Controlling multiphase flow

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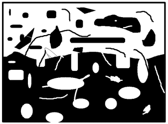
Introduction and Motivation

Analysis

Numerics

- $\rho_0 = \rho_1 \chi_{\Omega_1} + \rho_2 \chi_{\Omega_2}$ mixture of two immiscible viscous incompressible fluids in a bounded domain in $\mathbb{R}^2$.
- Multi-phase flow evolution by Navier–Stokes Eq. (cf. [Lions, 1996])

$$(NSE) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{y} - \mu \Delta \boldsymbol{y} + \nabla p = \rho \boldsymbol{u} + \rho \boldsymbol{g}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y} \cdot \nabla]\rho = 0, & \rho(0) = \rho_0, \\ \operatorname{div} \boldsymbol{y} = 0 & + B.C. \end{cases}$$



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$$\text{Minimize } J(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t$$

subject to

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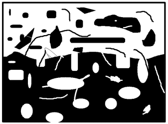

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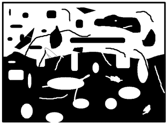

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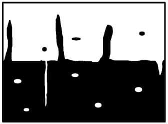

$\rho(t)$ **BAD**

Add additional term to functional to minimize the interface area!
$\Rightarrow$ Geometric functional!

$$\text{Minimize } J(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho) \, \mathrm{d}t$$

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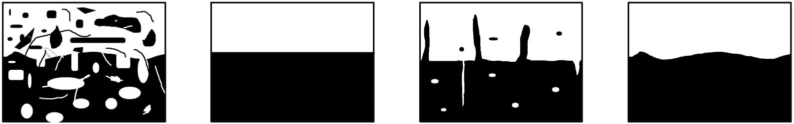
$$(NSE) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{y} - \mu \Delta \boldsymbol{y} + \nabla p = \rho \boldsymbol{u} + \rho \boldsymbol{g}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y} \cdot \nabla]\rho = 0, & \rho(0) = \rho_0, \\ \operatorname{div} \boldsymbol{y} = 0 & + B.C. \end{cases}$$



ρ_0 | σ | $\rho(t)$ **BAD** | $\rho(t)$ **GOOD**

Applications

- Air-water dynamics (air bubbles, water drops)
- Aluminium production (Al_2 and Al_2O_3)

Goals

- Existence of optimum
- Optimality conditions
- Numerical scheme with low order Finite Elements
- Convergence of the numerical scheme

Known result: Optimization (analysis, no numerics) of L^2-functional (no geometric term) subject to Stokes equation, cf. [Kunisch and Lu, 2011].

Analytical problems and strategy

Minimize

$$J(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho)$$

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- **Problem:** Not clear if blue term is w.l.s.c., and not clear if corresponding Lagrange multiplier to mass equation exists and is a function.
- **Solution:** Add artificial diffusion to equation and approximate Hausdorff measure ("Mortola-Modica", cf. [Braides, 1998])

$\Rightarrow$ **Phase-field formulation**

Analytical problems and strategy

Minimize

$$J_\delta(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 + \frac{\beta}{2} \left(\delta \int_{\Omega_T} |\nabla \rho|^2 + \frac{1}{4\delta} \int_{\Omega_T} W(\rho) \right)$$

subject to

$$(NSE_\varepsilon) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{y} - \mu \Delta \boldsymbol{y} + \nabla p = \rho \boldsymbol{u} + \rho \boldsymbol{g}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y} \cdot \nabla]\rho - \varepsilon \Delta \rho_t = 0, & \rho(0) = \rho_0, \\ \operatorname{div} \boldsymbol{y} = 0 & + B.C. \end{cases}$$

($W \geq 0$ double Well functional with $W(\rho) = 0$ iff $\rho = \rho_1$ or $\rho = \rho_2$)

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Theorem (Existence)

For $\delta, \varepsilon > 0$, there exists at least one minimum and the corresponding Lagrange multipliers belong to some $L^p(\Omega_T)$ for $p > 1$.

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Proof.

Lot of technical calculations. Key are a priori estimates and regularity:

- Use parabolic theory for regularity of ρ.
- Use [Lions, 1996] for regularity of $\boldsymbol{y}$.

