Convergent FE scheme for the two-fluid MHD equation

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AG Numerik (Tuebingen), 2010-08-03

Outline

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two fluid MHD equation, strong, [Baňas and Prohl, 2010]

Find $\boldsymbol{u}, p, \boldsymbol{b}, \rho \in ?$ s.t.

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(\rho \boldsymbol{u})_t + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div}(\eta(\rho)\boldsymbol{D}(\boldsymbol{u})) &= -\nabla p + \rho \boldsymbol{g} + \frac{1}{\bar{\mu}} \operatorname{curl} \boldsymbol{b} \times \boldsymbol{b}, \\
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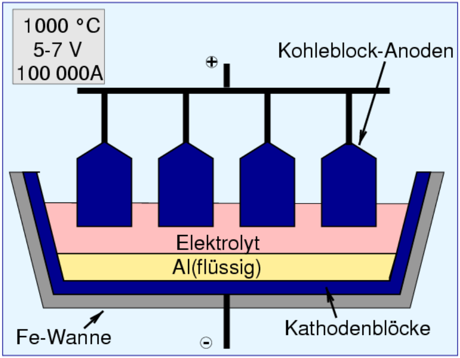
Application

Production of Al from Al_2O_3 by Electrolysis (cf. [Gerbeau et al., 2006, chap 6]):



(chemie.uni-freiburg.de)

Existence of weak solutions, [Gerbeau et al., 2006]

Under certain assumptions on the initial data, there exists a weak solution
$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^2 \cap \{\operatorname{div} \boldsymbol{u} = 0 \text{ weakly}\}) \cap L^2(0, T; \boldsymbol{W}_0^{1,2} \cap \{\operatorname{div} \boldsymbol{u} = 0 \text{ a.e.}\})$,
$\boldsymbol{b} \in L^\infty(0, T; \boldsymbol{L}^2 \cap \{\operatorname{div} \boldsymbol{b} = 0 \text{ weakly}\}) \cap L^2(0, T; \boldsymbol{X})$,
$\rho \in L^\infty((0, T) \times \Omega) \cap \mathcal{C}([0, T], L^p)$ which holds the property

$$|\{x \in \Omega : \ \alpha \leq \rho(x, t) \leq \beta| \text{ is constant in time for all } 0 \leq \alpha \leq \beta < \infty.$$

$\boldsymbol{X} := \boldsymbol{H}(\text{curl}) \cap \boldsymbol{H}_0(\text{div}) \cap \{\operatorname{div} \boldsymbol{b} = 0 \text{ a.e.}\}$

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Incredients for the proof

- Typical technical arguments related to one-fluid MHD.
- Use of DiPerna–Lions compactness, cf. [DiPerna and Lions, 1989] (for passing to a limit of ρ).

Outline

General setup

- $\mathcal{T}_h$ quasi-uniform trangulation of polyhedral domain $\Omega \subseteq \mathbb{R}^d$ ($d = 2, 3$).
- $V_h := \{\xi \in \mathcal{C}(\bar{\Omega}) : \ \xi \in P_1(T)\ \forall T \in \mathcal{T}_h\}$ FE space w.r.t. ρ.
- $\boldsymbol{V}_h \subseteq \boldsymbol{W}_0^{1,2}$ FE space w.r.t. $\boldsymbol{u}$ and $L_h \subseteq L_0^2$ FE space w.r.t. P, s.t. $(\boldsymbol{V}_h, L_h)$ holds inf-sup cond.
- $\boldsymbol{C}_h := \{\psi \in \boldsymbol{H}(\text{curl}) : \ \psi \in \mathcal{N}_j \text{ for some } j \geq 1\}$ (Nedelec) FE space w.r.t $\boldsymbol{b}$.
- $S_h \subseteq W^{1,2} \cap L_0^2$ s.t. inf-sup cond. holds for $(\boldsymbol{C}_h, S_h)$.

Idea of discretization

Reformulate:

$$(\rho \boldsymbol{u})_t + \text{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \frac{1}{2}\Big(\rho \boldsymbol{u}_t + (\rho \boldsymbol{u}.\nabla)\boldsymbol{u} + (\rho \boldsymbol{u})_t + \text{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u})\Big)$$

(true, since $\rho_t + \text{div}(\rho \boldsymbol{u}) = 0$).

