_h$ is defined piecewise on each element. We set

$$X_h(q_h, t) = \sum_{j=1}^{N} x_j(t)\, \phi_j[\mathbf{x}(0)](q_h), \qquad q_h \in \Gamma_h^0,$$

which has the properties that $X_h(q_j, t) = x_j(t)$ for $j = 1, \dots, N$, that $X_h(q_h, 0) = q_h$ for all $q_h \in \Gamma_h^0$, and

$$\Gamma_h[\mathbf{x}(t)] = \Gamma(X_h(\cdot, t)).$$

The *discrete velocity* $v_h(x, t) \in \mathbb{R}^3$ at a point $x = X_h(q_h, t) \in \Gamma(X_h(\cdot, t))$ is given by

$$\partial_t X_h(q_h, t) = v_h(X_h(q_h, t), t).$$

In view of the transport property of the basis functions [DE07],

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\phi_j[\mathbf{x}(t)](X_h(q_h, t))\Big) = 0,$$

the discrete velocity equals, for $x \in \Gamma_h[\mathbf{x}(t)]$,

$$v_h(x,t) = \sum_{j=1}^{N} v_j(t)\, \phi_j[\mathbf{x}(t)](x) \qquad \text{with} \;\; v_j(t) = \dot{x}_j(t),$$

where the dot denotes the time derivative $\mathrm{d}/\mathrm{d}t$. Hence, the nodal vector of the discrete velocity is $\mathbf{v} = \dot{\mathbf{x}}$.

2.4 ESFEM spatial semi-discretization of the evolving-surface problem

The finite element spatial semi-discretization of the problem (2.2) reads as follows: Find the unknown nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$ and the unknown finite element function $v_h(\cdot,t) \in S_h[\mathbf{x}(t)]^3$ such that, for all $\psi_h(\cdot,t) \in S_h[\mathbf{x}(t)]^3$,

$$\begin{aligned} \int_{\Gamma_h[\mathbf{x}]} v_h \cdot \psi_h \;+ \alpha \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} v_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \psi_h & \\ + \beta \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h[\mathbf{x}]} X_h \cdot \nabla_{\Gamma_h[\mathbf{x}]} \psi_h &= \int_{\Gamma_h[\mathbf{x}]} g\, \nu_{\Gamma_h[\mathbf{x}]} \cdot \psi_h, \end{aligned} \tag{2.3}$$

and

$$\partial_t X_h(q_h,t) = v_h(X_h(q_h,t),t), \qquad q_h \in \Gamma_h^0. \tag{2.4}$$

The initial values for the nodal vector $\mathbf{x}$ of the initial positions are taken as the exact initial values at the nodes x_j^0 of the triangulation of the given initial surface Γ^0:

$$x_j(0) = x_j^0, \qquad j = 1,\dots,N.$$

2.5 Matrix–vector formulation

We define the surface-dependent mass matrix $\mathbf{M}(\mathbf{x})$ and stiffness matrix $\mathbf{A}(\mathbf{x})$ on the surface determined by the nodal vector $\mathbf{x}$ (cf. [KLLP17, Section 2.5]):

$$\left.\begin{aligned} \mathbf{M}(\mathbf{x})|_{jk} &= \int_{\Gamma_h[\mathbf{x}]} \phi_j[\mathbf{x}]\phi_k[\mathbf{x}], \\ \mathbf{A}(\mathbf{x})|_{jk} &= \int_{\Gamma_h[\mathbf{x}]} \nabla_{\Gamma_h}\phi_j[\mathbf{x}] \cdot \nabla_{\Gamma_h}\phi_k[\mathbf{x}], \end{aligned}\right. \qquad (j,k = 1,\dots,N).$$

We further let (with the identity matrix $I_3 \in \mathbb{R}^{3\times 3}$)

$$\mathbf{M}^{[3]}(\mathbf{x}) = I_3 \otimes \mathbf{M}(\mathbf{x}) \quad \text{and} \quad \mathbf{A}^{[3]}(\mathbf{x}) = I_3 \otimes \mathbf{A}(\mathbf{x}),$$

and then define

$$\mathbf{K}(\mathbf{x}) = \mathbf{M}^{[3]}(\mathbf{x}) + \alpha \mathbf{A}^{[3]}(\mathbf{x}). \tag{2.5}$$

When no confusion can arise, we write in the following $\mathbf{M}(\mathbf{x})$ for $\mathbf{M}^{[3]}(\mathbf{x})$, $\mathbf{A}(\mathbf{x})$ for $\mathbf{A}^{[3]}(\mathbf{x})$ and $\|\cdot\|_{H^1(\Gamma)}$ for $\|\cdot\|_{H^1(\Gamma)^3}$, etc.

The right-hand side vector $\mathbf{g}(\mathbf{x},t) \in \mathbb{R}^{3N}$ is given by

$$\mathbf{g}(\mathbf{x},t)|_{j+N(\ell-1)} = \int_{\Gamma_h[\mathbf{x}]} g(\cdot,t)\,\big(\nu_{\Gamma_h[\mathbf{x}]}\big)_\ell\,\phi_j[\mathbf{x}],$$

for $j = 1,\dots,N$ and $\ell = 1,2,3$.

We then obtain from (2.3)–(2.4) the following system of ordinary differential equations (ODEs) for the nodal vectors $\mathbf{x}(t) \in \mathbb{R}^{3N}$:

$$\mathbf{K}(\mathbf{x})\dot{\mathbf{x}} + \beta\mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{g}(\mathbf{x},t). \tag{2.6}$$

2.6 Linearly implicit BDF time discretization

We apply a p-step linearly implicit backward difference formula (BDF) for $p \le 6$ as a time discretization to the ODE system (2.6). For a step size $\tau > 0$, and with $t_n = n\tau \le T$, we determine the approximation $\mathbf{x}^n$ to $\mathbf{x}(t_n)$ by the fully discrete system of linear equations

$$\begin{aligned}\mathbf{K}(\widetilde{\mathbf{x}}^n)\mathbf{v}^n + \beta\mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{x}^n &= \mathbf{g}(\widetilde{\mathbf{x}}^n,t_n),\\ \mathbf{v}^n &= \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{x}^{n-j}, \qquad n \ge p,\end{aligned} \tag{2.7}$$

where the extrapolated position vector $\widetilde{\mathbf{x}}^n$ is defined by

$$\widetilde{\mathbf{x}}^n = \sum_{j=0}^{p-1}\gamma_j\mathbf{x}^{n-1-j}, \qquad n \ge p. \tag{2.8}$$

The starting values $\mathbf{x}^0,\mathbf{x}^1,\dots,\mathbf{x}^{p-1}$ are assumed to be given. They can be precomputed in a way as is usual with multistep methods: using lower-order methods with smaller step sizes or using an implicit Runge–Kutta method.

The coefficients are given by $\delta(\zeta) = \sum_{j=0}^{p}\delta_j\zeta^j = \sum_{\ell=1}^{p}\frac{1}{\ell}(1-\zeta)^\ell$ and $\gamma(\zeta) = \sum_{j=0}^{p-1}\gamma_j\zeta^j = (1-(1-\zeta)^p)/\zeta$. The classical BDF method is known to be zero-stable for $p \le 6$ and to have order p; see [HW96, Chapter V]. This order is retained by the linearly implicit variant using the above coefficients γ_j; cf. [AL15, ALL17].

We note that the method requires solving a linear system with the symmetric positive definite matrix $\frac{\delta_0}{\tau}\mathbf{K}(\widetilde{\mathbf{x}}^n) + \beta\mathbf{A}(\widetilde{\mathbf{x}}^n)$ in the nth time step.

From the vectors $\mathbf{x}^n = (x_j^n)$ and $\mathbf{v}^n = (v_j^n)$ we obtain position and velocity approximations to $X(\cdot,t_n)$ and $v(\cdot,t_n)$ as

$$\begin{aligned}X_h^n(q_h) &= \sum_{j=1}^{N} x_j^n\,\phi_j[\mathbf{x}(0)](q_h) \qquad \text{for } q_h \in \Gamma_h^0,\\ v_h^n(x) &= \sum_{j=1}^{N} v_j^n\,\phi_j[\mathbf{x}^n](x) \qquad \text{for } x \in \Gamma_h[\mathbf{x}^n].\end{aligned} \tag{2.9}$$

2.7 Lifts

Here we recapitulate [KLLP17, Section 2.6]. In the error analysis we need to compare functions on three different surfaces: the *exact surface* $\Gamma(t) = \Gamma(X(\cdot,t))$, the *discrete surface* $\Gamma_h(t) = \Gamma_h[\mathbf{x}(t)]$, and the *interpolated surface* $\Gamma_h^*(t) = \Gamma_h[\mathbf{x}_*(t)]$, where $\mathbf{x}_*(t)$ is the nodal vector collecting the grid points $x_{*,j}(t) = X(q_j,t)$ on the exact surface. In the following definitions we omit the argument t in the notation.

For a finite element function $w_h : \Gamma_h \to \mathbb{R}^m$ ($m = 1$ or 3) on the discrete surface, with nodal values w_j, we denote by $\widehat{w}_h : \Gamma_h^* \to \mathbb{R}^m$ the finite element function on the interpolated surface that has the same nodal values:

$$\widehat{w}_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}_*].$$

The transition between the interpolated surface and the exact surface is done by the *lift operator*, which was introduced for linear surface approximations in [Dzi88]; see also [DE07, DE13]. Higher-order generalizations have been studied in [Dem09]. The lift operator l maps a function on the interpolated surface Γ_h^* to a function on the exact surface Γ, provided that Γ_h^* is sufficiently close to Γ.

The exact regular surface $\Gamma(X(\cdot,t))$ can be represented by a (sufficiently smooth) signed distance function $d : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$, cf. [DE07, Section 2.1], such that $\Gamma(X(\cdot,t)) = \big\{x \in \mathbb{R}^3 \mid d(x,t) = 0\big\} \subset \mathbb{R}^3$. Using this distance function, the lift of a continuous function $\eta_h \colon \Gamma_h^* \to \mathbb{R}^m$ is defined as

$$\eta_h^l(y) := \eta_h(x), \qquad x \in \Gamma_h^*,$$

where for every $x \in \Gamma_h^*$ the point $y = y(x) \in \Gamma$ is uniquely defined via $y = x - \nu(y)d(x)$.

