CONSERVATION LAWS DRIVEN BY LÉVY WHITE NOISE

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Abstract. We consider multidimensional conservation laws perturbed by multiplicative Lévy noise. We establish existence and uniqueness results for entropy solutions. The entropy inequalities are formally obtained by the Itó-Lévy chain rule. The multidimensionality requires a generalized interpretation of the entropy inequalities to accommodate Young measure-valued solutions. We first establish the existence of entropy solutions in the generalized sense via the vanishing viscosity method, and then establish the L^1-contraction principle. Finally, the L^1 contraction principle is used to argue that the generalized entropy solution is indeed the classical entropy solution.

Contents

1. Introduction

We are interested in stochastic perturbations of nonlinear conservation laws. A conservation law with source term (balance law) is an equation of the type

$$\frac{\partial u(t,x)}{\partial t} + \mathrm{div}_x F(u(t,x)) = q(t,x,u(t,x)), \qquad t>0,\ x\in\mathbb{R}^d, \tag{1.1}$$

where F is known as the flux function. In a deterministic context the source term $q(t,x,u)$ is given by a nicely behaved function and Kružkov's entropy solution framework provides a comprehensive understanding of the related Cauchy problem. There are multiple ways of interpreting q and we are particularly interested in the scenario where the source $q(t,x,u)$ represents a multiplicative white noise. This would make (1.1) a stochastic balance law and this equation has attracted significant attention in recent years. However, all studies have been limited to the case where the source $q(t,x,u)$ represents a Brownian multiplicative white noise i.e $q(t,x,u)=\sigma(t,x,u)\frac{dB_t}{dt}$, where $(B_t)_{t\ge 0}$ is a Brownian motion.

In this paper, we intend to study the Cauchy problem related to (1.1) where the source term $q(t,x,u)$ represents a multiplicative Lévy white noise. A more precise description of our problem is as follows. Let

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$\big(\Omega, P, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}\big)$ be a filtered probability space satisfying the usual hypothesis. We are looking for a $L^2(\mathbb{R}^d)$-valued predictable process $u(t)$ satisfying

$$du(t,x) + \mathrm{div}_x F(u(t,x))\,dt = \int_{|z|>0} \eta(x, u(t,x); z)\,\tilde{N}(dz, dt), \quad t > 0,\ x \in \mathbb{R}^d, \tag{1.2}$$

with the initial condition

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d. \tag{1.3}$$

In (1.2), $F : \mathbb{R} \to \mathbb{R}^d$ is a given nonlinear flux function, and $\tilde{N}(dz, dt) = N(dz, dt) - m(dz)\,dt$, where N is a Poisson random measure on $\mathbb{R} \times (0, \infty)$ with intensity measure $m(dz)$ such that $\int (1 \wedge |z|^2)\,m(dz) < +\infty$. Moreover, $\eta(x, u; z)$ is a real valued function defined on the domain $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}$. We point out that adding a Brownian component to the white noise term on the right hand side of (1.2) would make it more general, and the results of this paper are still valid under appropriate conditions.

The equation (1.2) becomes a multidimensional deterministic conservation law if $\eta = 0$. It is well-documented that solutions of deterministic conservation laws develop discontinuities (shocks) in finite time. Therefore the solutions must be interpreted in the weak sense and a so-called entropy condition is required to identify the physically relevant (unique) solution [5, 12].

The study of stochastic balance laws has so far been limited to equations driven by Brownian white noise. For some first results in that direction, see Holden and Risebro [14]. E, Khanin, Mazel, and Sinai [8] described the statistical properties of the Burgers equation with Brownian noise. Kim [15] extended the Kružkov well-posedness theory to one dimensional balance laws that are driven by additive Brownian noise. This approach does not apply to the multiplicative noise case. Indeed, a straightforward adaptation of the deterministic "doubling technique" leads to anticipating stochastic integrands, and so the standard route leading to the L^1-contraction principle cannot be followed. In a recent work, Feng and Nualart [11] came up with a way to address this issue, giving raise to what they referred to as *strong* entropy solutions, which in turn are intimately connected to vanishing viscosity solutions. In [11], the authors established the uniqueness of strong entropy solutions in a multidimensional L^p-framework. The existence, however, was restricted to one space dimension. We refer to Vovelle and Debussche [6] (see also Chen *et al.* [4]) for an existence result in the multidimensional case. In [6] the authors obtain the existence via the kinetic formulation, while [4] uses the BV framework. Another recent contribution to the multidimensional problem is Bauzet, Vallet, and Wittbold [2], where the question of existence is settled via the Young measure approach. We also mention the very recent contributions [16, 17] by Lions, Perthame, and Souganidis on conservation laws with rough (stochastic) fluxes.

During the last decade there has been many contributions in the larger area of stochastic partial differential equations that are driven by Lévy noise. An worthy reference on this subject is [21]. However, there are few results on the specific problem of conservation laws with Lévy noise. The present article marks a first step in our endeavor to build a comprehensive theory of mixed hyperbolic-parabolic equations driven by noise containing both diffusion and jump effects. We draw inspiration from [2, 10, 20] and the notion of *entropy process solutions* when utilizing the theory of Young measures as a tool to prove the existence of entropy solutions to Lévy driven conservation laws. The presence of Lévy noise asks for solutions that have discontinuous sample paths. Also, the entropy inequalities will have non-localities in them as a consequence of the Itô-Lévy chain rule. As a result the "strong entropy" approach of Feng and Nualart [11] seems difficult to adapt to the present situation.