Algorithm ((Scheme A), [Baňas and Prohl, 2010])

Find $(\boldsymbol{U}^n, P^n, \boldsymbol{B}^n, R^n, \rho^n)$ *s.t. for all* $(\chi, \boldsymbol{W}, \psi)$*:*

$$0 = (d_t\rho^n, \chi)_h + (\boldsymbol{U}^n.\nabla\rho^n, \chi) + \frac{1}{2}(\mathrm{div}(\boldsymbol{U}^n)\rho^n, \chi)$$

$$(\rho^{n-1}\boldsymbol{g}^n, \boldsymbol{W}) = \frac{1}{2}\left\{(\rho_+^{n-1} d_t\boldsymbol{U}^n, \boldsymbol{W})_* + \left(d_t(\rho_+^n\boldsymbol{U}^n), \boldsymbol{W}\right)_* + \left((\rho^{n-1}\boldsymbol{U}^{n-1}.\nabla)\boldsymbol{U}^n, \boldsymbol{W}\right) - \left((\rho^{n-1}\boldsymbol{U}^{n-1}.\nabla)\boldsymbol{W}, \boldsymbol{U}^n\right)\right\}$$
$$+ (\eta^{n-1}\boldsymbol{D}(\boldsymbol{U}^n), \boldsymbol{D}(\boldsymbol{W})) - (P^n, \mathrm{div}\,\boldsymbol{W}) + \frac{1}{\bar{\mu}}(\boldsymbol{B}^{n-1} \times \mathrm{curl}\,\boldsymbol{B}^n, \boldsymbol{W})$$

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for some non negative constants α
($\eta^{n-1} := \eta(\rho^{n-1})$ *and* $\xi^{n-1} := \xi(\rho^{n-1})$*)*

- α term $\Rightarrow$ M-Matrix property of the Scheme

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- β_1 term $\Rightarrow$ $\boldsymbol{L}^2$-strong convergence of div $\boldsymbol{U}^n$

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$$+ \beta_2 h^{-\beta_1}(\mathrm{div}\,\boldsymbol{U}^n, \mathrm{div}\,\boldsymbol{W}) + \beta_2 h^{\beta_2}(\nabla d_t\boldsymbol{U}^n, \nabla\boldsymbol{W})$$

$$0 = (d_t\boldsymbol{B}^n, \psi) + \frac{1}{\bar{\mu}}\left(\frac{1}{\xi^{n-1}}\mathrm{curl}\,\boldsymbol{B}^n, \mathrm{curl}\,\psi\right) - (\boldsymbol{U}^n\times\boldsymbol{B}^{n-1}, \mathrm{curl}\,\psi) - (\nabla R^n, \psi)$$

for some non negative constants α *,* β_1 *,* β_2
($\eta^{n-1} := \eta(\rho^{n-1})$ *and* $\xi^{n-1} := \xi(\rho^{n-1})$*)*

- α term $\Rightarrow$ M-Matrix property of the Scheme
- β_1 term $\Rightarrow$ $\boldsymbol{L}^2$-strong convergence of div $\boldsymbol{U}^n$
- β_2 term $\Rightarrow$ $\boldsymbol{W}^{1,2}$-Boundness of $\boldsymbol{U}^n$

Algorithm ((Scheme A), [Baňas and Prohl, 2010])

Find $(\boldsymbol{U}^n, P^n, \boldsymbol{B}^n, R^n, \rho^n)$ *s.t. for all* $(\chi, \boldsymbol{W}, \psi)$*:*

$$
\begin{aligned}
0 &= (d_t\rho^n, \chi)_h + (\boldsymbol{U}^n.\nabla\rho^n, \chi) + \frac{1}{2}(\mathrm{div}(\boldsymbol{U}^n)\rho^n, \chi) + \alpha h^\alpha(\nabla\rho^n, \nabla\chi) \\
(\rho^{n-1}\boldsymbol{g}^n, \boldsymbol{W}) &= \frac{1}{2}\left\{(\rho_+^{n-1} d_t\boldsymbol{U}^n, \boldsymbol{W})_* + \left(d_t(\rho_+^n\boldsymbol{U}^n), \boldsymbol{W}\right)_* + \left((\rho^{n-1}\boldsymbol{U}^{n-1}.\nabla)\boldsymbol{U}^n, \boldsymbol{W}\right) - \left((\rho^{n-1}\boldsymbol{U}^{n-1}.\nabla)\boldsymbol{W}, \boldsymbol{U}^n\right)\right\} \\
&\quad + (\eta^{n-1}\boldsymbol{D}(\boldsymbol{U}^n), \boldsymbol{D}(\boldsymbol{W})) - (P^n, \mathrm{div}\,\boldsymbol{W}) + \frac{1}{\bar{\mu}}(\boldsymbol{B}^{n-1}\times\mathrm{curl}\,\boldsymbol{B}^n, \boldsymbol{W}) \\
&\quad + \beta_2 h^{-\beta_1}(\mathrm{div}\,\boldsymbol{U}^n, \mathrm{div}\,\boldsymbol{W}) + \beta_2 h^{\beta_2}(\nabla d_t\boldsymbol{U}^n, \nabla\boldsymbol{W}) + \beta_3 h^{\beta_3}(\Delta_h\boldsymbol{U}^n, \Delta_h\boldsymbol{W}) \\
0 &= (d_t\boldsymbol{B}^n, \psi) + \frac{1}{\bar{\mu}}\left(\frac{1}{\xi^{n-1}}\mathrm{curl}\,\boldsymbol{B}^n, \mathrm{curl}\,\psi\right) - (\boldsymbol{U}^n\times\boldsymbol{B}^{n-1}, \mathrm{curl}\,\psi) - (\nabla R^n, \psi)
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$$