We denote the composed lift L from finite element functions on Γ_h to functions on Γ via Γ_h^* by

$$w_h^L = (\widehat{w}_h)^l.$$

3 Statement of the main result: fully discrete error bound

We formulate the main result of this paper, which yields optimal-order error bounds for the ESFEM / BDF full discretization of the surface-evolution equation (1.1), for finite elements of polynomial degree $k \geq 2$ and BDF methods of order $p \leq 6$. We denote by $\Gamma(t_n) = \Gamma(X(\cdot,t_n))$ the exact surface and by $\Gamma_h^n = \Gamma(X_h^n) = \Gamma_h[\mathbf{x}^n]$ the discrete surface at time t_n. For the lifted position function we introduce the notation

$$(x_h^n)^L(x) = (X_h^n)^L(q) \in \Gamma_h^n \qquad \text{for} \quad x = X(q,t_n) \in \Gamma(t_n).$$

Theorem 3.1 *Consider the ESFEM / BDF linearly implicit full discretization* (2.7) *of the surface-evolution equation* (1.1)*, using finite elements of polynomial degree* $k \geq 2$ *and BDF methods of order* $p \leq 6$*. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for* X*. Suppose that the problem admits an exact solution* (X, v) *that is sufficiently smooth (say, of class* $C([0,T], H^{k+1}) \cap C^{p+1}([0,T], W^{1,\infty})$*) on the time interval* $0 \leq t \leq T$*, and that the flow map* $X(\cdot,t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ *is non-degenerate for* $0 \leq t \leq T$*, so that* $\Gamma(t)$ *is a regular surface. Suppose further that the starting values are sufficiently accurate:*

$$\|(X_h^i)^L - X(\cdot, i\tau)\|_{H^1(\Gamma^0)^3} \leq C_0(h^k + \tau^p), \qquad i = 0, 1, \ldots, p-1.$$

Then, there exist $h_0 > 0$*,* $\tau_0 > 0$ *and* $c_0 > 0$ *such that for all mesh widths* $h \leq h_0$ *and step sizes* $\tau \leq \tau_0$ *satisfying the mild stepsize restriction*

$$\tau^p \leq c_0 h,$$

the following error bounds hold over the exact surface $\Gamma(t_n) = \Gamma(X(\cdot, t_n))$ *uniformly for* $0 \leq t_n = n\tau \leq T$*:*

$$\begin{aligned}
\|(x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\|_{H^1(\Gamma(t_n))^3} &\leq C(h^k + \tau^p), \\
\|(v_h^n)^L - v(\cdot, t_n)\|_{H^1(\Gamma(t_n))^3} &\leq C(h^k + \tau^p).
\end{aligned}$$

The constant C *is independent of* h *and* τ *and* n *with* $n\tau \leq T$*, but depends on bounds of higher derivatives of the solution* (X, v)*, and on the length* T *of the time interval.*

We note that the first error bound is equivalent to

$$\|(X_h^n)^L - X(\cdot, t_n)\|_{H^1(\Gamma^0)^3} \leq C'(h^k + \tau^p),$$

and we mention that the remarks after Theorem 3.1 in [KLLP17] (the convergence theorem of the ESFEM semi-discretization) apply also to the fully discretized situation considered here.

The proof of Theorem 3.1 is given in the course of the next four sections.

4 Preparation: Estimates relating different surfaces

In our previous work [KLLP17, Section 4] we have shown some auxiliary results relating different finite element surfaces, which we recapitulate here.

The finite element matrices of Section 2.5 induce discrete versions of Sobolev norms. For any $\mathbf{w} = (w_j) \in \mathbb{R}^N$ with corresponding finite element function $w_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}] \in S_h[\mathbf{x}]$ we note

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x})}^2 = \mathbf{w}^T \mathbf{M}(\mathbf{x}) \mathbf{w} = \|w_h\|_{L^2(\Gamma_h[\mathbf{x}])}^2, \tag{4.1}$$

$$\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x})}^2 = \mathbf{w}^T \mathbf{A}(\mathbf{x}) \mathbf{w} = \|\nabla_{\Gamma_h[\mathbf{x}]} w_h\|_{L^2(\Gamma_h[\mathbf{x}])}^2. \tag{4.2}$$

We use the following setting. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3N}$ be two nodal vectors defining discrete surfaces $\Gamma_h[\mathbf{x}]$ and $\Gamma_h[\mathbf{y}]$, respectively. We let $\mathbf{e} = (e_j) = \mathbf{x} - \mathbf{y} \in \mathbb{R}^{3N}$. For $\theta \in [0,1]$, we consider the intermediate surface $\Gamma_h^\theta = \Gamma_h[\mathbf{y} + \theta\mathbf{e}]$ and the corresponding finite element functions given as

$$e_h^\theta = \sum_{j=1}^N e_j \phi_j[\mathbf{y} + \theta\mathbf{e}]$$

and in the same way, for any vectors $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$,

$$w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta\mathbf{e}] \quad \text{and} \quad z_h^\theta = \sum_{j=1}^N z_j \phi_j[\mathbf{y} + \theta\mathbf{e}].$$

The following lemma collects results from [KLLP17, Section 4].

Lemma 4.1 *(i) In the above setting the following identities hold:*

$$\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{y}))\mathbf{z} = \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta (\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) z_h^\theta \, \mathrm{d}\theta,$$
$$\mathbf{w}^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\mathbf{y}))\mathbf{z} = \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} w_h^\theta \cdot (D_{\Gamma_h^\theta} e_h^\theta) \nabla_{\Gamma_h^\theta} z_h^\theta \, \mathrm{d}\theta,$$

with $D_{\Gamma_h^\theta} e_h^\theta = \mathrm{trace}(E) I_3 - (E + E^T)$ *for* $E = \nabla_{\Gamma_h^\theta} e_h^\theta \in \mathbb{R}^{3\times 3}$.

(ii) If $\|\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \le \mu$ *and* $\|D_{\Gamma_h^\theta} e_h^\theta\|_{L^\infty(\Gamma_h^\theta)} \le \rho$ *for* $0 \le \theta \le 1$*, then* $\|\mathbf{w}\|_{\mathbf{M}(\mathbf{y}+\theta\mathbf{e})} \le e^{\mu/2} \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y})}$ *and* $\|\mathbf{w}\|_{\mathbf{A}(\mathbf{y}+\theta\mathbf{e})} \le e^{\rho/2} \|\mathbf{w}\|_{\mathbf{A}(\mathbf{y})}$.

(iii) If $\|\nabla_{\Gamma_h[\mathbf{y}]} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])} \le \frac{1}{2}$*, then, for* $0 \le \theta \le 1$*, the function* $w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta\mathbf{e}]$ *on* $\Gamma_h^\theta = \Gamma_h[\mathbf{y} + \theta\mathbf{e}]$ *is bounded by*

$$\|\nabla_{\Gamma_h^\theta} w_h^\theta\|_{L^p(\Gamma_h^\theta)} \le c_p \, \|\nabla_{\Gamma_h^0} w_h^0\|_{L^p(\Gamma_h^0)} \quad \text{for} \quad 1 \le p \le \infty,$$

where c_p *depends only on* p *(we have* $c_\infty = 2$*).*

(iv) Let $y_h^\theta \in \Gamma_h^\theta$ *be defined as* $y_h^\theta = \sum_{j=1}^N (y_j + \theta e_j)\phi_j[\mathbf{y}](q_h)$ *for* $q_h \in \Gamma_h[\mathbf{y}]$*. If* $\|\nabla_{\Gamma_h[\mathbf{y}]} e_h^0\|_{L^\infty(\Gamma_h[\mathbf{y}])} \le \frac{1}{2}$*, then the corresponding unit normal vectors differ by no more than*

$$|\nu_{\Gamma_h^\theta}(y_h^\theta) - \nu_{\Gamma_h^0}(y_h^0)| \le C\theta |\nabla_{\Gamma_h^0} e_h^0(y_h^0)|,$$

where C *is independent of* h *and of* $q_h \in \Gamma_h[\mathbf{y}]$*.*

The following result is shown in Lemma 4.1 of [DLM12].

Lemma 4.2 *Let* $\Gamma(t) = \Gamma(X(\cdot,t))$*,* $t \in [0,T]$*, be a smoothly evolving family of smooth closed surfaces, and let the vector* $\mathbf{x}_*(t) \in \mathbb{R}^{3N}$ *collect the nodes* $x_j^*(t) = X(q_j,t)$*. Then, for* $0 \le s,t \le T$ *and for all* $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$*,*

$$\mathbf{w}^T \big(\mathbf{M}(\mathbf{x}_*(t)) - \mathbf{M}(\mathbf{x}_*(s))\big)\mathbf{z} \le C(t-s) \|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*(t))} \|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}_*(t))},$$
$$\mathbf{w}^T \big(\mathbf{A}(\mathbf{x}_*(t)) - \mathbf{A}(\mathbf{x}_*(s))\big)\mathbf{z} \le C(t-s) \|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*(t))} \|\mathbf{z}\|_{\mathbf{A}(\mathbf{x}_*(t))}$$

and the norms for different times are uniformly equivalent for $0 \le s, t \le T$*:*

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*(t))} \le C\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*(s))}, \qquad \|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*(t))} \le C\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*(s))}.$$

The constant C *depends only on a bound of the* $W^{1,\infty}$ *norm of the surface velocity.*

We also need a result which compares the finite element surfaces with exact and extrapolated nodes.

Lemma 4.3 *Let* $\Gamma(t) = \Gamma(X(\cdot,t))$*,* $t \in [0,T]$*, be a smoothly evolving family of smooth closed surfaces. We denote the nodal vectors of exact solution values by* $\mathbf{x}_*^n = \mathbf{x}_*(t_n)$ *and of the extrapolated values by* $\widetilde{\mathbf{x}}_*^n = \sum_{j=0}^{p-1} \gamma_j \mathbf{x}_*^{n-1-j}$*. Then, the following estimates hold for all* $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$*:*

$$\mathbf{w}^T(\mathbf{M}(\widetilde{\mathbf{x}}_*^n) - \mathbf{M}(\mathbf{x}_*^n))\mathbf{z} \le C\tau^p \, \|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*^n)} \|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}_*^n)},$$
$$\mathbf{w}^T(\mathbf{A}(\widetilde{\mathbf{x}}_*^n) - \mathbf{A}(\mathbf{x}_*^n))\mathbf{z} \le C\tau^p \, \|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*^n)} \|\mathbf{z}\|_{\mathbf{A}(\mathbf{x}_*^n)},$$

where C *is independent of* h*,* τ *and* n *with* $0 \le n\tau \le T$*.*

Proof For the extrapolated value $\widetilde{X}(q,t) = \sum_{j=0}^{p-1} \gamma_j X(q, t-(j+1)\tau)$, we use the error formula with Peano kernel representation, see e.g. [Gau97, Section 3.2.6],

$$\widetilde{X}(q,t) - X(q,t) = \tau^p \int_0^p \kappa_p(\lambda)\, \partial_t^{p+1} X(q, t-\lambda\tau)\, \mathrm{d}\lambda \tag{4.3}$$

with a bounded Peano kernel κ_p. We note that we have

$$\widetilde{x}_{*,j}^n - x_{*,j}^n = \widetilde{X}(q_j, t_n) - X(q_j, t_n).$$

Since X is assumed smooth, we obtain from the above error formula that for $0 \le \theta \le 1$, the finite element function $\widetilde{e}_h^{n,\theta}$ in $S_h(\Gamma_h^\theta)$ with the nodal vector $\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n$, for $\Gamma_h^\theta = \Gamma_h[\mathbf{x}_*^n + \theta(\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n)]$, has a gradient bounded in the maximum norm by $c\tau^p$, where c is independent of τ and h. So we have the bound

$$\|\nabla_{\Gamma_h[\mathbf{x}_*^n]} \cdot \widetilde{e}_h^{n,0}\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])} \le c\tau^p.$$