The remaining part of the paper is organized as follows. We state the assumptions, detail the technical framework, and state the main results in Section 2. In Section 3, we establish the wellposedness and derive apriori estimates for the viscous approximations. Section 4 deals with the existence of entropy solutions via Young measure valued limits of viscous approximations. Finally, Section 5 is devoted to the question of uniqueness of entropy solutions.

2. Technical framework and statements of the main results

Here and in the sequel we use the letters C, K, etc. to denote various generic constants. There are situations where constants may change from line to line, but the notation is kept unchanged so long as it does not impact the central idea. The Euclidean norm on any $\mathbb{R}^d$-type space is denoted by $|\cdot|$. The space $C^n(\mathbb{R}^d)$ consist of the real valued functions on $\mathbb{R}^d$ that are n-times continuously differentiable.

For a constant $T > 0$, the space-time cylinder $[0,T) \times \mathbb{R}^d$ is denoted by Π_T and the symbol Π_∞ stands for $[0,\infty) \times \mathbb{R}^d$. The spaces $C_c^{1,2}(\Pi_T)$ and $C_c^{1,2}(\Pi_\infty)$ contain the compactly supported functions on Π_T and Π_∞, respectively, which are continuously differentiable in the time variable and twice continuously differentiable in the space variable.

2.1. **Entropy inequalities.** We begin this section with a formal derivation of the entropy inequalities á la Kružkov, keeping in mind the need to replace the traditional chain rule of deterministic calculus by the Itô-Lévy chain rule. Let $0 \le \beta \in C^2(\mathbb{R})$ be a real valued convex function, and ζ be such that $\zeta'(r) = \beta'(r)F'(r)$. For a small positive number $\varepsilon > 0$, assume that the parabolic perturbation

$$du(t,x) + \mathrm{div}_x F(u(t,x))\,dt = \int_{|z|>0} \eta(x,u(t,x);z)\,\tilde{N}(dz,dt) + \varepsilon \Delta u(t,x)\,dt$$

of (1.2) has a strong (predictable) solution $u_\varepsilon(t,x)$. Now we apply the Itô-Lévy formula to $\beta(u_\varepsilon(t,x))$, yielding

$$\begin{aligned}
&d\beta(u_\varepsilon(t,x)) + \mathrm{div}_x\zeta(u_\varepsilon(t,x))\,dt \\
&= \int_{|z|>0} \Big(\beta(u_\varepsilon(t,x) + \eta(x,u_\varepsilon(t,x);z)) - \beta(u_\varepsilon(t,x))\Big)\tilde{N}(dz,dt) \\
&\quad + \int_{|z|>0} \Big(\beta(u_\varepsilon(t,x) + \eta(x,u_\varepsilon(t,x);z)) - \beta(u_\varepsilon(t,x)) - \eta(x,u_\varepsilon(t,x);z)\beta'(u_\varepsilon(t,x))\Big) m(dz)\,dt \\
&\quad + \Big(\varepsilon\Delta_{xx}\beta(u_\varepsilon(t,x)) - \varepsilon\beta''(u_\varepsilon(t,x))|\nabla_x u_\varepsilon(t,x)|^2\Big)\,dt.
\end{aligned}$$

Given a nonnegative test function $\psi \in C_c^{1,2}([0,\infty) \times \mathbb{R}^d)$, we apply the Itó-Lévy product rule to $\beta(u_\varepsilon(t,\cdot))\psi(t,\cdot)$, arriving at

$$\begin{aligned}
&d\big[\beta(u_\varepsilon(t,x))\psi(t,x)\big] = \partial_t\psi(t,x)\beta(u_\varepsilon(t,x))\,dt - \psi(t,x)\mathrm{div}_x\zeta(u_\varepsilon(t,x))\,dt \\
&+ \int_{|z|>0} \psi(t,x)\Big(\beta(u_\varepsilon(t,x) + \eta(x,u_\varepsilon(t,x);z)) - \beta(u_\varepsilon(t,x))\Big)\tilde{N}(dz,dt) \\
&+ \int_{|z|>0} \psi(t,x)\Big(\beta(u_\varepsilon(t,x) + \eta(x,u_\varepsilon(t,x);z)) - \beta(u_\varepsilon(t,x)) - \eta(x,u_\varepsilon(t,x);z)\beta'(u_\varepsilon(t,x))\Big)m(dz)\,dt \\
&+ \psi(t,x)\Big(\varepsilon\Delta_{xx}\beta(u_\varepsilon(t,x)) - \varepsilon\beta''(u_\varepsilon(t,x))|\nabla_x u_\varepsilon(t,x)|^2\Big)\,dt.
\end{aligned}$$