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- β_3 term $\Rightarrow$ strong $\boldsymbol{L}^2$-convergence of $\boldsymbol{U}^n$ (for $d = 3$; for $d = 2$ this is free due Sobolev imbeddings)

Lemma (Existence, energy law, maximum principle, [Baňas and Prohl, 2010])

Let $\alpha, \beta_1, \beta_2, \beta_3 \geq 0$, $\sqrt{\beta_2} h^{\beta_2} 2 \|\nabla \boldsymbol{U}^0\| \leq C$. *Then there exists a solution* $(\boldsymbol{U}^n, \boldsymbol{B}^n, \rho^n, P^n, R^n)$ *of the numerical scheme, which holds the discrete energy law*

$$\int_\Omega \rho^{n-1} \boldsymbol{g}^n \boldsymbol{U}^n = \frac{1}{2}\partial_t \Big(\|\sqrt{\rho_+^n}\boldsymbol{U}^n\|_*^2 + \beta_2 h^{\beta_2} \|\nabla \boldsymbol{U}^n\|^2 + \frac{1}{\bar{\mu}}\|\boldsymbol{B}^n\|^2 \Big) + \|\sqrt{\eta^{n-1}}\boldsymbol{D}(\boldsymbol{U}^n)\|^2 + \beta_1 h^{-\beta_1} \|\operatorname{div} \boldsymbol{U}^n\|^2$$
$$+ \beta_3 h^{\beta_3} \|\Delta_h \boldsymbol{U}^n\|^2 + \frac{1}{\bar{\mu}^2}\left\| \frac{1}{\sqrt{\xi^{n-1}}} \operatorname{curl} \boldsymbol{B}^n \right\|^2 + \frac{k}{2}\Big(\|\sqrt{\rho_+^{n-1}} d_t \boldsymbol{U}^n\|_*^2 + \beta_2 h^{\beta_2}\|\nabla d_t \boldsymbol{U}^n\|^2 + \|d_t \boldsymbol{B}^n\|^2 \Big),$$
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Let $V_h \cap L_0^2 \subseteq L_h$, $\mathcal{T}_h$ *be a strongly acute triangulation,* $\alpha, \beta_> 0$ *and* $0 < \alpha + \frac{\beta_2}{2} < \frac{6-d}{6}$. *Then* $0 < \bar{\rho}_1 \leq \rho^n \leq \bar{\varrho}_2 < \infty$ *(discrete maximum principle).*

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Sketch of the proof

- *Suppose*
$$\max_{0 \leq \ell \leq n-1} \Big(\|\rho^\ell\|_h^2 + \|\sqrt{\rho_+^\ell} \boldsymbol{U}^\ell\|_*^2 + \beta_2 h^{\beta_2} \|\nabla \boldsymbol{U}^\ell\|^2 + \frac{1}{\bar{\mu}} \|\boldsymbol{B}^\ell\|^2 \Big) \leq C,$$
(fullfilled for $n = 1$ *by asumption).*
- *Define* $\mathcal{F}^n([\rho, \boldsymbol{U}, \boldsymbol{B}], [\chi, \boldsymbol{W}, \psi]) :=$ *Scheme A – right hand side (terms with* R^n, P^n *vanish).*
- *Show:* $\mathcal{F}^n([\rho, \boldsymbol{U}, \boldsymbol{B}], [\rho, \boldsymbol{U}, \boldsymbol{B}]] \geq 0 \Rightarrow \exists\ \rho^n, \boldsymbol{U}^n, \boldsymbol{B}^n$ *with Brouwer. Boundness of* ρ^ℓ*, etc. is fullfilled for* $n+1$ *by Brouwer.*