Together with Lemma 4.1 and an $L^2 - L^\infty - L^2$ estimate, we thus obtain

$$\begin{aligned}
\mathbf{w}^T(\mathbf{M}(\widetilde{\mathbf{x}}_*^n) - \mathbf{M}(\mathbf{x}_*^n))\mathbf{z} &= \int_0^1 \int_{\Gamma_h^{n,\theta}} w_h^\theta (\nabla_{\Gamma_h^{n,\theta}} \cdot \widetilde{e}_h^{n,\theta}) z_h^\theta \mathrm{d}\theta \\
&\le \int_0^1 \|w_h^\theta\|_{L^2(\Gamma_h^{n,\theta})} \|\nabla_{\Gamma_h^{n,\theta}} \cdot \widetilde{e}_h^{n,\theta}\|_{L^\infty(\Gamma_h^{n,\theta})} \|z_h^\theta\|_{L^2(\Gamma_h^{n,\theta})} \mathrm{d}\theta \\
&\le c\tau^p \|w_h^0\|_{L^2(\Gamma_h^{0,n})} \|z_h^0\|_{L^2(\Gamma_h^{0,n})} \\
&\le c\tau^p \|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*^n)} \|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}_*^n)}.
\end{aligned}$$

The second estimate is proved in the same way. □

The above lemma immediately implies the following norm equivalence, for sufficiently small step size τ,

$$\tfrac{1}{2}\|\mathbf{w}\|^2_{\mathbf{K}(\mathbf{x}_*^n)} \le \|\mathbf{w}\|^2_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)} \le \tfrac{3}{2}\|\mathbf{w}\|^2_{\mathbf{K}(\mathbf{x}_*^n)}. \tag{4.4}$$

5 Stability

We denote by

$$\mathbf{x}_*(t) = \big(x_{*,j}(t)\big) \in \mathbb{R}^{3N} \quad \text{with} \quad x_{*,j}(t) = X(q_j, t), \qquad (j = 1, \dots, N)$$

the nodal vector of the *exact* positions on the surface $\Gamma(X(\cdot,t))$. This defines a discrete surface $\Gamma_h[\mathbf{x}_*(t)]$ that interpolates the exact surface $\Gamma(X(\cdot,t))$.

We consider the interpolated exact velocity

$$v_{*,h}(\cdot,t) = \sum_{j=1}^{N} v_{*,j}(t)\phi_j[\mathbf{x}_*(t)] \qquad \text{with} \qquad v_{*,j}(t) = \dot{x}_{*,j}(t),$$

with the corresponding nodal vector

$$\mathbf{v}_*(t) = \big(v_{*,j}(t)\big) = \dot{\mathbf{x}}_*(t) \in \mathbb{R}^{3N}.$$

We write

$$\mathbf{x}_*^n = \mathbf{x}_*(t_n), \quad \mathbf{v}_*^n = \mathbf{v}_*(t_n).$$

The errors of the numerical solution values $\mathbf{x}^n$ and $\mathbf{v}^n$ are marked with their respective subscript, hence are denoted by

$$\mathbf{e}_\mathbf{v}^n = \mathbf{v}^n - \mathbf{v}_*^n, \qquad \mathbf{e}_\mathbf{x}^n = \mathbf{x}^n - \mathbf{x}_*^n.$$

5.1 Error equations

The nodal vectors of the exact solution satisfy the equations of the linearly implicit BDF method only up to defects $\mathbf{d}_\mathbf{v}^n$ and $\mathbf{d}_\mathbf{x}^n$ that, for $n \geq p$, are defined by the equations

$$\begin{aligned} \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\mathbf{v}_*^n + \beta\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\mathbf{x}_*^n &= \mathbf{g}(\widetilde{\mathbf{x}}_*^n, t_n) + \mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n, \\ \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{x}_*^{n-j} &= \mathbf{v}_*^n + \mathbf{d}_\mathbf{x}^n. \end{aligned} \tag{5.1}$$

We subtract (5.1) from (2.7) to obtain the error equations

$$\begin{aligned} \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\mathbf{e}_\mathbf{v}^n + {}& \beta\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\mathbf{e}_\mathbf{x}^n \\ &= -\big(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{e}_\mathbf{v}^n - \big(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{v}_*^n \\ &\quad - \beta\big(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{e}_\mathbf{x}^n - \beta\big(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{x}_*^n \\ &\quad + \mathbf{g}(\widetilde{\mathbf{x}}^n, t_n) - \mathbf{g}(\widetilde{\mathbf{x}}_*^n, t_n) - \mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n, \\ \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{e}_\mathbf{x}^{n-j} = {}& \mathbf{e}_\mathbf{v}^n - \mathbf{d}_\mathbf{x}^n. \end{aligned} \tag{5.2}$$

5.2 Stability bound

We recall that the matrix $\mathbf{K}(\mathbf{x}_*)$ defines a norm which is equivalent to the H^1 norm on $\Gamma_h[\mathbf{x}_*]$. The defect $\mathbf{d_v} \in \mathbb{R}^{3N}$ will be measured in the dual norm defined by

$$\|\mathbf{d}\|_{\star,\mathbf{x}_*}^2 := \mathbf{d}^T\mathbf{M}(\mathbf{x}_*)\mathbf{K}(\mathbf{x}_*)^{-1}\mathbf{M}(\mathbf{x}_*)\mathbf{d},$$

which is such that for the finite element function $d_h \in S_h[\mathbf{x}^*]^3$ with nodal vector $\mathbf{d}$ we have, from [LMV13, Proof of Theorem 5.1] or [KLLP17, Formula (5.5)],

$$\|\mathbf{d}\|_{\star,\mathbf{x}_*} = \|d_h\|_{H_h^{-1}(\Gamma_h[\mathbf{x}^*])} := \sup_{0\neq\psi_h\in S_h[\mathbf{x}^*]^3} \frac{\int_{\Gamma_h[\mathbf{x}^*]} d_h\cdot\psi_h}{\|\psi_h\|_{H^1(\Gamma_h[\mathbf{x}^*])^3}}. \tag{5.3}$$

In these norms we have the following stability result.

Proposition 5.1 *Suppose that the defects of the p-step linearly implicit BDF method are bounded as follows, with a sufficiently small $\vartheta > 0$ (that is independent of h and τ and n): for $n \geq p$ with $n\tau \leq T$,*

$$\|\mathbf{d}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^k)} \leq \vartheta h \quad \textit{and} \quad \|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^k} \leq \vartheta h \quad \text{for } k\tau \leq T. \tag{5.4}$$

Further, assume that the initial values are chosen such that

$$\|\mathbf{e}_\mathbf{x}^k\|_{\mathbf{K}(\mathbf{x}_*^k)} \leq \vartheta h \quad \textit{and} \quad \|\mathbf{e}_\mathbf{v}^k\|_{\mathbf{K}(\mathbf{x}_*^k)} \leq \vartheta h \quad \text{for } k = 0,\dots,p-1. \tag{5.5}$$

Then, the following error bounds hold, for $n \geq p$ such that $n\tau \leq T$,

$$\begin{aligned}
\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 &\leq C\tau\sum_{j=p}^n\Big(\|\mathbf{d}_\mathbf{x}^j\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \|\mathbf{d}_\mathbf{v}^j\|_{\star,\mathbf{x}_*^j}^2\Big) + C\sum_{i=0}^{p-1}\|\mathbf{e}_\mathbf{x}^i\|_{\mathbf{K}(\mathbf{x}_*^i)}^2,\\
\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 &\leq C\tau\sum_{j=p}^n\Big(\|\mathbf{d}_\mathbf{x}^j\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \|\mathbf{d}_\mathbf{v}^j\|_{\star,\mathbf{x}_*^j}^2\Big) + C\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n}^2 + C\sum_{i=0}^{p-1}\|\mathbf{e}_\mathbf{x}^i\|_{\mathbf{K}(\mathbf{x}_*^i)}^2,
\end{aligned} \tag{5.6}$$

where C is independent of h, τ and n with $n\tau \leq T$, but depends on T.

In Section 6 we will show that the defects obtained on inserting the exact solution values into the BDF scheme satisfy the bounds

$$\|\mathbf{d}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} \leq C(h^k+\tau^p), \qquad \|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n} \leq C(h^k+\tau^p).$$

Hence, condition (5.4) is satisfied under the mild stepsize restriction

$$\tau^p \leq c_0 h \tag{5.7}$$

for a sufficiently small c_0 that is independent of h and τ. We note that the error functions $e_x^n, e_v^n \in S_h[\mathbf{x}_*^n]^3$ with nodal vectors $\mathbf{e}_\mathbf{x}^n$ and $\mathbf{e}_\mathbf{v}^n$, respectively, are then bounded by

$$\left.\begin{aligned}
\|e_x^n\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} &\leq C(h^k+\tau^p),\\
\|e_v^n\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} &\leq C(h^k+\tau^p),
\end{aligned}\right. \qquad \text{for } n\tau \leq T.$$

Proof The proof is based on energy estimates for the matrix–vector formulation of the error equations (5.2) and relies on the results of Section 4. In the proof, c will be a generic constant independent of h and τ and n with $n\tau \le T$, which assumes different values on different occurrences. For many estimates we use similar techniques of proof as for the corresponding time-continuous results in [KLLP17]. However, to keep the paper fairly self-contained we include some detailed arguments.

In view of the condition in (iii) of Lemma 4.1 for $\mathbf{y} = \widetilde{\mathbf{x}}_*^n$ and $\mathbf{x} = \widetilde{\mathbf{x}}^n$, we need to control the $W^{1,\infty}$ norm of the position error $\widetilde{e}_x^n$. Let us assume that the error estimate (5.6) holds for $p, \dots, n-1$. Then, using an inverse inequality and the norm equivalence (4.4) and the definition of $\widetilde{\mathbf{e}}_\mathbf{x}^n$ (cf. (2.8)), we obtain

$$\begin{aligned}\|\nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{e}_x^n\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} &\le ch^{-1}\|\nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{e}_x^n\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\ &\le ch^{-1}\|\widetilde{\mathbf{e}}_\mathbf{x}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)} \le ch^{-1}\|\widetilde{\mathbf{e}}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} \\ &\le ch^{-1}\sum_{j=1}^{p}\|\mathbf{e}_\mathbf{x}^{n-j}\|_{\mathbf{K}(\mathbf{x}_*^n)} \\ &\le ch^{-1}\cdot c\vartheta h \le c\vartheta,\end{aligned} \tag{5.8}$$

where the last but one estimate follows from (5.6) for the past, and the assumption on small defects (5.4). For sufficiently small ϑ, we are thus in the position to use the bounds given in Lemma 4.1.

We estimate the two error equations (5.2) separately, and then combine them to yield the final estimate.

(a) *Estimates for the velocity law.* By testing the first line of the error equations (5.2) with $\mathbf{e}_\mathbf{v}^n$ we obtain

$$\begin{aligned}&\tfrac{1}{2}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \le \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}^2 \\ =&\ -(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{v}_*^n - (\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{e}_\mathbf{v}^n \\ &-\beta(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{A}(\widetilde{\mathbf{x}}^n)-\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{x}_*^n - \beta(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{A}(\widetilde{\mathbf{x}}^n)-\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{e}_\mathbf{x}^n \\ &+(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{g}(\widetilde{\mathbf{x}}^n,t_n)-\mathbf{g}(\widetilde{\mathbf{x}}_*^n,t_n)\big) - \beta(\mathbf{e}_\mathbf{v}^n)^T\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\mathbf{e}_\mathbf{x}^n - (\mathbf{e}_\mathbf{v}^n)^T\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n,\end{aligned}$$

where the inequality follows from (4.4). To bound the right-hand side, we use arguments of the proof of Proposition 10.1 (and that of Proposition 5.1) of [KLLP17], using the results of Lemma 4.1.