We integrate the above equality with respect to (t,x) and use $\langle\cdot,\cdot\rangle$ to denote inner product in $L^2(\mathbb{R}^d)$. The result is

$$\begin{aligned}
0 &\le \langle \beta(u_\varepsilon(T,.)), \psi(T,\cdot)\rangle \\
&\le \langle \beta(u_\varepsilon(0,.)), \psi(0,\cdot)\rangle + \int_0^T \langle \zeta(u_\varepsilon(r,.)), \nabla_x\psi(r,\cdot)\rangle\,dr \\
&+ \int_0^T \langle \beta(u_\varepsilon(r,\cdot)), \partial_t\psi(r,\cdot)\rangle\,dr + \mathcal{O}(\varepsilon) \\
&+ \int_0^T \int_{|z|>0} \langle \beta(u_\varepsilon(r,.) + \eta(.,u_\varepsilon(r,.);z)) - \beta(u_\varepsilon(r,.)), \psi(r,\cdot)\rangle \tilde{N}(dz,dr) \\
&+ \int_0^T \int_{|z|>0} \langle \beta(u_\varepsilon(r,\cdot) + \eta(.,u_\varepsilon(r,.);z)) - \beta(u_\varepsilon(r,.)) - \eta(.,u_\varepsilon(r,.);z)\beta'(u_\varepsilon(r,.)), \psi(r,\cdot)\rangle\, m(dz)\,dr.
\end{aligned} \tag{2.1}$$

The notation $\mathcal{O}(\varepsilon)$ is used to denote quantities that depend on ε and are bounded above by $C\varepsilon$. Clearly, the above inequality is stable under the limit $\varepsilon \to 0$, if the family $\{u_\varepsilon\}_{\varepsilon>0}$ has L^p_{loc}-type stability. Just as the deterministic equations, the above inequality (2.1) provides us with the entropy condition. We now formally define the entropy solutions.

Definition 2.1 (entropy flux pair)**.** A pair (β,ζ) is called an entropy flux pair if $\beta \in C^2(\mathbb{R})$ and $\beta \ge 0$, and $\zeta = (\zeta_1, \zeta_2, ...\zeta_d) : \mathbb{R} \mapsto \mathbb{R}^d$ is a vector field satisfying $\zeta'(r) = \beta'(r)F'(r)$ for all r. An entropy flux pair (β,ζ) is called convex if $\beta''(\cdot) \ge 0$.

Definition 2.2 (entropy solution)**.** A $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $u(t) = u(t,x)$ is called a stochastic entropy solution of (1.2) if
(1) For each $T > 0$, $p = 2, 3, 4, \ldots$,

$$\sup_{0\le t\le T} E\Big[||u(t)||_p^p\Big] < \infty.$$

(2) Given any non-negative test function $\psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$ and any convex entropy pair (β,ζ) with β' bounded, it holds that

$$\begin{aligned}
&\langle \psi(0,\cdot), \beta(u(0,\cdot))\rangle + \int_{t=0}^T \langle \partial_t\psi(t,\cdot), \beta(u(t,\cdot))\rangle\,dt + \int_{r=0}^T \langle \zeta(u(r,.)), \nabla_x\psi(r,\cdot)\rangle\,dr \\
&\quad + \int_{r=0}^T \int_{|z|>0} \langle \beta(u(r,.) + \eta(.,u(r,.);z)) - \beta(u(r,.)), \psi(r,\cdot)\rangle \tilde{N}(dz,dr) \\
&\quad + \int_{r=0}^T \int_{|z|>0} \langle \beta(u(r,.) + \eta(.,u(r,.);z)) - \beta(u(r,.)) - \eta(.,u(r,.);z)\,\beta'(u(r,.)), \psi(r,\cdot)\rangle\, m(dz)\,dr \\
&\qquad \ge 0 \quad P\text{-almost surely.}
\end{aligned}$$

The aim of this paper is to establish the existence and uniqueness of entropy solutions according to Definition 2.2, and we will do so under the following assumptions:

($\mathbf{A}$.1) For $k = 1, 2, \ldots, d$, the functions $F_k(s) \in C^2(\mathbb{R})$; and $F_k(s)$, $F_k'(s)$, and $F_k''(s)$ have at most polynomial growth in s.

($\mathbf{A}$.2) There exist positive constants $K > 0$ and $\lambda^* \in [0,1)$ such that

$$|\eta(x,u;z) - \eta(y,v;z)| \le (\lambda^*|u-v| + K|x-y|)(|z|\wedge 1) \text{ for all } x,y\in\mathbb{R}^d;\ u,v\in\mathbb{R};\ z\in\mathbb{R}.$$

($\mathbf{A}$.3) The Lévy measure $m(dz)$ is a Radon measure on $\mathbb{R}\backslash\{0\}$ with a possible singularity at $z = 0$, which satisfies

$$\int_{\mathbb{R}_z} (|z|^2\wedge 1)\, m(dz) < \infty.$$

($\mathbf{A}$.4) There exists a nonnegative function $g \in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ such that

$$|\eta(x,u;z)| \le g(x)(1+|u|)(|z|\wedge 1)$$

for all $(x,u,z)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}$.

Remark. We are able to accommodate polynomially growing flux function as a result of the requirement that the entropy solutions satisfy L^p bounds for all $p \ge 2$. This in turn forces us to choose initial data that are in L^p for all p. It is possible to accommodate initial conditions which are only L^2, but we would then require the flux function to be globally Lipschitz. Furthermore, the assumption ($\mathbf{A}$.2) is needed to handle the nonlocal nature of the entropy inequalities.

2.2. **Generalized entropy solutions.** The focus of this paper is well-posedness for multidimensional problems. Contrary to one dimensional problems [3, 11], compensated compactness is not applicable and securing proper compactness for vanishing viscosity approximations requires an alternative viewpoint. One option is to further weaken the notion of entropy solutions to accommodate solutions that are parametrized measures (Young measures). However, in view of [2, 20] (and Lemma 4.3), we can equivalently look for generalized entropy solutions that are $L^2(\mathbb{R}^d\times(0,1))$-valued processes.