Then direct application of Lagrange multiplier theorem. □

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- For $\delta \searrow 0$ (and no side constraints), it is known ([Braides, 1998]) that $J_\delta(\rho, \boldsymbol{u}) \xrightarrow{\Gamma} J(\rho, \boldsymbol{u})$ (Γ-convergence), i.e.,
  1. For every sequence $(\rho_\delta, \boldsymbol{u}_\delta) \to (\rho, \boldsymbol{u})$ (for $\delta \to 0$) we have
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Open question: How to combine both results? How to choose $\delta = \delta(\varepsilon)$?

Optimality Conditions

$$
\begin{aligned}
0 &= -\rho \boldsymbol{z}_t - \rho_t \boldsymbol{z} - \mu \Delta \boldsymbol{z} + \rho(\nabla \boldsymbol{y}) \cdot \boldsymbol{z} + \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{z} + (\nabla \rho \cdot \boldsymbol{y})\boldsymbol{z} + \eta \nabla \rho, \\
0 &= \lambda(\rho - \tilde{\rho}) - \beta\delta\Delta\rho + \frac{\beta}{8\delta} W'(\rho) + \boldsymbol{y}_t \cdot \boldsymbol{z} \\
&\quad - \big([\boldsymbol{y} \cdot \nabla]\boldsymbol{y}\big) \cdot \boldsymbol{z} - \boldsymbol{u} \cdot \boldsymbol{z} - \eta_t - [\boldsymbol{y} \cdot \nabla]\eta - \varepsilon \Delta \eta_t, \\
0 &= \alpha \boldsymbol{u} - \rho \boldsymbol{z}, \\
0 &= \rho \boldsymbol{y}_t - \rho[\boldsymbol{y} \cdot \nabla]\boldsymbol{y} - \mu(\rho)\Delta \boldsymbol{y} - \rho \boldsymbol{u}, \\
0 &= \rho_t + [\boldsymbol{y} \cdot \nabla]\rho - \varepsilon \Delta \rho.
\end{aligned}
$$

Strategy for discretization

- Use first discritize, then optimize ansatz.
- Convergent and unconditionally stable scheme known for density depedent Navier–Stokes, cf. [Bañas and Prohl, 2010].
- Due to strong coupling of primal and dual variables in the adjoint equation, we need bounds on higher bounds of the primal variables.
- Here: Fix $\delta, \varepsilon > 0$. Still open: Interplay between δ, ε and numerical parameters (time step size *k* and grid size *h*)?

Numerical framework

- Density space R_h: standard piecewise linear FE space.
- Velocity/pressure space $\boldsymbol{V}_h/M_h$: standard inf-sup-stable FE spaces (e.g., Taylor–Hood, MINI).
- Time discretization: Implicit Euler.

Find $(\boldsymbol{Y}^n, P^n, R^n) \in \boldsymbol{V}_h \times M_h \times R_h$ such that for all $(\boldsymbol{Z}, \Pi, E) \in \boldsymbol{V}_h \times M_h \times R_h$:

$$(d_t R^n, E) + \varepsilon(d_t \nabla R^n, \nabla E) + ([\boldsymbol{Y}^n \cdot \nabla] R^n, E) + \frac{1}{2}(R^n \operatorname{div} \boldsymbol{Y}^n, E) = 0,$$

$$\frac{1}{2}(R^{n-1} d_t \boldsymbol{Y}^n, \boldsymbol{Z}) + \frac{1}{2}(d_t(R^n \boldsymbol{Y}^n), \boldsymbol{Z}) + \frac{1}{2}([R^{n-1}\boldsymbol{Y}^{n-1} \cdot \nabla]\boldsymbol{Y}^n, \boldsymbol{Z})$$
$$- \frac{1}{2}([R^{n-1}\boldsymbol{Y}^{n-1} \cdot \nabla]\boldsymbol{Z}, \boldsymbol{Y}^n) + \mu(\nabla \boldsymbol{Y}^n, \nabla \boldsymbol{Z}) + (\nabla P^n, \boldsymbol{Z}) = (R^{n-1}\boldsymbol{U}^n, \boldsymbol{Z}),$$

$$(\operatorname{div} \boldsymbol{Y}^n, \Pi) = 0.$$

Comments

Scheme is modification of scheme in [Bañas and Prohl, 2010].

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Comments

First line becomes skew symmetric.