(i) For $0 \le \theta \le 1$, we denote $\Gamma_h^{n,\theta} = \Gamma_h[\widetilde{\mathbf{x}}_*^n + \theta\widetilde{\mathbf{e}}_\mathbf{x}^n]$, where $\widetilde{\mathbf{e}}_\mathbf{x}^n = \widetilde{\mathbf{x}}^n - \widetilde{\mathbf{x}}_*^n = \sum_{j=0}^{p-1}\gamma_j\mathbf{e}_\mathbf{x}^{n-p+j}$. We denote the finite element functions in $S_h(\Gamma_h^{n,\theta})^3$ with nodal vectors $\widetilde{\mathbf{e}}_\mathbf{x}^n$, $\mathbf{e}_\mathbf{v}^n$ and $\mathbf{v}_*^n$ by $\widetilde{e}_x^{n,\theta}$,$e_v^{n,\theta}$ and $v_*^{n,\theta}$, respectively. The definition (2.8) and Lemma 4.1 then give us

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{v}_*^n =&\ \int_0^1\int_{\Gamma_h^{n,\theta}} e_v^{n,\theta}\cdot\big(\nabla_{\Gamma_h^{n,\theta}}\cdot\widetilde{e}_x^{n,\theta}\big)v_*^{n,\theta}\,\mathrm{d}\theta \\ &+\alpha\int_0^1\int_{\Gamma_h^{n,\theta}}\nabla_{\Gamma_h^{n,\theta}}e_v^{n,\theta}\cdot\big(D_{\Gamma_h^{n,\theta}}\widetilde{e}_x^{n,\theta}\big)\nabla_{\Gamma_h^{n,\theta}}v_*^{n,\theta}\,\mathrm{d}\theta.\end{aligned}$$

Using the Cauchy–Schwarz inequality, we estimate the integral with the product of the $L^2 - L^2 - L^\infty$ norms of the three factors. We thus have

$$\begin{aligned}
&(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{v}_*^n \\
&\leq \int_0^1 \|e_v^{n,\theta}\|_{L^2(\Gamma_h^{n,\theta})} \, \|\nabla_{\Gamma_h^{n,\theta}} \cdot \widetilde{e}_x^{n,\theta}\|_{L^2(\Gamma_h^{n,\theta})} \, \|v_*^{n,\theta}\|_{L^\infty(\Gamma_h^{n,\theta})} \, \mathrm{d}\theta \\
&\quad + \alpha \int_0^1 \|\nabla_{\Gamma_h^{n,\theta}} e_v^{n,\theta}\|_{L^2(\Gamma_h^{n,\theta})} \, \|D_{\Gamma_h^{n,\theta}} \widetilde{e}_x^{n,\theta}\|_{L^2(\Gamma_h^{n,\theta})} \, \|\nabla_{\Gamma_h^{n,\theta}} v_*^{n,\theta}\|_{L^\infty(\Gamma_h^{n,\theta})} \, \mathrm{d}\theta \\
&\leq c \int_0^1 \|e_v^{n,\theta}\|_{H^1(\Gamma_h^{n,\theta})} \, \|\widetilde{e}_x^{n,\theta}\|_{H^1(\Gamma_h^{n,\theta})} \, \|v_*^{n,\theta}\|_{W^{1,\infty}(\Gamma_h^{n,\theta})} \, \mathrm{d}\theta.
\end{aligned}$$

By (5.8) and Lemma 4.1, this is bounded by

$$\begin{aligned}
&(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{v}_*^n \\
&\leq c\|e_v^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \, \|\widetilde{e}_x^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \, \|v_*^n\|_{W^{1,\infty}(\Gamma_h[\widetilde{\mathbf{x}}_*^n])},
\end{aligned}$$

where the last factor is bounded independently of h and τ. By Young's inequality, we thus obtain

$$\begin{aligned}
(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{v}_*^n &\leq \tfrac{1}{48}\|e_v^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}^2 + c\sum_{j=1}^p \|e_x^{n-j}\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}^2 \\
&= \tfrac{1}{48}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}^2 + c\sum_{j=1}^p \|\mathbf{e}_\mathbf{x}^{n-j}\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}^2 \\
&\leq \tfrac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\sum_{j=1}^p \|\mathbf{e}_\mathbf{x}^{n-j}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2,
\end{aligned}$$

where the last inequality follows from the norm equivalence (4.4).
(ii) Similarly, estimating the three factors in the integrals by $L^2 - L^\infty - L^2$, we obtain

$$\begin{aligned}
(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{e}_\mathbf{v}^n &\leq c\|e_v^n\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}^2 \, \|\nabla_{\Gamma_h} \cdot \widetilde{e}_x^n\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\
&\quad + c\|\nabla_{\Gamma_h} e_v^n\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}^2 \, \|D_{\Gamma_h} \widetilde{e}_x^n\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\
&\leq c\vartheta\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \leq \tfrac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2,
\end{aligned}$$

where we used the estimate (5.8) in the last but one inequality.
(iii)–(iv) The estimates involving the mean curvature term $\beta\mathbf{A}$ (in view of (2.5)) can be shown analogously as (i) and (ii):

$$\begin{aligned}
&(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{x}_*^n + (\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)\mathbf{e}_\mathbf{x}^n \\
&\leq \tfrac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\sum_{j=1}^p \|\mathbf{e}_\mathbf{x}^{n-j}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2, \\
&(\mathbf{e}_\mathbf{v}^n)^T\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\mathbf{e}_\mathbf{x}^n \leq \tfrac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2.
\end{aligned}$$

(v) Similarly as in (i) we rewrite

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{g}(\widetilde{\mathbf{x}}^n,t_n)-\mathbf{g}(\widetilde{\mathbf{x}}_*^n,t_n)\big) &= \int_{\Gamma_h^{1,n}} g^n\nu_{\Gamma_h^{1,n}}\cdot e_v^{1,n} - \int_{\Gamma_h^{0,n}} g^n\nu_{\Gamma_h^{0,n}}\cdot e_v^{0,n} \\ &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\Gamma_h^{n,\theta}} g^n\nu_{\Gamma_h^{n,\theta}}\cdot e_v^{n,\theta}\mathrm{d}\theta.\end{aligned}$$

We use the Leibniz formula and $\partial_\theta^\bullet e_v^{0,n}=0$ just as in (iii) of the proof of [KLLP17, Proposition 5.1], to finally obtain

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T\big(\mathbf{g}(\widetilde{\mathbf{x}}^n,t_n)-\mathbf{g}(\widetilde{\mathbf{x}}_*^n,t_n)\big) &\le c\|e_v^n\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\,\|\widetilde{e}_x^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\ &\le c\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}\|\widetilde{\mathbf{e}}_\mathbf{x}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}^2 \\ &\le \tfrac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\sum_{j=1}^p \|\mathbf{e}_\mathbf{x}^{n-j}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2.\end{aligned}$$

(vi) The term with the defect is estimated as

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n &= (\mathbf{e}_\mathbf{v}^n)^T\mathbf{K}(\mathbf{x}_*^n)^{1/2}\mathbf{K}(\mathbf{x}_*^n)^{-1/2}\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n \\ &\le \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n} \le \tfrac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n}^2.\end{aligned}$$

Finally, by combining all these estimates, using multiple absorptions, with sufficiently small ϑ we finally obtain

$$\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \le c\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\sum_{j=1}^p \|\mathbf{e}_\mathbf{x}^{n-j}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n}^2. \tag{5.9}$$

(b) *Estimates for ODE.* We rewrite the second equation of (5.2) as

$$\frac{1}{\tau}\sum_{j=p}^n \delta_{n-j}\mathbf{e}_\mathbf{x}^j = \mathbf{e}_\mathbf{v}^n - \widehat{\mathbf{d}_\mathbf{x}}^n,$$

with $\delta_j=0$ for $j>p$ and

$$\widehat{\mathbf{d}_\mathbf{x}}^n = \mathbf{d}_\mathbf{x}^n + \frac{1}{\tau}\sum_{j=0}^{p-1}\delta_{n-j}\mathbf{e}_\mathbf{x}^j,$$

where we note that $\widehat{\mathbf{d}_\mathbf{x}}^n = \mathbf{d}_\mathbf{x}^n$ for $n\ge 2p$. With the coefficients of the power series

$$\mu(\zeta)=\sum_{n=0}^\infty \mu_n\zeta^n = \frac{1}{\delta(\zeta)}$$

we then have, for $n\ge p$,

$$\mathbf{e}_\mathbf{x}^n = \tau\sum_{j=p}^n \mu_{n-j}(\mathbf{e}_\mathbf{v}^j - \widehat{\mathbf{d}_\mathbf{x}}^j).$$

By the zero-stability of the BDF method of order $p \le 6$ (which states that all zeros of $\delta(\zeta)$ are outside the unit circle with the exception of the simple zero at $\zeta = 1$), the coefficients μ_n are bounded: $|\mu_n| \le c$ for all n.

Taking the $K(\mathbf{x}_*^n)$ norm on both sides and recalling that by Lemma 4.2 all these norms are uniformly equivalent for $0 \le n\tau \le T$, we obtain with the Cauchy–Schwarz inequality

$$\begin{aligned}\|\mathbf{e}_\mathbf{x}^n\|_{K(\mathbf{x}_*^n)}^2 &\le c\tau \sum_{j=p}^{n} \|\mathbf{e}_\mathbf{v}^j - \widehat{\mathbf{d}_\mathbf{x}}^j\|_{K(\mathbf{x}_*^j)}^2 \\ &\le c\tau \sum_{j=p}^{n} \|\mathbf{e}_\mathbf{v}^j\|_{K(\mathbf{x}_*^j)}^2 + c\tau \sum_{j=p}^{n} \|\mathbf{d}_\mathbf{x}^j\|_{K(\mathbf{x}_*^j)}^2 + c \sum_{i=0}^{p-1} \|\mathbf{e}_\mathbf{x}^i\|_{K(\mathbf{x}_*^i)}^2.\end{aligned}$$

Combining this inequality with (5.9) and using a discrete Gronwall inequality then yields the result. □

6 Consistency error

In this section we show that the consistency errors, that is, the defects defined by (5.1) and obtained by inserting the interpolated exact solution into the numerical method, are bounded in the required norms by $C(h^k + \tau^p)$ for the finite element method of polynomial degree k and the p-step BDF method.