Definition 2.3 (Generalized entropy solution)**.** An $L^2(\mathbb{R}^d\times(0,1))$-valued $\{\mathcal{F}_t : t \ge 0\}$-predictable stochastic process $v(t) = v(t,x,\alpha)$ is called a generalized stochastic entropy solution of (1.2) if
(1) for each $T > 0$, $p = 2, 3, 4, \ldots$,

$$\sup_{0\le t\le T} E\Big[||v(t,\cdot,\cdot)||_p^p\Big] < \infty.$$

(2) For $0 \le \psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$ and each convex entropy pair (β,ζ) with β' bounded, it holds that

$$\langle \psi(0,\cdot), \beta(v(0,\cdot))\rangle + \int_0^T\int_{\alpha=0}^1 \langle \partial_t\psi(r,\cdot), \beta(v(r,\cdot,\alpha))\rangle\,d\alpha\,dr + \int_0^T\int_{\alpha=0}^1 \langle \zeta(v(r,.,\alpha)), \nabla_x\psi(r,\cdot)\rangle\,d\alpha\,dr$$

$$
\begin{aligned}
&+ \int_0^T \int_{|z|>0} \int_{\alpha=0}^1 \langle \beta(v(r,.,\alpha) + \eta(.,v(r,.,\alpha);z)) - \beta(v(r,.,\alpha)), \psi(r,\cdot)\rangle \, d\alpha \, \tilde{N}(dz,dr) \\
&+ \int_0^T \int_{|z|>0} \int_{\alpha=0}^1 \langle \beta(v(r,.,\alpha) + \eta(.,v(r,.,\alpha);z)) - \beta(v(r,.,\alpha)) \\
&\qquad\qquad - \eta(.,v(r,.,\alpha);z)\beta'(v(r,.,\alpha)), \psi(r,\cdot)\rangle \, d\alpha \, m(dz)\, dr \\
&\geq 0 \quad P-\text{a.s}
\end{aligned}
$$

We can now state the main results of this paper.

Theorem 2.1 (existence)**.** *Suppose assumptions* (**A**.1)-(**A**.4) *hold, and that the* $\bigcap_{p=1,2,\dots} L^p(\mathbb{R}^d)$*-valued* $\mathcal{F}_0$*-measurable random variable* u_0 *satisfies*

$$
E\Big[||u_0||_p^p + ||u_0||_2^p\Big] < \infty, \qquad \text{for } p = 1,2,\dots\ . \tag{2.2}
$$

Then there exists a generalized entropy solution of (1.2)-(1.3) *in the sense of Definition* 2.3*.*

Theorem 2.2 (uniqueness)**.** *Suppose assumptions* (**A**.1)-(**A**.4) *hold, and that the* $\bigcap_{p=1,2,\dots} L^p(\mathbb{R}^d)$*-valued* $\mathcal{F}_0$*-measurable random variable* u_0 *satisfies* (2.2)*. Then the generalized entropy solution of* (1.2)-(1.3) *is unique. Moreover, it is the unique stochastic entropy solution.*

The above definitions do not say anything explicit about how a solution satisfies the initial condition. However, it follows after simple considerations that it satisfies the initial condition in a certain weak sense (see [19, 23]).

Lemma 2.3. *Any generalized entropy solution* $u(t,\cdot,\cdot)$ *of* (1.2)-(1.3) *satisfies the initial condition in the following sense: for every non negative test function* $\psi \in C_c^2(\mathbb{R}^d)$ *such that* $\operatorname{supp}(\psi) = K$*,*

$$
\lim_{h\to 0} E\Big[\frac{1}{h}\int_{t=0}^h \int_K \int_{\lambda=0}^1 |u(t,x,\lambda) - u_0(x)|\psi(x)\, d\lambda\, dx\, dt\Big] = 0
$$

Proof. Since K is of finite measure, it is enough to prove

$$
\lim_{h\to 0} E\Big[\frac{1}{h}\int_{t=0}^h \int_K \int_{\lambda=0}^1 |u(t,x,\lambda) - u_0(x)|^2\psi(x)\, d\lambda\, dx\, dt\Big] = 0.
$$

For $\delta \in (0,1)$, let $K_\delta = \{x : \operatorname{dist}(x,K) \leq \delta\}$. Note that, for any $\delta > 0$,

$$
\begin{aligned}
&E\Big[\int_K \int_{\lambda=0}^1 |u(t,x,\lambda) - u_0(x)|^2 \psi(x)\, d\lambda\, dx\Big] \\
&\quad \leq 2E\Big[\int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^1 |u(t,x,\lambda) - u_0(y)|^2\psi(x)\varrho_\delta(x-y)\, d\lambda\, dx\, dy\Big] \\
&\qquad + 2E\Big[\int_{y\in K_\delta}\int_{x\in K} |u_0(y) - u_0(x)|^2\psi(x)\varrho_\delta(x-y)\, dx\, dy\Big],
\end{aligned}
$$

where $\{\varrho_\delta\}_{\delta>0}$ is the sequence of standard mollifiers in $\mathbb{R}^d$. In other words,

$$
\begin{aligned}
&E\Big[\frac{1}{h}\int_{t=0}^h \int_K \int_{\lambda=0}^1 |u(t,x,\lambda) - u_0(x)|^2\psi(x)\, d\lambda\, dx\, dt\Big] \\
&\quad \leq 2E\Big[\frac{1}{h}\int_{t=0}^h \int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^1 |u(t,x,\lambda) - u_0(y)|^2\psi(x)\varrho_\delta(x-y)\, d\lambda\, dx\, dy\, dt\Big] \\
&\qquad + 2E\Big[\int_{y\in K_\delta}\int_{x\in K} |u_0(y) - u_0(x)|^2\psi(x)\varrho_\delta(x-y)\, dx\, dy\Big].
\end{aligned} \tag{2.3}
$$