Find $(\boldsymbol{Y}^n, P^n, R^n) \in \boldsymbol{V}_h \times M_h \times R_h$ such that for all $(\boldsymbol{Z}, \Pi, E) \in \boldsymbol{V}_h \times M_h \times R_h$:

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$$(\operatorname{div} \boldsymbol{Y}^n, \Pi) = 0.$$

Comments

Second line becomes skew symmetric as (cf. [Liu and Walkington, 2007])

$$\rho(\boldsymbol{y}_t + [\boldsymbol{y} \cdot \nabla] \boldsymbol{y}) = \frac{1}{2}\Big(\rho \boldsymbol{y}_t + \rho[\boldsymbol{y} \cdot \nabla] \boldsymbol{y} + (\rho \boldsymbol{y})_t + \operatorname{div}(\rho \boldsymbol{y} \otimes \boldsymbol{y})\Big).$$

Find $(\boldsymbol{Y}^n, P^n, R^n) \in \boldsymbol{V}_h \times M_h \times R_h$ such that for all $(\boldsymbol{Z}, \Pi, E) \in \boldsymbol{V}_h \times M_h \times R_h$:

$$
\begin{aligned}
&(d_t R^n, E) + \varepsilon(d_t \nabla R^n, \nabla E) + ([\boldsymbol{Y}^n \cdot \nabla] R^n, E) + \frac{1}{2}(R^n \operatorname{div} \boldsymbol{Y}^n, E) = 0,\\
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&\qquad (\operatorname{div} \boldsymbol{Y}^n, \Pi) = 0.
\end{aligned}
$$

Comments

Assume: Triangulation is strongly acute (iff angles of interior edges are bdd away from 90°)

$\Rightarrow$ M-matrix property for first line

$\Rightarrow$ lower bound for R^n

Find $(\boldsymbol{Y}^n, P^n, R^n) \in \boldsymbol{V}_h \times M_h \times R_h$ such that for all $(\boldsymbol{Z}, \Pi, E) \in \boldsymbol{V}_h \times M_h \times R_h$:

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- \frac{1}{2}([R^{n-1} \boldsymbol{Y}^{n-1} \cdot \nabla] \boldsymbol{Z}, \boldsymbol{Y}^n) + \mu(\nabla \boldsymbol{Y}^n, \nabla \boldsymbol{Z}) + (\nabla P^n, \boldsymbol{Z}) &= (R^{n-1} \boldsymbol{U}^n, \boldsymbol{Z}),\\
(\operatorname{div} \boldsymbol{Y}^n, \Pi) &= 0.
\end{aligned}
$$

Comments

Assume: $R_h \cap L^2_0 \subseteq M_h$

$\Rightarrow$ Upper bound for R^n

Lemma (Bounds for primal variables)

There exists a solution $\{(R^n, \boldsymbol{Y}^n, P^n)\}$ of the discrete equation with the property

$$0 < \rho_1 \leq R^n \leq \rho_2 < \infty$$

and for the time interpolant of the solution $(\mathcal{R}, \boldsymbol{\mathcal{Y}}, \mathcal{P})$ there is a constant $C = C(\varepsilon, \delta, T)$ independent of k, h with

$$\sup_{t\in[0,T]} \left[\|\nabla\boldsymbol{\mathcal{Y}}(t)\|^2 + \|\Delta_h\mathcal{R}(t)\|^2\right] + \int_0^T \|\Delta_h\boldsymbol{\mathcal{Y}}(t)\|^2 + \|d_t\boldsymbol{\mathcal{Y}}(t)\|^2 + \|d_t\nabla\mathcal{R}(t)\|^2 \,\mathrm{d}t \leq C.$$