Let us first recall the formula for the defect of the spatial semi-discretization $d_{h,v}(\cdot,t)$ from Section 8 of [KLLP17], for $\psi_h \in S_h[\mathbf{x}_*(t)]^3$:

$$\begin{aligned}\int_{\Gamma_h[\mathbf{x}_*(t)]} d_{h,v}(\cdot,t)\cdot\psi_h = &\int_{\Gamma_h[\mathbf{x}_*(t)]} \widetilde{I}_h v(\cdot,t)\cdot\psi_h + \alpha \int_{\Gamma_h[\mathbf{x}_*(t)]} \nabla_{\Gamma_h}\widetilde{I}_h v(\cdot,t)\cdot\nabla_{\Gamma_h}\psi_h \\ &+ \beta \int_{\Gamma_h[\mathbf{x}_*(t)]} \nabla_{\Gamma_h}\widetilde{I}_h X(\cdot,t)\cdot\nabla_{\Gamma_h}\psi_h - \int_{\Gamma_h[\mathbf{x}_*(t)]} g(\cdot,t)\,\nu_{\Gamma_h[\mathbf{x}_*(t)]}\cdot\psi_h,\end{aligned}$$

which satisfies the following bounds.

Lemma 6.1 [KLLP17, Lemma 8.1] *Let the surface X and its velocity v be sufficiently smooth. Then there exists a constant $c > 0$ (independent of t) such that for all $h \le h_0$, with a sufficiently small $h_0 > 0$, and for all $t \in [0,T]$, the defects $d_{h,v}$ of the kth-degree finite element interpolation are bounded as*

$$\|d_{h,v}(\cdot,t)\|_{H_h^{-1}(\Gamma(X_h^*))} \le c h^k.$$

We will now bound the defect of the full discretization.

Lemma 6.2 *Let the surface X and its velocity v be sufficiently smooth. Then there exist $h_0 > 0$ and $\tau_0 > 0$ such that for all $h \le h_0$ and for all $\tau \le \tau_0$, the consistency errors are bounded as*

$$\begin{aligned}\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n} &= \|d_v^n\|_{H_h^{-1}(\Gamma(X_h^*(t_n)))} \le c\big(\tau^p + h^k\big), \\ \|\mathbf{d}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} &= \|d_x^n\|_{H^1(\Gamma(X_h^*(t_n)))} \le c\tau^p,\end{aligned}$$

where c is independent of h, τ and n with $n\tau \le T$.

Proof For the defect in v, the corresponding finite element function $d_v^n \in S_h[\widetilde{\mathbf{x}}_*^n]$ with nodal values $\mathbf{d}_\mathbf{v}^n$ satisfies the following: for all finite element functions $\bar\psi_h \in S_h[\mathbf{x}_*^n]$ and the corresponding $\psi_h \in S_h[\widetilde{\mathbf{x}}_*^n]$ with the same nodal values,

$$\begin{aligned}\int_{\Gamma_h[\mathbf{x}_*^n]} d_v^n \cdot \bar\psi_h = &\int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h + \alpha \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \nabla_{\Gamma_h}\widetilde{I}_h v(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h \\ &+ \beta\int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \nabla_{\Gamma_h}\widetilde{I}_h X(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h - \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} g(\cdot,t_n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\cdot\psi_h,\end{aligned} \tag{6.1}$$

where $\widetilde{I}_h v(\cdot,t_n), \widetilde{I}_h X(\cdot,t_n) \in S_h[\widetilde{\mathbf{x}}_*^n]^3$ denote the finite element interpolation of $v(\cdot,t_n)$ and $X(\cdot,t_n)$, respectively, on $\Gamma_h[\widetilde{\mathbf{x}}_*^n]$. Let us first rewrite (6.1), by subtracting the weak form of the problem (2.2). For the first term on the right-hand side, by adding and subtracting, this yields

$$\begin{aligned}&\int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h - \int_{\Gamma(X(t_n))} v(\cdot,t_n)\cdot\psi_h^l \\ &= \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h - \int_{\Gamma_h[\mathbf{x}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h \\ &\quad + \int_{\Gamma_h[\mathbf{x}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h - \int_{\Gamma(X(t_n))} v(\cdot,t_n)\cdot\psi_h^l.\end{aligned}$$

Note that the last pair is simply a spatial defect, therefore repeating the same process for all four terms, and using the spatial defect $d_{h,v}$ from Section 8 of [KLLP17], we obtain

$$\begin{aligned}\int_{\Gamma_h[\mathbf{x}_*^n]} d_v^n\cdot\psi_h = &\int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h - \int_{\Gamma_h[\mathbf{x}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h \\ &+ \alpha\int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \nabla_{\Gamma_h}\widetilde{I}_h v(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h - \alpha\int_{\Gamma_h[\mathbf{x}_*^n]} \nabla_{\Gamma_h}\widetilde{I}_h v(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h \\ &+ \beta\int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \nabla_{\Gamma_h}\widetilde{I}_h X(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h - \beta\int_{\Gamma_h[\mathbf{x}_*^n]} \nabla_{\Gamma_h}\widetilde{I}_h X(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h \\ &- \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} g(\cdot,t_n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\cdot\psi_h + \int_{\Gamma_h[\mathbf{x}_*^n]} g(\cdot,t_n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\cdot\psi_h \\ &+ \int_{\Gamma_h[\mathbf{x}_*^n]} d_{h,v}(\cdot,t_n)\cdot\psi_h.\end{aligned}$$

We estimate the defect d_v^n pairwise, using similar tools as in part (a) of the proof of Proposition 5.1 and recalling (5.3).

For the first pair, we use the setting of Lemma 4.3, and then a Cauchy–Schwarz inequality and an $L^2 - L^2 - L^\infty$ estimate yield

$$\begin{aligned}
&\Big| \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h - \int_{\Gamma_h[\mathbf{x}_*^n]} \widetilde{I}_h v(\cdot,t_n)\cdot\psi_h \Big| \\
&= \Big| \int_0^1 \int_{\Gamma_h^{n,\theta}} \psi_h^\theta (\nabla_{\Gamma_h^{n,\theta}} \cdot \widetilde{e}_h^{n,\theta}) v_{*,h}^{n,\theta} \mathrm{d}\theta \Big| \\
&\le \int_0^1 \|\psi_h^\theta\|_{L^2(\Gamma_h^{n,\theta})} \|\nabla_{\Gamma_h^{n,\theta}} \cdot \widetilde{e}_h^{n,\theta}\|_{L^2(\Gamma_h^{n,\theta})} \|v_{*,h}^{n,\theta}\|_{L^\infty(\Gamma_h^{n,\theta})} \mathrm{d}\theta \\
&\le c\|\psi_h^0\|_{L^2(\Gamma_h^{0,n})} \|\widetilde{e}_h^{n,0}\|_{H^1(\Gamma_h^{0,n})} \|v_{*,h}^{n,0}\|_{L^\infty(\Gamma_h^{0,n})} \\
&\le c\|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \|\widetilde{e}_h^n\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \\
&\quad \cdot \Big(\|v_*(\cdot,t_n)\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])} + \|v_{*,h}(\cdot,t_n) - v_*(\cdot,t_n)\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])} \Big) \\
&\le c\|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \|\widetilde{e}_h^n\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} (1 + ch^2) \|v_*(\cdot,t_n)\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*^n])} \\
&\le c\|\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n\|_{\mathbf{K}(\mathbf{x}_*^n)} \|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \\
&\le c\tau^p \|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])},
\end{aligned}$$

where we used a $W^{1,\infty}$ interpolation estimate from [Dem09, Proposition 2.7], and the last inequality follows from (4.3).

The other three pairs are again estimated similarly as above, and we finally obtain the bounds

$$\begin{aligned}
&\Big| \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \nabla_{\Gamma_h} \widetilde{I}_h v(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h - \int_{\Gamma_h[\mathbf{x}_*^n]} \nabla_{\Gamma_h} \widetilde{I}_h v(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h \Big| \\
&\qquad\qquad \le c\tau^p \|\psi_h\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \\
&\Big| \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} \nabla_{\Gamma_h} \widetilde{I}_h X(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h - \int_{\Gamma_h[\mathbf{x}_*^n]} \nabla_{\Gamma_h} \widetilde{I}_h X(\cdot,t_n)\cdot\nabla_{\Gamma_h}\psi_h \Big| \\
&\qquad\qquad \le c\tau^p \|\psi_h\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \\
&\Big| \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} g(\cdot,t_n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\cdot\psi_h - \int_{\Gamma_h[\mathbf{x}_*^n]} g(\cdot,t_n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\cdot\psi_h \Big| \\
&\qquad\qquad \le c\tau^p \|\psi_h\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}.
\end{aligned}$$

Furthermore, as shown in Lemma 8.1 of [KLLP17], the spatial defect $d_{h,v}(\cdot,t_n)$ is bounded by

$$\int_{\Gamma_h[\mathbf{x}_*^n]} d_{h,v}(\cdot,t_n)\cdot\psi_h \le ch^k \|\psi_h\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}.$$

Combining the above estimates, we obtain the bound $\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n} \le c\big(\tau^p + h^k\big)$.
The defect in X is given by

$$\mathbf{d}_\mathbf{x}^n = \frac{1}{\tau}\sum_{j=0}^p \delta_j \mathbf{x}_*(t_{n-j}) - \dot{\mathbf{x}}_*(t_n)$$

and is solely due to temporal discretization. The bound $\|\mathbf{d}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} \le c\tau^p$ then follows by Taylor expansion. □

7 Proof of Theorem 3.1

The errors are decomposed using interpolations and the definition of lifts from Section 2.7. We denote by $\widehat I_h v \in S_h[\mathbf{x}_*]$ the finite element interpolation of v on the interpolated surface $\Gamma_h[\mathbf{x}_*]$ and by $I_h v = (\widehat I_h v)^l$ its lift to the exact surface $\Gamma(X)$. We write

$$\begin{aligned} (v_h^n)^L - v(\cdot,t_n) &= \big(\widehat v_h^n - \widehat I_h v(\cdot,t_n)\big)^l + \big(I_h v(\cdot,t_n) - v(\cdot,t_n)\big), \\ (X_h^n)^L - X(\cdot,t_n) &= \big(\widehat X_h^n - \widehat I_h X(\cdot,t_n)\big)^l + \big(I_h X(\cdot,t_n) - X(\cdot,t_n)\big). \end{aligned}$$

The last terms in these formulas can be bounded in the $H^1(\Gamma)$ norm by Ch^k, using the interpolation bounds of [Kov17].

To bound the first terms on the right-hand sides, we first use the defect bounds of Lemma 6.2, which then, under the mild stepsize restriction, together with the stability estimate of Proposition 5.1 proves the result, since by the norm equivalences from Lemma 4.1 and equations (4.1)–(4.2) we have

$$\begin{aligned} \|\big(\widehat v_h^n - \widehat I_h v(\cdot,t_n)\big)^l\|_{L^2(\Gamma(\cdot,t_n))} &\le c\|\widehat v_h^n - \widehat I_h v(\cdot,t_n)\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \\ &= c\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{M}(\mathbf{x}_*^n)}, \\ \|\nabla_\Gamma\big(\widehat v_h^n - \widehat I_h v(\cdot,t_n)\big)^l\|_{L^2(\Gamma_h[\mathbf{x}_*^n]))} &\le c\|\nabla_{\Gamma_h^*}\big(\widehat v_h^n - \widehat I_h v(\cdot,t_n)\big)\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \\ &= c\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)}, \end{aligned}$$

and similarly for $\widehat X_h^n - \widehat I_h X(\cdot,t_n)$.