Let $\psi(t,x) = \gamma(t)\psi(x)\varrho_\delta(x-y)$, where $\gamma(t) = \frac{h-t}{h}$ for $0 \leq t \leq h$. Now, let $\beta(u) = (u - u_0(y))^2$ and $\xi(u) = \int_0^u 2(r - u_0(y))F'(r)\, dr = 2\int_0^u rF'(r)\, dr - 2u_0(y)(F(u) - F(0)) \leq C(1 + |u_0(y)|^2 + |u|^p)$ for some positive integer p. We now apply Definition 2.3 with the entropy flux pair (β,ξ), obtaining

$$
0 \leq E\Big[\int_{y\in K_\delta}\int_{x\in K} |u_0(y) - u_0(x)|^2\psi(x)\varrho_\delta(x-y)\, dx\, dy\Big]
$$

$$
\begin{aligned}
&- E\Big[\frac{1}{h}\int_{t=0}^{h}\int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^{1} |u(t,x,\lambda)-u_0(y)|^2\psi(x)\varrho_\delta(x-y)\,d\lambda\,dx\,dy\,dt\Big] \\
&\qquad + C\delta^{-2}\int_{r=0}^{h} E\Big[\int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^{1}(1+|u(r,x,\lambda)|^p+|u_0(y)|^2)\,d\lambda\,dx\,dy\Big]\,dr \\
&+ \frac{C''}{\delta}\int_{r=0}^{h} E\Big[\int_{x\in K}\int_{\lambda=0}^{1}\int_{|z|>0}|\eta(x,u(r,x,\lambda),z)|^2\,m(dz)\,d\lambda\,dx\Big]\,dr,
\end{aligned}
$$

and so

$$
\begin{aligned}
&E\Big[\frac{1}{h}\int_{t=0}^{h}\int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^{1} |u(t,x,\lambda)-u_0(y)|^2\psi(x)\varrho_\delta(x-y)\,d\lambda\,dx\,dy\,dt\Big] \\
&\quad \le E\Big[\int_{y\in K_\delta}\int_{x\in K}|u_0(y)-u_0(x)|^2\psi(x)\varrho_\delta(x-y)\,dx\,dy\Big] \\
&\qquad + C\delta^{-2}\int_{r=0}^{h} E\Big[\int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^{1}(1+|u(r,x,\lambda)|^p+|u_0(y)|^2)\,d\lambda\,dx\,dy\Big]\,dr \\
&\qquad + \frac{C''}{\delta}\int_{r=0}^{h} E\Big[\int_{x\in K}\int_{\lambda=0}^{1}\int_{|z|>0}|\eta(x,u(r,x,\lambda),z)|^2\,m(dz)\,d\lambda\,dx\Big]\,dr.
\end{aligned}
$$

Hence, by passing to the limit $h\to 0$,

$$
\begin{aligned}
&\limsup_{h\to 0} E\Big[\frac{1}{h}\int_{t=0}^{h}\int_{y\in K_\delta}\int_{x\in K}\int_{\lambda=0}^{1} |u(t,x,\lambda)-u_0(y)|^2\psi(x)\varrho_\delta(x-y)\,d\lambda\,dx\,dy\,dt\Big] \\
&\quad \le E\Big[\int_{y\in K_\delta}\int_{x\in K}|u_0(y)-u_0(x)|^2\psi(x)\varrho_\delta(x-y)\,dx\,dy\Big]. \qquad (2.4)
\end{aligned}
$$

Combining (2.3) and (2.4) yields

$$
\begin{aligned}
&\limsup_{h\to 0} E\Big[\frac{1}{h}\int_{t=0}^{h}\int_{K}\int_{\lambda=0}^{1} |u(t,x,\lambda)-u_0(x)|^2\psi(x)\,d\lambda\,dx\,dt\Big] \\
&\quad \le 4E\Big[\int_{y\in K_\delta}\int_{x\in K}|u_0(y)-u_0(x)|^2\psi(x)\varrho_\delta(x-y)\,dx\,dy\Big] \quad \text{for all } \delta>0. \qquad (2.5)
\end{aligned}
$$

We now let $\delta\to 0$ in the right-hand side of (2.5), which gives

$$
\limsup_{h\to 0} E\Big[\frac{1}{h}\int_{t=0}^{h}\int_{K}\int_{\lambda=0}^{1} |u(t,x,\lambda)-u_0(x)|^2\psi(x)\,d\lambda\,dx\,dt\Big] \le 0;
$$

the proof is complete since $\psi\ge 0$. □

Before concluding this section, we introduce a class of entropy functions. Let $\beta:\mathbb{R}\to\mathbb{R}$ be a C^∞ function satisfying

$$
\beta(0)=0,\quad \beta(-r)=\beta(r),\quad \beta'(-r)=-\beta'(r),\quad \beta''\ge 0,
$$

and

$$
\beta'(r)=\begin{cases} -1 & \text{when } r\le -1,\\ \in[-1,1] & \text{when } |r|<1,\\ +1 & \text{when } r\ge 1.\end{cases}
$$

For any $\vartheta>0$, define $\beta_\vartheta:\mathbb{R}\to\mathbb{R}$ by $\beta_\vartheta(r)=\vartheta\beta(\frac{r}{\vartheta})$. Then

$$
|r|-M_1\vartheta\le\beta_\vartheta(r)\le|r| \quad\text{and}\quad |\beta_\vartheta''(r)|\le\frac{M_2}{\vartheta}\mathbf{1}_{|r|\le\vartheta}, \qquad (2.6)
$$

where $M_1=\sup_{|r|\le 1}\big||r|-\beta(r)\big|$ and $M_2=\sup_{|r|\le 1}|\beta''(r)|$.