Discrete Optimality Conditions

$$
\begin{aligned}
0 = {} & \frac{1}{2}E^n\nabla R^n - \frac{1}{2}R^n\nabla E^n - \frac{1}{2}d_t R^n \boldsymbol{Z}^n - R^n d_t \boldsymbol{Z}^{n+1} + \frac{1}{2}R^n\nabla \boldsymbol{Y}^{n+1}\cdot \boldsymbol{Z}^{n+1} \\
& + \frac{1}{2}R^n\nabla \boldsymbol{Z}^{n+1}\cdot \boldsymbol{Y}^{n+1} - \frac{1}{2}(\nabla R^{n-1}\cdot \boldsymbol{Y}^{n-1})\boldsymbol{Z}^n - \frac{1}{2}R^{n-1}\operatorname{div}\boldsymbol{Y}^{n-1}\boldsymbol{Z}^n \\
& - \frac{1}{2}[R^{n-1}\boldsymbol{Y}^{n-1}\cdot\nabla]\boldsymbol{Z}^n - \mu\Delta_h \boldsymbol{Z}^n - \nabla Q^n, \\
0 = {} & -\operatorname{div}\boldsymbol{Z}^n, \\
0 = {} & -d_t E^{n+1} - [\boldsymbol{Y}^n\cdot\nabla]E^n - \frac{1}{2}(\operatorname{div}\boldsymbol{Y}^n)E^n + \varepsilon d_t\Delta_h E^{n+1} + \frac{1}{2}d_t\boldsymbol{Y}^{n+1}\cdot\boldsymbol{Z}^{n+1} \\
& - \frac{1}{2}\boldsymbol{Y}^n\cdot d_t\boldsymbol{Z}^{n+1} + \frac{1}{2}[\boldsymbol{Y}^n\cdot\nabla]\boldsymbol{Y}^{n+1}\cdot\boldsymbol{Z}^{n+1} - \boldsymbol{U}^{n+1}\cdot\boldsymbol{Z}^{n+1} \\
& - \frac{1}{2}[\boldsymbol{Y}^n\cdot\nabla]\boldsymbol{Z}^{n+1}\cdot\boldsymbol{Y}^{n+1} + \lambda(R^n - \tilde{\rho}(t_n)) - \beta\delta\Delta_h R^n + \frac{\beta}{8\delta}W'(R^n), \\
0 = {} & \alpha\boldsymbol{U}^n - R^{n-1}\boldsymbol{Z}^n.
\end{aligned}
$$

Lemma (Bounds for dual variables)

By the Lagrange multiplier theorem, there exist Lagrange multipliers $(\mathcal{Z}, \mathcal{Q}, \mathcal{E})$ and there exists a constant $C = C(\varepsilon, \delta, T)$ independent of k, h with

$$\sup_{t\in[0,T]} \left[\|\nabla\mathcal{E}\|^2 + \|\mathcal{Z}\|^2\right] + \int_0^T \|\nabla\mathcal{Z}\|^2 + \|d_t\mathcal{Z}\|^2 \,\mathrm{d}t \leq C.$$

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Proof.

Simultaneously test discrete optimality system with $\boldsymbol{Z}^n$, R^n and $d_t\boldsymbol{Z}^{n+1}$. □

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Theorem (Convergence)

There exist $\boldsymbol{y}, p, \rho; \boldsymbol{z}, q, \eta; \boldsymbol{u} : \Omega_T \to \mathbb{R}^{(2)}$, such that the solutions of the fully discrete optimality system converge to them in some norms (up to subsequences). The limit functions solve the original fully continuous optimality system.

Done

- Existence for optimization of geometric functional (with $\delta > 0$) s.t. NSE_ε ($\varepsilon > 0$).
- Optimality conditions for $\delta, \varepsilon > 0$.
- Discretization of optimality conditions.
- Convergence analysis with unconditionally stable scheme.

Outlook

- What happens for $\varepsilon, \delta \to 0$?
- Implementation.
- Compare model with corresponding graph formulation.

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THANK YOU FOR YOUR ATTENTION!

References I

Bañas, Ľ. and Prohl, A. (2010).
Convergent finite element discretization of the multi-fluid nonstationary incompressible magnetohydrodynamics equations.
Math. Comp., 79(272):1957–1999.

Braides, A. (1998).
Approximation of free discontinuity problems.
Number 1694 in Lecture notes in mathematics. Springer, Berlin.

Kunisch, K. and Lu, X. (2011).
Optimal control for multi-phase fluid stokes problems.
Nonlinear Anal., 74(2):585–599.

References II

Lions, P.-L. (1996).
Mathematical topics in fluid mechanics, volume 1: Incompressible models of *Oxford lecture series in mathematics and its applications*.
Clarendon Press.

Liu, C. and Walkington, N. J. (2007).
Convergence of numerical approximations of the incompressible Navier-Stokes equations with variable density and viscosity.
SIAM J. Numer. Anal., 45(3):1287–1304.