8 A dynamic velocity law

8.1 Weak formulation and ESFEM / BDF full discretization

We now consider the dynamic velocity law (1.2), viz.,

$$\partial^\bullet v + v\nabla_{\Gamma(X)}\cdot v - \alpha\Delta_{\Gamma(X)}v = g(\cdot,t)\,\nu_{\Gamma(X)},$$

where again $g : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is a given smooth function of (x,t), and $\alpha > 0$ is a fixed parameter. This problem is considered together with the ordinary differential equation (2.1) for the positions X determining the surface $\Gamma(X)$. Initial values are specified for X and v.

The weak formulation of the dynamic velocity law (1.2) reads as follows: Find $v(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))^3$ such that for all test functions $\psi(\cdot,t) \in H^1(\Gamma(X(\cdot,t)))^3$ with vanishing material derivative,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Gamma(X)} v\cdot\psi \;+\alpha\int_{\Gamma(X)}\nabla_{\Gamma(X)}v\cdot\nabla_{\Gamma(X)}\psi = \int_{\Gamma(X)} g\,\nu_{\Gamma(X)}\cdot\psi, \tag{8.1}$$

together with the ordinary differential equation (2.1) for the positions X determining the surface $\Gamma(X)$. The finite element space discretization is done in the usual way. We forego the straightforward formulation and immediately present the matrix–vector formulation of the semi-discretization. As in Section 2.5, the nodal vectors $\mathbf{v}(t) \in \mathbb{R}^{3N}$ of the finite element function $v_h(\cdot,t)$, together with the surface nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$ satisfy a system of ordinary differential equations with matrices and driving term as in Section 2.5:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\Big(\mathbf{M}(\mathbf{x})\mathbf{v}\Big) + \mathbf{A}(\mathbf{x})\mathbf{v} &= \mathbf{g}(\mathbf{x},t), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{8.2}$$

We apply a p-step linearly implicit BDF method to the above ODE system with a step size $\tau > 0$: with $t_n = n\tau \le T$ and with the extrapolated nodal vector $\widetilde{\mathbf{x}}_*^n$ defined by (2.8), the new nodal vectors of velocity and position, $\mathbf{v}^n$ and $\mathbf{x}^n$, respectively, are determined from the following system of linear equations:

$$\begin{aligned} \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})\mathbf{v}^{n-j} + \ \mathbf{A}(\widetilde{\mathbf{x}}^{n})\mathbf{v}^{n} &= \mathbf{g}(\widetilde{\mathbf{x}}^{n},t) \\ \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{x}^{n-j} = \ \mathbf{v}^{n}. & \end{aligned} \tag{8.3}$$

As in Section 2, the nodal vector $\mathbf{x}^n$ defines the discrete surface $\Gamma_h[\mathbf{x}^n] = \Gamma(X_h^n)$, which is to approximate the exact surface $\Gamma(X)$, and we obtain the position and velocity approximations (2.9).

8.2 Statement of the error bound

The following result is the analogue of Theorem 3.1 for the dynamic velocity law. We use the same notation for the lifted approximations.

Theorem 8.1 *Consider the ESFEM / BDF linearly implicit full discretization* (8.3) *of the dynamic velocity equation* (1.2), *using finite elements of polynomial degree $k \ge 2$ and BDF methods of order $p \le 5$. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for X. Suppose that the problem admits an exact solution X, v that is sufficiently smooth (say, of class $C([0,T],H^{k+1}) \cap C^{p+1}([0,T],W^{1,\infty})$) on the time interval $0 \le t \le T$, and that the flow map $X(\cdot,t) : \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ is non-degenerate for $0 \le t \le T$, so that $\Gamma(t)$ is a regular surface. Suppose further that the starting values are sufficiently accurate: for $i = 0,\dots,p-1$,*

$$\|(X_h^i)^L - X(\cdot,i\tau)\|_{H^1(\Gamma^0)^3} + \|(v_h^i)^L - v(\cdot,i\tau)\|_{H^1(\Gamma^0)^3} \le C_0(h^k + \tau^p).$$

Then, there exist $h_0 > 0$, $\tau_0 > 0$ and $c_0 > 0$ such that for all mesh widths $h \le h_0$ and step sizes $\tau \le \tau_0$ satisfying the mild stepsize restriction $\tau^p \le c_0 h$, *the following error bounds hold over the exact surface $\Gamma(t_n) = \Gamma(X(\cdot,t_n))$ uniformly for $0 \le t_n = n\tau \le T$:*

$$\|(x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\|_{H^1(\Gamma(t_n))^3} \le C(h^k + \tau^p),$$

$$\|(v_h^n)^L - v(\cdot,t_n)\|_{L^2(\Gamma(t_n))^3} + \Big(\sum_{j=p}^{n} \|(v_h^j)^L - v(\cdot,t_j)\|^2_{H^1(\Gamma(t_j))^3}\Big)^{1/2} \le C(h^k + \tau^p).$$

The constant C is independent of h and τ and n with $n\tau \le T$, but depends on bounds of higher derivatives of the solution (X, v), and on the length T of the time interval.

8.3 Auxiliary results by Dahlquist and Nevanlinna & Odeh

While the formulations of Theorems 3.1 and 8.1 are very similar, the proofs differ substantially in the stability analysis. In this subsection we recall two important results that combined permit us to use energy estimates for BDF methods up to order 5: the first result is from Dahlquist's G-stability theory, and the second one from the multiplier technique of Nevanlinna and Odeh. These results have previously been used in the error analysis of BDF methods for various parabolic problems in [AL15, ALL17, KP16, LMV13].

Lemma 8.1 (Dahlquist [Dah78]) *Let $\delta(\zeta) = \sum_{j=1}^p \delta_j \zeta^j$ and $\mu(\zeta) = \sum_{j=1}^p \mu_j \zeta^j$ be polynomials of degree at most p (at least one of them of degree p) that have no common divisor. Let $\langle \cdot,\cdot \rangle$ denote an inner product on $\mathbb{R}^N$. If*

$$\mathrm{Re}\frac{\delta(\zeta)}{\mu(\zeta)} > 0, \qquad \textit{for} \quad |\zeta| < 1,$$

then there exists a symmetric positive definite matrix $G = (g_{ij}) \in \mathbb{R}^{p\times p}$ such that for all $\mathbf{w}_0, \dots, \mathbf{w}_p \in \mathbb{R}^N$

$$\Big\langle \sum_{i=0}^p \delta_i \mathbf{w}_{p-i}, \sum_{i=0}^p \mu_i \mathbf{w}_{p-i} \Big\rangle \ge \sum_{i,j=1}^p g_{ij}\langle \mathbf{w}_i, \mathbf{w}_j\rangle - \sum_{i,j=1}^p g_{ij}\langle \mathbf{w}_{i-1}, \mathbf{w}_{j-1}\rangle.$$

In view of the following result, the choice $\mu(\zeta) = 1 - \eta\zeta$ together with the polynomial $\delta(\zeta)$ of the BDF methods will play an important role later on.

Lemma 8.2 (Nevanlinna & Odeh [NO81]) *If $p \le 5$, then there exists $0 \le \eta < 1$ such that for $\delta(\zeta) = \sum_{\ell=1}^p \frac{1}{\ell}(1-\zeta)^\ell$,*

$$\mathrm{Re}\,\frac{\delta(\zeta)}{1 - \eta\zeta} > 0, \qquad \textit{for} \quad |\zeta| < 1.$$

The smallest possible values of η are found to be $\eta = 0, 0, 0.0836, 0.2878, 0.8160$ for $p = 1, \dots, 5$, respectively.

8.4 Error equations

By using the same notations as in the previous sections for the nodal vectors of the exact positions $\mathbf{x}_*^n \in \mathbb{R}^{3N}$ and of the exact velocity $\mathbf{v}_*^n \in \mathbb{R}^{3N}$, and for their defects $\mathbf{d}_{\mathbf{v}}^n$ and $\mathbf{d}_{\mathbf{x}}^n$, we obtain that they fulfil the following equations:

$$\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j})\mathbf{v}_*^{n-j} + \mathbf{A}(\widetilde{\mathbf{x}}_*^n)\mathbf{v}_*^n = \mathbf{g}(\widetilde{\mathbf{x}}^n, t) + \mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_{\mathbf{v}}^n,$$

$$\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{x}_*^{n-j} = \mathbf{v}_*^n + \mathbf{d}_{\mathbf{x}}^n.$$

By subtracting the above equations from (8.3), we obtain the error equations for the surface nodes and velocity:

$$\begin{aligned}
&\mathbf{M}(\mathbf{x}_*^n)\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{e}_{\mathbf{v}}^{n-j} + \mathbf{A}(\mathbf{x}_*^n)\mathbf{e}_{\mathbf{v}}^n \\
&= -\frac{1}{\tau}\sum_{j=1}^{p}\delta_j\big(\mathbf{M}(\mathbf{x}_*^{n-j}) - \mathbf{M}(\mathbf{x}_*^n)\big)\mathbf{e}_{\mathbf{v}}^{n-j} - \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\big(\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}) - \mathbf{M}(\mathbf{x}_*^{n-j})\big)\mathbf{e}_{\mathbf{v}}^{n-j} \\
&\quad - \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\big(\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j})\big)(\mathbf{v}_*^{n-j} + \mathbf{e}_{\mathbf{v}}^{n-j}) \\
&\quad - \big(\mathbf{A}(\widetilde{\mathbf{x}}_*^n) - \mathbf{A}(\mathbf{x}_*^n)\big)\mathbf{e}_{\mathbf{v}}^n - \big(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n)\big)(\mathbf{v}_*^n + \mathbf{e}_{\mathbf{v}}^n) \\
&\quad + \mathbf{g}(\widetilde{\mathbf{x}}^n, t_n) - \mathbf{g}(\widetilde{\mathbf{x}}_*^n, t_n) - \mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_{\mathbf{v}}^n \\
&\frac{1}{\tau}\sum_{j=0}^{p}\delta_j \mathbf{e}_{\mathbf{x}}^{n-j} = \mathbf{e}_{\mathbf{v}}^n - \mathbf{d}_{\mathbf{x}}^n.
\end{aligned} \tag{8.4}$$

8.5 Stability

We then have the following stability result.

Proposition 8.1 *Under the smallness assumptions of Proposition 5.1 for the defects and the errors in the initial values, the following error bound holds for BDF methods of order $p \le 5$ for $n\tau \le T$:*

$$\begin{aligned}
&\|\mathbf{e}_{\mathbf{x}}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \|\mathbf{e}_{\mathbf{v}}^n\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 + \tau\sum_{j=p}^{n}\|\mathbf{e}_{\mathbf{v}}^j\|_{\mathbf{A}(\mathbf{x}_*^j)}^2 \\
&\le C\tau\sum_{j=p}^{n}\Big(\|\mathbf{d}_{\mathbf{x}}^j\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \|\mathbf{d}_{\mathbf{v}}^j\|_{\star,\mathbf{x}_*^j}^2\Big) + c\|\mathbf{d}_{\mathbf{v}}^n\|_{\star,\mathbf{x}_*^n}^2 \\
&\quad + C\sum_{i=0}^{p-1}\Big(\|\mathbf{e}_{\mathbf{x}}^i\|_{\mathbf{K}(\mathbf{x}_*^i)}^2 + \|\mathbf{e}_{\mathbf{v}}^i\|_{\mathbf{M}(\mathbf{x}_*^i)}^2\Big).
\end{aligned} \tag{8.5}$$

The constant C is independent of h, τ and n, but depends on T.