By simply dropping ϑ, for $\beta=\beta_\vartheta$ we define

$$
\begin{aligned}
F_k^\beta(a,b)&=\int_b^a\beta'(\sigma-b)F_k'(\sigma)\,d(\sigma),\\
F^\beta(a,b)&=(F_1^\beta(a,b),F_2^\beta(a,b),\dots,F_d^\beta(a,b)),
\end{aligned}
$$

$$F_k(a,b) = \operatorname{sign}(a-b)(F_k(a) - F_k(b)),$$
$$F(a,b) = (F_1(a,b), F_2(a,b), \dots, F_d(a,b)).$$

3. Existence and a-priori estimates for the viscous problem

The entropy inequalities, and the corresponding well-posedness result, are reliant on the fact that one can (spatially) regularize the solution of (1.2) by adding small diffusion operator. Therefore, in this section, we will provide a detailed analysis of the following viscous problem:

$$du(t,x) + \operatorname{div}_x F(u(t,x))\,dt = \int_{|z|>0} \eta(x, u(t,x); z)\,\tilde{N}(dz,\,dt) + \varepsilon \Delta_{xx} u\,dt, \quad t>0,\ x\in\mathbb{R}^d, \tag{3.1}$$

with initial condition (1.3). To the best of our knowledge, the answers to the wellposedness questions for Lévy driven SPDEs are not readily available in its full generality to cover (3.1). However, a relevant reference is [7], where the one-dimensional viscous Burgers equation with Lévy noise is studied.

Throughout this section, we impose the following regularity assumptions:

- (**B**.1) The function $F:\mathbb{R}\to\mathbb{R}^d$ is smooth, i.e., $F_k\in C^\infty$, and the n-th derivative satisfies $|\partial_u^n F_k(u)| \le K_n$ for some constant K_n and for all $n\in\mathbb{N}$ and $k=1,\dots,d$.
- (**B**.2) For every $n\in\mathbb{N}$, $\partial_u^n \eta(x,u;z)$ and $D_x^n \eta(x,u;z)$ exist and are continuous. Moreover, $\eta(\cdot,u;z)\in\mathcal{S}(\mathbb{R}^d)$.
- (**B**.3) For every $n\in\mathbb{N}$, there exists $K_n(x)\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that
$$|\partial_u^n \eta(x,u;z)| + |D_x^n \eta(x,u,z)| \le K_n(x)(1\wedge|z|).$$
- (**B**.4) The initial condition u_0 belongs to $\mathcal{S}(\mathbb{R}^d)$.

It is implied by (**B**.4) that $E[||u_0||_2^2] < \infty$, and we introduce the following Picard-type iterates: for any natural number $n\ge 0$, define

$$\begin{aligned} &u^0(t,x) = u_0(x),\\ &du^n(t,x) + \operatorname{div}_x F(u^{n-1}(t,x))dt = \varepsilon\Delta u^n(t,x)dt + \int_{|z|>0} \eta(x,u^{n-1}(t,x);z)\,\tilde{N}(dz,dt)\\ &u^n(0,x) = u_0(x). \end{aligned} \tag{3.2}$$

Let $G_\varepsilon(t,x)$ be the heat kernel associated with operator $\varepsilon\Delta_{xx}$ i.e

$$G(t,x) \equiv G_\varepsilon(t,x) = \frac{1}{(4\pi\varepsilon t)^{\frac{d}{2}}} e^{\frac{-|x|^2}{4\varepsilon t}}, \quad t>0.$$

We are looking for a $L^2(\mathbb{R}^d)$-valued predictable process $u^n(t,x)$ that qualifies as the mild solution to (3.2). In other words, we want a predictable process $u^n(t,x)$ that satisfies

$$\begin{aligned} u^n(t,x) = &\int_{\mathbb{R}^d_y} G(t,x-y)u_0(y)\,dy - \int_{s=0}^t \int_{\mathbb{R}^d_y} G(t-s,x-y)\sum_{i=1}^d \partial_{y_i} F_i(u^{n-1}(s,y))\,dy\,ds\\ &+ \int_{s=0}^t \int_{|z|>0}\int_{\mathbb{R}^d_y} G(t-s,x-y)\eta(y,u^{n-1}(s,y);z)\,dy\,\tilde{N}(dz,ds), \end{aligned} \tag{3.3}$$

almost surely, for every t. Note that the cádlág solution $v(t,x)$ to (3.2) is given by

$$\begin{aligned} v(t,x) = &\int_{\mathbb{R}^d_y} G(t,x-y)u_0(y)\,dy - \int_{s=0}^t \int_{\mathbb{R}^d_y} G(t-s,x-y)\sum_{i=1}^d \partial_{y_i} F_i(u^{n-1}(s,y))\,dy\,ds\\ &+ \int_{s=0}^t \int_{|z|>0}\int_{\mathbb{R}^d_y} G(t-s,x-y)\eta(y,u^{n-1}(s,y);z)\,dy\,\tilde{N}(dz,ds). \end{aligned} \tag{3.4}$$

Moreover, the martingale term on the right-hand side of (3.4) is stochastically continuous and cádlág. Therefore, $u^n(t,\cdot) = v(t-,\cdot)$ would definitely exist and for any fixed t, $u^n(t,x) = v(t-,x)$ almost surely. In other words, $u^n(t,\cdot) = v(t-,\cdot)$ is cáglád (hence predictable) and satisfies (3.3).

