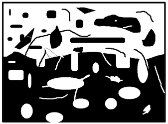
Optimal control of a geometric functional under the density-dependent Navier-Stokes equation

Markus Klein (U Tübingen)

Caputh, 2011-09-15

Contents

- ρ_0 mixture of two fluids in a domain in $\mathbb{R}^2$.



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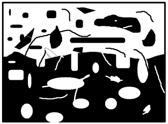
- ρ_0 mixture of two fluids in a domain in $\mathbb{R}^2$.
- Flow evolution by Navier–Stokes Eq.

$$(NSE)\begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y}\cdot\nabla]\boldsymbol{y} - \mu\Delta\boldsymbol{y} = \rho\boldsymbol{u}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y}\cdot\nabla]\rho = 0, & \rho(0) = \rho_0 \end{cases}$$



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- Find external force s.t. $\int_{\Omega_T} |\rho(t) - \sigma|^2$ is small, where σ is given.

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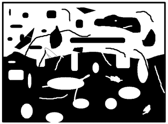

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Minimize

$$J(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 \,\mathrm{d}x \,\mathrm{d}t + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}x \,\mathrm{d}t$$

subject to

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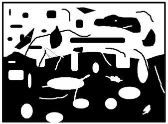

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ρ_0 σ $\rho(t)$ **GOOD**

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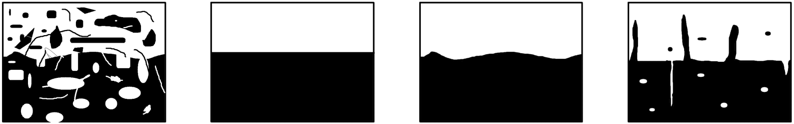

ρ_0 $\qquad$ σ $\qquad$ $\rho(t)$ GOOD $\qquad$ $\rho(t)$ BAD

Add additional term to functional to minimize the interface area!

Minimize

$$J(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho)\,\mathrm{d}t$$

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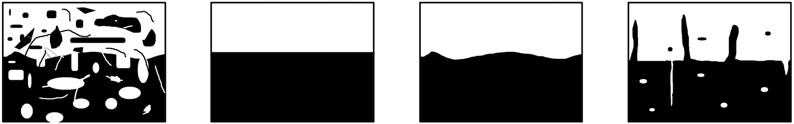

ρ_0 | σ | $\rho(t)$ **GOOD** | $\rho(t)$ **BAD**

Minimize

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Questions:

- Existence of a minimum?

Minimize

$$J(\rho, \boldsymbol{u}) = \int_{\Omega_T} |\rho(t) - \sigma|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2} \int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2} \int_0^T \mathcal{H}^1(S_\rho)\,\mathrm{d}t$$

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Questions:

- Existence of a minimum? yes, w/o red term. Otherwise not clear!

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- Optimality conditions?

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Questions:

- Existence of a minimum? yes, w/o red term. Otherwise not clear!
- Optimality conditions? Not clear. Even if, Lagrange multiplier would very irregular.
- Numerical Approximation?

Minimize

$$J_\delta(\rho,\boldsymbol{u}) = \int_{\Omega_T} |\rho(t)-\sigma|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\alpha}{2}\int_{\Omega_T} |\boldsymbol{u}|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2}\delta \int_{\Omega_T} |\nabla\rho|^2 + \frac{\beta}{8\delta}\int_{\Omega_T} W(\rho)$$

($W \geq 0$ double well potential, $W = 0$ iff ρ attends densities of initial mixture)

$$\text{subject to } (NSE_\varepsilon) \begin{cases} \rho \boldsymbol{y}_t + \rho[\boldsymbol{y}\cdot\nabla]\boldsymbol{y} - \mu\Delta\boldsymbol{y} = \rho\boldsymbol{u}, & \boldsymbol{y}(0) = \boldsymbol{y}_0, \\ \rho_t + [\boldsymbol{y}\cdot\nabla]\rho - \varepsilon\Delta\rho = 0, & \rho(0) = \rho_0 \end{cases}$$

with $\delta = \delta(\varepsilon)$? [$\rho$ is phase-field approximation (Mortola-Modica)]

Minimize

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- Optimality conditions: Yes, and the Lagrange multiplier are in some $L^p(\Omega_T)$!

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- Numerical approximation: In work.

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Thank you for your attention!