Proof We test the first error equation in (8.4) with $\mathbf{e}_\mathbf{v}^n - \eta \mathbf{e}_\mathbf{v}^{n-1}$ to obtain

$$(\mathbf{e}_\mathbf{v}^n - \eta \mathbf{e}_\mathbf{v}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \frac{1}{\tau} \sum_{j=0}^{p} \delta_j \mathbf{e}_\mathbf{v}^{n-j} + (\mathbf{e}_\mathbf{v}^n - \eta \mathbf{e}_\mathbf{v}^{n-1})^T \mathbf{A}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{v}^n = \rho^n,$$

where the right-hand term ρ^n can be estimated by the same arguments as in part (a) of the proof of Proposition 5.1. On the left-hand side we have a term containing the stiffness matrix $\mathbf{A}(\mathbf{x}_*^n)$, which is estimated from below as follows using Lemmas 4.2 and 4.3:

$$\begin{aligned}
(\mathbf{e}_\mathbf{v}^n - \eta \mathbf{e}_\mathbf{v}^{n-1})^T \mathbf{A}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{v}^n &\geq \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \eta \|\mathbf{e}_\mathbf{v}^{n-1}\|_{\mathbf{A}(\mathbf{x}_*^n)} \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)} \\
&\geq \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \eta(1 + c\tau) \|\mathbf{e}_\mathbf{v}^{n-1}\|_{\mathbf{A}(\mathbf{x}_*^{n-1})} \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)} \\
&\geq (1 - \tfrac{1}{2}\eta - c\tau) \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - (\tfrac{1}{2}\eta + c\tau) \|\mathbf{e}_\mathbf{v}^{n-1}\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 .
\end{aligned}$$

The other term on the left-hand side, which contains the mass matrix $\mathbf{M}(\mathbf{x}_*^n)$, is estimated from below using Lemmas 8.1 and 8.2. Let us introduce

$$\mathbf{E}_\mathbf{v}^n = \big(\mathbf{e}_\mathbf{v}^{n-p+1}, \dots, \mathbf{e}_\mathbf{v}^{n-1}, \mathbf{e}_\mathbf{v}^n\big)$$

and the norm

$$|\mathbf{E}_\mathbf{v}^n|_{G,\mathbf{x}_*^n}^2 = \sum_{i,j=1}^{p} g_{ij} (\mathbf{e}_\mathbf{v}^{n-p+i})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{v}^{n-p+j},$$

which satisfies the norm equivalence relation

$$\lambda_{\min} \sum_{i=1}^{p} \|\mathbf{e}_\mathbf{v}^{n-p+i}\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 \leq |\mathbf{E}_\mathbf{x}^n|_{G,\mathbf{x}_*^n}^2 \leq \lambda_{\max} \sum_{i=1}^{p} \|\mathbf{e}_\mathbf{v}^{n-p+i}\|_{\mathbf{M}(\mathbf{x}_*^n)}^2, \tag{8.6}$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the smallest and largest eigenvalue of the symmetric positive definite matrix $G = (g_{ij})$ of Lemma 8.1. Hence we obtain from Lemmas 8.1 and 8.2

$$(\mathbf{e}_\mathbf{v}^n - \eta \mathbf{e}_\mathbf{v}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \sum_{j=0}^{p} \delta_j \mathbf{e}_\mathbf{v}^{n-j} \geq |\mathbf{E}_\mathbf{v}^n|_{G,\mathbf{x}_*^n}^2 - |\mathbf{E}_\mathbf{v}^{n-1}|_{G,\mathbf{x}_*^n}^2 ,$$

where we note that by Lemma 4.2,

$$|\mathbf{E}_\mathbf{v}^{n-1}|_{G,\mathbf{x}_*^n}^2 \leq (1 + c\tau) |\mathbf{E}_\mathbf{v}^{n-1}|_{G,\mathbf{x}_*^{n-1}}^2,$$

so that altogether we have

$$\begin{aligned}
&|\mathbf{E}_\mathbf{v}^n|_{G,\mathbf{x}_*^n}^2 - (1 + c\tau) |\mathbf{E}_\mathbf{v}^{n-1}|_{G,\mathbf{x}_*^{n-1}}^2 \\
&+ \tau(1 - \tfrac{1}{2}\eta - c\tau) \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \tau(\tfrac{1}{2}\eta + c\tau) \|\mathbf{e}_\mathbf{v}^{n-1}\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 \leq \tau \rho^n.
\end{aligned}$$

Using these inequalities from 1 to n yields for sufficiently small τ, with a positive constant γ,

$$|\mathbf{E}_{\mathbf{v}}^n|_{G,\mathbf{x}_*^n}^2 + \gamma\tau \sum_{j=0}^{n} e^{c(n-j)\tau} \|\mathbf{e}_{\mathbf{v}}^j\|_{\mathbf{A}(\mathbf{x}_*^j)}^2 \leq e^{cn\tau} |\mathbf{E}_{\mathbf{v}}^0|_{G,\mathbf{x}_*^0}^2 + \tau \sum_{j=0}^{n} e^{c(n-j)\tau} \rho^j.$$

Using this bound together with estimates for ρ^j and $\mathbf{e}_{\mathbf{x}}^j$ obtained in the same way as in the proof of Proposition 5.1 then yields the stated result. $\square$

Together with bounds for the consistency errors $\mathbf{d}_{\mathbf{v}}^n$ and $\mathbf{d}_{\mathbf{x}}^n$, which are proven in the same way as in Section 6, the stability bounds of Proposition 8.1 then yield the $O(h^k + \tau^p)$ error bounds of Theorem 8.1.

9 Coupling with diffusion on the surface

Let us now turn to the parabolic surface PDE coupled with the regularised velocity law. We consider the following coupled problem of an evolving surface driven by diffusion on the surface, for which the ESFEM semi-discretization was studied in [KLLP17]:

$$\begin{aligned} \partial^\bullet u + u \nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u &= f(u, \nabla_{\Gamma(X)} u), \\ v - \alpha \Delta_{\Gamma(X)} v + \beta H_{\Gamma(X)} \nu_{\Gamma(X)} &= g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)} \\ \partial_t X(q,t) &= v(X(q,t),t), \end{aligned} \tag{9.1}$$

with $\alpha > 0$ and $\beta \geq 0$. The weak formulation and the ESFEM spatial semi-discretization, also in its matrix–vector formulation, are given in Section 2 of [KLLP17]. The finally obtained coupled system of differential-algebraic equations for the vectors of nodal values $\mathbf{u}(t) \in \mathbb{R}^N$, $\mathbf{v}(t) \in \mathbb{R}^{3N}$, and $\mathbf{x}(t) \in \mathbb{R}^{3N}$ reads, with the matrices of Section 2.5:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \Big(\mathbf{M}(\mathbf{x}) \mathbf{u} \Big) + \mathbf{A}(\mathbf{x}) \mathbf{u} &= \mathbf{f}(\mathbf{x}, \mathbf{u}), \\ \mathbf{K}(\mathbf{x}) \mathbf{v} + \beta \mathbf{A}(\mathbf{x}) \mathbf{x} &= \mathbf{g}(\mathbf{x}, \mathbf{u}), \\ \dot{\mathbf{x}} &= \mathbf{v}. \end{aligned} \tag{9.2}$$

The right-hand side vectors are defined slightly differently from Section 2.5. They are given by

$$\begin{aligned} \mathbf{f}(\mathbf{x}, \mathbf{u})|_j &= \int_{\Gamma_h[\mathbf{x}]} f(u_h, \nabla_{\Gamma_h} u_h)\, \phi_j[\mathbf{x}], \\ \mathbf{g}(\mathbf{x}, \mathbf{u})|_{3(j-1)+\ell} &= \int_{\Gamma_h[\mathbf{x}]} g(u_h, \nabla_{\Gamma_h} u_h) \left(\nu_{\Gamma_h[\mathbf{x}]} \right)_\ell \phi_j[\mathbf{x}], \end{aligned}$$

for $j = 1, \ldots, N$, and $\ell = 1, 2, 3$.

The linearly implicit BDF discretization then reads as follows: with the extrapolated position vectors $\widetilde{\mathbf{x}}^n$ defined by (2.8),

$$\begin{aligned}\frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})\mathbf{u}^{n-j}+\mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{u}^n &= \mathbf{f}(\widetilde{\mathbf{x}}^n,\widetilde{\mathbf{u}}^n),\\ \mathbf{K}(\widetilde{\mathbf{x}}^n)\mathbf{v}^n+\beta\mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{x}^n &= \mathbf{g}(\widetilde{\mathbf{x}}^n,\widetilde{\mathbf{u}}^n),\\ \frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{x}^{n-j} &= \mathbf{v}^n.\end{aligned} \tag{9.3}$$

Full discretizations using BDF methods of parabolic PDEs on an evolving surface with a *given* velocity have been studied in [LMV13]. The combination of the proofs of Lemma 4.1 and Theorem 5.1 of [LMV13] with the error analysis of the ESFEM semi-discretization in [KLLP17] and with the proof of Theorem 3.1 in the present paper yields the following convergence theorem. We omit the details of the proof.

Theorem 9.1 *Consider the ESFEM / BDF linearly implicit full discretization* (9.3) *of the coupled surface-evolution equation* (9.1)*, using finite elements of polynomial degree* $k \geq 2$ *and BDF methods of order* $p \leq 5$*. We assume quasi-uniform admissible triangulations of the initial surface and initial values chosen by finite element interpolation of the initial data for* X*. Suppose that the problem admits an exact solution* u, X, v *that is sufficiently smooth (say, of class* $C([0,T],H^{k+1})\cap C^{p+1}([0,T],W^{1,\infty})$*) on the time interval* $0 \leq t \leq T$*, and that the flow map* $X(\cdot,t): \Gamma_0 \to \Gamma(t) \subset \mathbb{R}^3$ *is non-degenerate for* $0 \leq t \leq T$*, so that* $\Gamma(t)$ *is a regular surface. Suppose further that the starting values are sufficiently accurate. Then, there exist* $h_0 > 0$*,* $\tau_0 > 0$ *and* $c_0 > 0$ *such that for all mesh widths* $h \leq h_0$ *and step sizes* $\tau \leq \tau_0$ *satisfying the mild stepsize restriction* $\tau^p \leq c_0 h$*, the following error bounds hold over the exact surface* $\Gamma(t_n) = \Gamma(X(\cdot,t_n))$ *uniformly for* $0 \leq t_n = n\tau \leq T$*:*

$$\begin{aligned}\|(u_h^n)^L - u(\cdot,t_n)\|_{L^2(\Gamma(t_n))^3} + \Big(\sum_{j=p}^{n}\|(u_h^j)^L - u(\cdot,t_j)\|^2_{H^1(\Gamma(t_j))^3}\Big)^{1/2}&\\ \leq C(h^k+\tau^p)&,\\ \|(v_h^n)^L - v(\cdot,t_n)\|_{H^1(\Gamma(t_n))^3} \leq C(h^k+\tau^p)&,\\ \|(x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\|_{H^1(\Gamma(t_n))^3} \leq C(h^k+\tau^p)&.\end{aligned}$$