If $u^0(x)$ is assumed to be smooth, then the first iterate $u^1(t,x)$ is immediately well defined. However, in order to make sense of $u^n(t,x)$ for any n, one needs to establish some essential regularity properties for u^{n-1}. The assumptions ($\mathbf{B}$.1)-($\mathbf{B}$.4) will be used for this purpose.

Lemma 3.1. *Let $h = h(s) = h(s,x)$ be a predictable process with trajectories in $L^2\big([0,T]; H^p(\mathbb{R}^d)\big)$, for all $p = 1, 2, \ \ldots$ and $h(s,\cdot) \in \mathcal{S}(\mathbb{R}^d)$. Furthermore, let*

$$V(t,x) = \int_{s=0}^{t} \int_{\mathbb{R}^d_y} G(t-s, x-y) h(s,y)\, dy\, ds$$

Then $V(t) = V(t,x)$ is a predictable process with paths in $L^2\big([0,T]; H^p(\mathbb{R}^d)\big) \cap C\big([0,T]; H^p(\mathbb{R}^d)\big)$. In particular,

$$\partial_{x_k} V(t,x) = \int_{s=0}^{t} \int_{\mathbb{R}^d_y} G(t-s, x-y) \partial_{y_k} h(s,y)\, dy\, ds.$$

Proof. The proof is a simple consequence of properties of convolution. □

In addition, for a predictable process $g(\cdot,\cdot;z) \in L^2([0,T]; H^p(\mathbb{R}^d))$ with $p = 1, 2, \ldots$, and

$$E\Big[\int_{s=0}^{T} \int_{\mathbb{R}^d_y} \int_{|z|>0} \big(g^2(s,y;z) + |D^n_y g(s,y;z)|^2\big)\, m(dz)\, dy\, ds\Big] < \infty,$$

for every $n \in \mathbb{N}$ and $g(s,\cdot,z) \in \mathcal{S}(\mathbb{R}^d)$, the quantity

$$N_L(t,x) = \int_{s=0}^{t} \int_{|z|>0} \int_{\mathbb{R}^d_y} G(t-s, x-y) g(s,y;z)\, dy\, \tilde{N}(dz, ds)$$

satisfies the following property for each $T > 0$.

Lemma 3.2. *The process $N_L(t,x) \in L^2([0,T]; H^p(\mathbb{R}^d))$ for $p = 1, 2, 3, \ldots$. Moreover, $N_L(t,x)$ is cádlág and stochastically continuous and hence admits predictable version. Furthermore,*

$$\partial_{x_k} N_L(t,x) = \int_{s=0}^{t} \int_{|z|>0} \int_{\mathbb{R}^d_y} G(t-s, x-y) \partial_{y_k} g(s,y;z)\, dy\, \tilde{N}(dz, ds) \tag{3.5}$$

and $N_L(t,\cdot) \in C^\infty(\mathbb{R}^d)$.

Proof. Once the representation (3.5) is established, the proof of the fact that $N_L(t,x) \in L^2([0,T]; H^p(\mathbb{R}^d))$ is a straightforward application of Itô-Lévy isometry. Moreover, the cádlág property of the right-hand side is the direct inheritance of being a stochastic integral, and the stochastic continuity is a direct consequence of the Itô-Lévy isometry.

In order to prove the representation (3.5), we have to show that the distributional derivative coincides with the right hand side. Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a test function. As a consequence of Fubini's theorem and Itô-Lévy isometry

$$\begin{aligned}
&E\Big[\Big|\int_{\mathbb{R}^d_x} \int_{s=0}^{t} \int_{|z|>0} \big\{ -G(t-s) *_x g)(s,x;z) \partial_{x_k} \varphi(x) \\
&\qquad\qquad - G(t-s) *_x (\partial_{x_k} g)(s,x;z) \varphi(x) \big\} \tilde{N}(dz, ds)\, dx \Big|^2\Big] \\
&= E\Big[\int_{s=0}^{t} \int_{|z|>0} \Big| \int_{\mathbb{R}^d_x} \big\{ \partial_{x_k} G(t-s) *_x g)(s,x;z) \varphi(x) \\
&\qquad\qquad - G(t-s) *_x (\partial_{x_k} g)(s,x;z) \varphi(x) \big\}\, dx \Big|^2\, m(dz)\, ds\Big] \\
&= 0,
\end{aligned}$$

where we have used the integration by parts along with properties of convolution. In the above, $*_x$ signifies convolution in x only. This representation shows that $\partial_{x_k} N_L(t,\cdot)$ has trajectories in $L^2([0,T]; L^2(\mathbb{R}^d))$ and it has a predictable version.