Lemma (Existence, energy law, maximum principle, [Baňas and Prohl, 2010])

Let $\alpha, \beta_1, \beta_2, \beta_3 \geq 0$, $\sqrt{\beta_2} h^{\beta_2} 2 \|\nabla \boldsymbol{U}^0\| \leq C$. *Then there exists a solution* $(\boldsymbol{U}^n, \boldsymbol{B}^n, \rho^n, P^n, R^n)$ *of the numerical scheme, which holds the discrete energy law*

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Let $V_h \cap L_0^2 \subseteq L_h$, $\mathcal{T}_h$ *be a strongly acute triangulation,* $\alpha, \beta_> 0$ *and* $0 < \alpha + \frac{\beta_2}{2} < \frac{6-d}{6}$. *Then* $0 < \bar{\rho}_1 \leq \rho^n \leq \bar{\varrho}_2 < \infty$ *(discrete maximum principle).*

Sketch of the proof

By asumption the discrete inf-sup-*cond. is fullfilled* $\Rightarrow \exists R^n, P^n$.

Lemma (Existence, energy law, maximum principle, [Bañas and Prohl, 2010])

Let $\alpha, \beta_1, \beta_2, \beta_3 \geq 0$, $\sqrt{\beta_2} h^{\beta_2} 2 \|\nabla \boldsymbol{U}^0\| \leq C$. Then there exists a solution $(\boldsymbol{U}^n, \boldsymbol{B}^n, \rho^n, P^n, R^n)$ of the numerical scheme, which holds the discrete energy law

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Sketch of the proof

Use $\psi := \frac{1}{\bar{\mu}}\boldsymbol{B}^n$ and $\boldsymbol{W} := \boldsymbol{U}^n$ in (Scheme A) and direct calculation.

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$0 < \bar{\rho}_1 \leq \rho^n \leq \bar{\varrho}_2 < \infty$ *(discrete maximum principle).*

Sketch of the proof

- *Write (Scheme A) as $\mathcal{A}^n \boldsymbol{U}^n = \mathcal{G}^n$ and show that $\mathcal{A}^n$ is a M-Matrix.*
- *By the strongly acute triangulation the off-diagonal entries of the stiffness matrix are negative. Use standard techniques of FEM theory (e.g. inverse estimates) and assumption of constants to proof this.*
- *Lower bound for ρ^n follows from M-Matrix property.*
- *Upper bound for ρ^n: direct calculation.*

Passing to the (weak) limit

Let $\mathcal{U}, \mathcal{B}$, etc. be the (time-) interpolant of $\boldsymbol{U}^n$, $\boldsymbol{B}^n$, etc. and all asumptions hold from the last slide. There exists
$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{L}^2 \cap \{\operatorname{div} \boldsymbol{u} = 0 \text{ weakly}\}) \cap L^2(0, T; \boldsymbol{W}_0^{1,2} \cap \{\operatorname{div} \boldsymbol{u} = 0 \text{ a.e.}\})$,
$\boldsymbol{b} \in L^\infty(0, T; \boldsymbol{L}^2 \cap \{\operatorname{div} \boldsymbol{u} = 0 \text{ weakly}\}) \cap L^2(0, T; \boldsymbol{X})$ and $\rho \in L^\infty(0, T; L^\infty)$ s.t.

$$
\begin{aligned}
\mathcal{U} &\rightharpoonup^* \boldsymbol{u} && \text{in } L^\infty(0, T; \boldsymbol{L}^2),\\
\mathcal{U} &\rightharpoonup \boldsymbol{u} && \text{in } L^2(0, T; \boldsymbol{W}^{1,2}),\\
\operatorname{div} \mathcal{U} &\to 0 && \text{in } L^2(0, T; L^2) \quad (\beta_1 > 0),\\
\mathcal{B} &\rightharpoonup^* \boldsymbol{b} && \text{in } L^\infty(0, T; \boldsymbol{L}^2),\\
\mathcal{B} &\rightharpoonup \boldsymbol{b} && \text{in } L^2(0, T; \boldsymbol{H}(\text{curl})),\\
\sigma &\rightharpoonup^* \rho && \text{in } L^\infty(0, T; L^\infty).
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Sketch of the proof

- *Boundness of follows direct from the energy law.*
- $\operatorname{div} \boldsymbol{u} = 0$ *a.e. follows directly.*
- *As* $\operatorname{div} \mathcal{B} = 0$*, we have* $\operatorname{div} \boldsymbol{b} = 0$ *by a result of Kikuchi, cf. [Hiptmair, 2002, Theorem 4.9].*

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Under certain assumptions for the constants and initial data, $\mathcal{U}^+ \to \boldsymbol{u}$ *in* $L^2(0,T;\boldsymbol{L}^2)$, $\sigma^+ \rightharpoonup^* \rho$ *in* $L^\infty(0,T;L^2)$*, we have*

- ρ *is unique weak solution of* $\rho_t + \operatorname{div}(\rho\boldsymbol{u}) = 0$,
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- $\sigma \to \rho$ *in* $L^2(0, T; L^2)$.