The constant C *is independent of* h *and* τ *and* n *with* $n\tau \leq T$*, but depends on bounds of higher derivatives of the solution* (u,v,X)*, and on the length* T *of the time interval.*

10 Numerical experiments

10.1 Forced mean curvature flow

We performed numerical experiments for the velocity law (1.1): for $x = X(q,t) \in \Gamma(t)$ with $q \in \Gamma_0$,

$$\begin{aligned} v(x,t) - \alpha \Delta_{\Gamma(t)} v(x,t) = {} & -\beta H_{\Gamma(t)}(x)\, \nu_{\Gamma(t)}(x) + g\big(x,t\big)\, \nu_{\Gamma(t)}(x), \\ \partial_t X(q,t) = {} & v(X(q,t),t), \end{aligned} \tag{10.1}$$

where the inhomogeneity $g : R^3 \times [0,T] \to \mathbb{R}$ is chosen such that the exact solution is $X(q,t) = r(t)q$, with q on the unit sphere Γ_0. The function r satisfies the logistic differential equation:

$$\begin{aligned} \dot r(t) &= \Big(1 - \tfrac{r_1}{r(t)}\Big) r(t), \qquad t \in [0,T], \\ r(0) &= r_0, \end{aligned}$$

with $r_1 \geq r_0 = 1$, i.e. $r(t) = r_0 r_1 \big(r_0(1-e^{-t}) + r_1 e^{-t}\big)^{-1}$.

Therefore, the velocity is simply given by, for $x(t) = X(q,t)$,

$$v(x(t),t) = \dot x(t) = \dot r(t) p = \Big(1 - \tfrac{r_1}{r(t)}\Big) r(t) p = \Big(1 - \tfrac{r_1}{r(t)}\Big) x(t).$$

The numerical experiments were performed in Matlab, using a quadratic approximation of the initial surface Γ_0 and using the quadratic ESFEM implementation from [Kov17], and linearly implicit BDF methods of various orders.

Let $(\mathcal{T}_k)_{k=1,2,\dots,m}$ and $(\tau_k)_{k=1,2,\dots,n}$ be a series of quadratic initial meshes and time steps, respectively, such that $2\tau_k = \tau_{k-1}$, with $\tau_1 = 0.1$, where the meshes are generated independently.

We computed the fully discrete numerical solution of the above problem, with parameters $\alpha = 1$ and $\beta = 1$, for each mesh and stepsize using the second order BDF method and second order ESFEM. In Figures 10.1 and 10.2 we report on the following errors of the quadratic ESFEM / BDF2 full discretization

$$\|(x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\|_{L^2(\Gamma(t_n))^3} \quad \text{and} \quad \|\nabla_\Gamma\big((x_h^n)^L - \mathrm{id}_{\Gamma(t_n)}\big)\|_{L^2(\Gamma(t_n))^3}$$

at time $T = N\tau = 5$. The logarithmic plots show the errors against time step size τ (in Figure 10.1), and against the mesh width h (in Figure 10.2).

The different lines correspond to different mesh refinements and to different time step sizes in Figure 10.1 and Figure 10.2, respectively. In both figures we can observe two regions: In Figure 10.1, a region where the temporal discretization error dominates, matching to the $O(\tau^2)$ order of convergence of our theoretical result, and a region, with small stepsizes, where the space discretization error dominates (the error curves are flattening out). In Figure 10.2, the same description applies, but with reversed roles. First the space discretization error dominates, while for finer meshes the temporal error dominates. The convergence in time, see Figure 10.1, can be nicely observed in agreement with the theoretical results (note the reference line), whereas we observe better L^2 norm convergence rates ($O(h^3)$) for the space discretization, see Figure 10.2, than shown in Theorem 3.1 for the H^1 norm (only $O(h^2)$). This phenomenon is due to the fact that in the defect estimates we use the interpolation instead of a Ritz projection (which is hard to define in this setting), therefore have a defect estimate of order two. However, the classical optimal L^2 norm convergence rates of $O(h^3)$ are nevertheless observed.

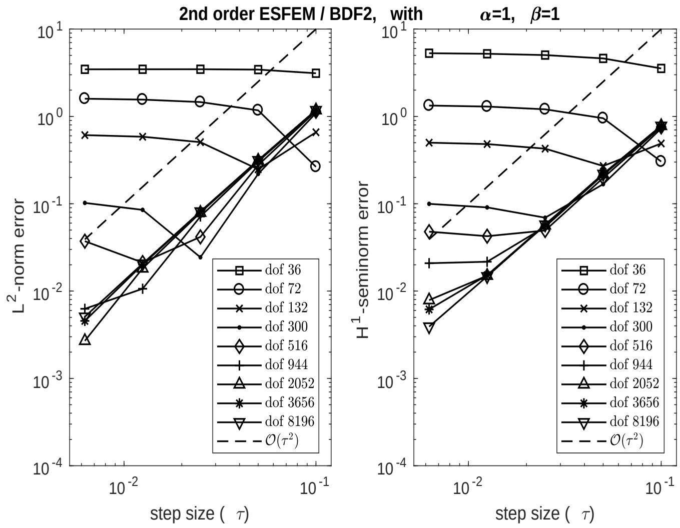

Fig. 10.1: Temporal convergence of the BDF2 / quadratic ESFEM discretization for the surface-evolution equation (10.1)

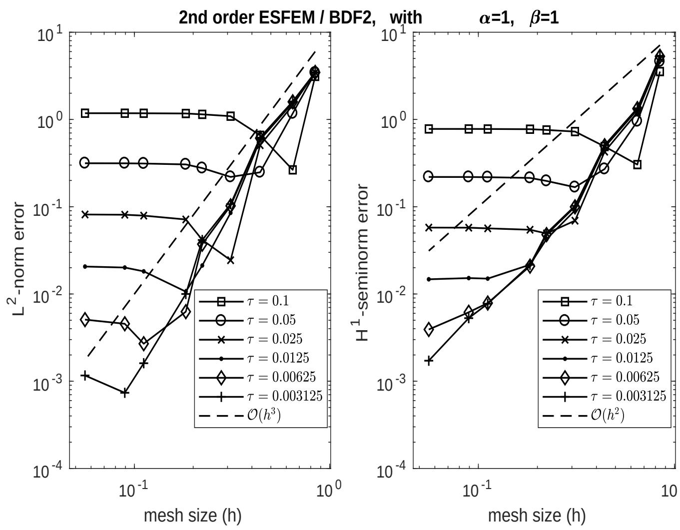

Fig. 10.2: Spatial convergence of the BDF2 / quadratic ESFEM discretization for the surface-evolution equation (10.1)

Figure 10.3 shows the same errors for the BDF method of order 4. It is clearly seen that in this problem the BDF4 method gives much better accuracy than BDF2, at nearly the same computational cost.

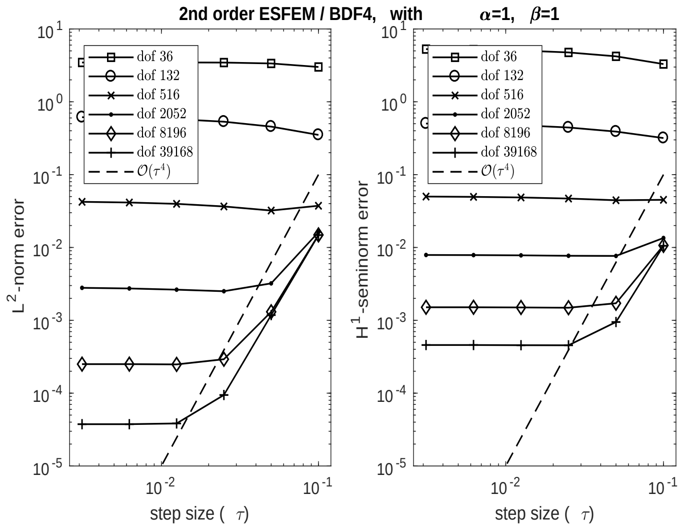

Fig. 10.3: Temporal convergence of the BDF4 / quadratic ESFEM discretization for the surface-evolution equation (10.1)

Numerical experiments for a semi-linear parabolic PDE system coupled to a velocity law on a surface with less symmetry, illustrating the coupled problem of Theorem 9.1, are discussed in detail in our previous work [KLLP17], where linearly implicit BDF methods have also been used.

10.2 Mean curvature flow

We also performed some numerical experiments, using mean curvature flow (MCF), to illustrate the effect of the elliptic regularisation. We again consider the problem (10.1), however without a forcing term, i.e. the following form of mean curvature flow:

$$\begin{aligned} v(x,t) - \alpha \varDelta_{\Gamma(t)} v(x,t) &= -\beta H_{\Gamma(t)}(x)\, \nu_{\Gamma(t)}(x), \\ \partial_t X(q,t) &= v(X(q,t),t). \end{aligned} \tag{10.2}$$

The initial surface is a rounded cube, the parameter β is fixed to one. Figure 10.4 shows the results of different numerical experiments (using quadratic finite elements and BDF method of order 4) at times $t = 0, 0.2, 0.4, 0.5$ from top to bottom, while the parameter α is set to $0.1, 0.01, 0.001$ and 0, from left to right, respectively. We note that our convergence results apply only to the case of a fixed positive α, but the numerical experiments show good behaviour also for $\alpha \to 0$.

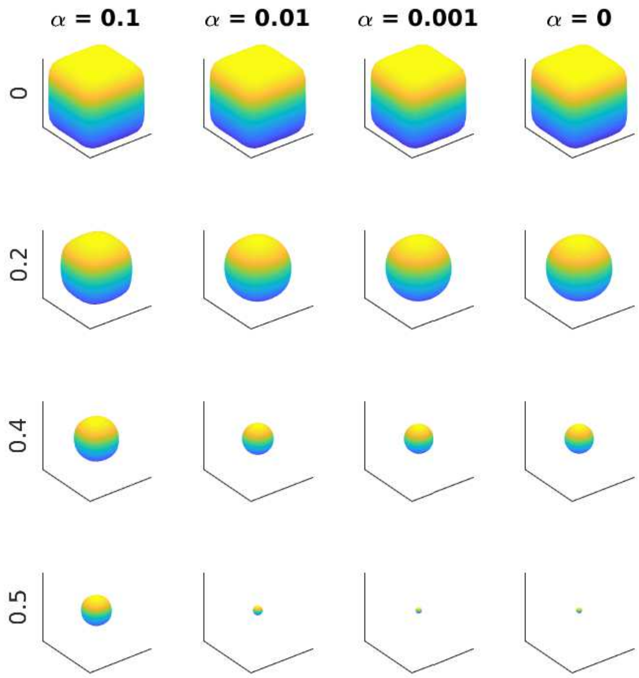

Fig. 10.4: MCF with different values of α at different times

Acknowledgement

This work is supported by Deutsche Forschungsgemeinschaft, SFB 1173.

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