Sketch of the proof

- *Weak convergence is assured by last lemma. Show that* $\|\sigma^+\| \to \|\rho\|$.
- *Use (variant (cf. [Walkington, 2004]) of) DiPerna–Lions compactness argument ([DiPerna and Lions, 1989]) and standard arguments (weak continuity of the norm, assumptions, Fatou, energy law, etc.) to conclude*

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Lemma (Compactness result, [Lions and Magenes, 1972])

If there exists $C > 0$, $\alpha > 0$, s.t. for all $0 < \delta \leq T$

$$\int_{\delta}^{T} \|v_h(t) - v_h(t - \delta)\|_{L^2} \leq C\delta^{\alpha},$$

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Let $(.,.)_* = (.,.)_h$, $V_h \cap L^2 \subseteq L_h$ *and* $0 < \alpha, \beta_1, \beta_2, \beta_3$ *s.t.* $0 < \alpha + \frac{\beta_2}{2} < \frac{6-d}{6}$ *and* $\alpha \geq \frac{\beta_3}{4}$ *(for* $d = 3$*). Then there exists* $C > 0$, $\alpha > 0$ *s.t.*

$$\int_{\delta}^{T} \|\boldsymbol{\mathcal{U}}(t,.) - \boldsymbol{\mathcal{U}}(t-\delta,.)\|_{\boldsymbol{L}^2}^2 + \|\boldsymbol{\mathcal{B}}(t,.) - \boldsymbol{\mathcal{B}}(t-\delta,.)\|_{\boldsymbol{L}^2}^2 \, dt \leq C\delta^k \quad \forall \delta \in [0, T].$$

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Sketch of the proof

- *much direct, technical calculation. Frequently use of energy law, standard estimates of FEM theory.*
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Corollary

The weak limits $\boldsymbol{u}, \boldsymbol{b}, \rho$ and p, R solve the weak two-fluid MHD equation.

Discontinuous Galerkin Approach

In [Liu and Walkington, 2007] they propose a discontinuous Galerkin scheme for the denstity depend Navier–Stokes eq. with p.w. constant FE space w.r.t. ρ and aver. divergence zero w.r.t. $\boldsymbol{u}$.

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Continuous Galerkin

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- gen: direct use of appropriate testfunctions $\Rightarrow$ energy law.
- here: M-matrix property.
- here: here: Taylor–Hood, MINI elements are admitted.

Disadvantage

- gen: prob. high effort near the boundary.
- here: stabilazation terms required for convergence.

Discontinuous Galerkin

Advantage

- gen: Possibility of higher polynomial degrees.
- gen: Flexibility in grid design.
- here: Monotonicity of the iterates.

Disadvantage

- gen: High grow of dof for computition, bad condition number of stiffness matrix.
- here: control of jump terms difficult.

Outline

- Solve (Scheme A) by a fixed point scheme which decouples $\boldsymbol{u}$ and $\boldsymbol{b}$ (right hand side of the eq. old iterate). This scheme converges to weak solution of the two-fluid MHD eq.
- Implementation of NSE with Taylor–Hood elements.

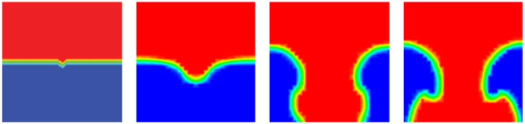
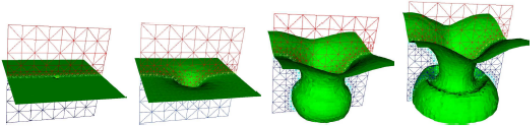


FIGURE 2. Density at $x = 0.5$ at times $t = 0, 4, 14, 20$.



Outline

Summary

- Discretization based on FEM (via continuous Galerkin method!) and Euler scheme for time. Calculation simplifies with splitting scheme.
- No convergence rates!
- Mostly advatages in contrast to discontinuous Galerkin methods (M-matrix property, no jump terms, etc.)

Thanks

Thank you for your attention!

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