Replace g by $\partial_{x_k} g$, and repeat the above argument to conclude that N_L has trajectories in $L^2([0,T]; H^p(\mathbb{R}^d))$ and $N_L(t,\cdot) \in C^p(\mathbb{R}^d)$ for $p = 1, 2, \ldots$. □

Lemma 3.3. *$N_L(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ almost surely for all $t \in [0,T]$.*

Proof. From Lemma 3.2, we already know that $N_L(t,\cdot) \in C^\infty(\mathbb{R}^d)$. All we have to show is

$$\sup_{x\in\mathbb{R}^d} \big(|x|^n |N_L(t,x)|\big) < \infty \quad a.s.$$

for all $n \in \mathbb{N}$. On one hand, by Morrey's inequality there exists a universal constant $C > 0$ and $p > d$ such that

$$\sup_{x\in\mathbb{R}^d} \big(|x|^n |N_L(t,x)|\big) \le C||| \cdot |^n |N_L(t,\cdot)|||_{W^{1,p}},$$

for every positive integer n. On the other hand, direct computation reveals that, for $t > 0$, there exist n-th order polynomials $C_j(t)$ of t and a non-zero constant C_0 such that

$$t^n \partial^n_{x_k} G(t, x-y) = \big(C_0 x_k^n + C_1(t) x_k^{n-1} + \cdots + C_n(t)\big) G(t,x-y).$$

Therefore, by induction, it is sufficient to show that for all $j = 0, 1, \ldots, n$,

$$\begin{aligned}
&\Big\| \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_y} |t-s|^j \partial^n_{x_k} G(t-s, x-y) g(s,y;z)\, dy\, \tilde{N}(dz,ds) \Big\|_{W^{1,p}_x} \\
&\quad = \Big\| \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_y} |t-s|^j G(t-s,x-y) \partial^n_{y_k} g(s,y;z)\, dy\, \tilde{N}(dz,ds) \Big\|_{W^{1,p}_x} < \infty \quad a.s.,
\end{aligned}$$

for some $p > d$. It follows from the Sobolev inequality that if $\ell > \frac{d}{2}$ then there is $p > d$ such that

$$\| \cdot \|_{W^{1,p}} \le C \| \cdot \|_{H^{1+\ell}},$$

along with the fact that, for all $\ell \in \mathbb{N}$,

$$\begin{aligned}
&E\Big[\int_{\mathbb{R}^d_x} \Big| \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_y} |t-s|^j G(t-s,x-y) \partial^{\ell+1}_{y_k} g(s,y;z)\, dy\, \tilde{N}(dz,ds) \Big|^2 dx \Big] \\
&\quad = \int_{\mathbb{R}^d_x} \int_{s=0}^t \int_{|z|>0} E\Big[|t-s|^{2j} |G(t-s,\cdot) *_x \partial^{\ell+1}_{y_k} g(s,\cdot;z)|^2\, m(dz)\, ds\, dx \Big] \\
&\quad \le C \int_{s=0}^t \int_{|z|>0} E\Big[\big\| \partial^{\ell+1}_{y_k} g(s,y;z) \big\|_2^2 \Big]\, m(dz)\, ds < \infty.
\end{aligned}$$

In the above we have used Young's inequality for convolution. □

We finally conclude:

Lemma 3.4. *For each $n = 1, 2, \ldots$, the process $u^n(t,\cdot) \in L^2\big([0,T]; H^p(\mathbb{R}^d)\big)$ for $p = 1, 2, \ldots$ and $u^n(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$. Moreover, $u^n(t,\cdot)$ is stochastically continuous and has a cáglád (hence predictable) version.*

3.1. **Equivalence of mild, weak, and strong solutions.** It is well-known in the context of SPDEs governed by diffusions that, under moderate conditions, mild solutions coincide with weak solutions. For SPDEs driven by jump-diffusions, mild solutions can also be shown to coincide with weak solutions under moderate conditions. Moreover, Lemma 3.4 ensures that $u^n(t,x)$ has the sufficient smoothness to be the strong solution of (3.2). In our context, the next lemma states this fact. A detailed proof can be given, for example, by adapting the arguments given in [7].

Lemma 3.5. *For each $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$\begin{aligned}
&\langle u^n(t), \varphi\rangle - \langle u^n(0), \varphi\rangle \\
&\quad = \int_{s=0}^t \langle F(u^{n-1}(s)), \nabla\varphi\rangle ds + \int_{r=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_x} \eta(x, u^{n-1}(r,x); z) \varphi(x)\, dx\, \tilde{N}(dz,dr) \\
&\qquad + \varepsilon \int_{r=0}^t \langle \Delta\varphi, u^n(r)\rangle dr,
\end{aligned}$$

almost surely, for almost all $t \in [0,T]$.

As in Feng and Nualart [11], we also define the energy functional $e_{2r} : L^2(\mathbb{R}^d) \mapsto [0, \infty]$ as follows:

$$e_{2r}(u) = \frac{1}{2}||\Delta^r u||_2^2, \qquad r = 0, 1, 2, 3, \ldots.$$

Lemma 3.6. *There exists a finite constant $C_{\varepsilon,r,T} > 0$, independent of n, such that*

$$E\left[e_{2r}(u^n(t))\right] \le C_{\varepsilon,r,T}\left(1 + \sum_{k=0}^{r} E\left[e_{2k}(u_0)\right]\right), \qquad t \le T. \tag{3.6}$$

Proof. We have seen that, for each $n = 1, 2, 3, \ldots$, $u^n(t, \cdot) \in \mathcal{S}(\mathbb{R}^d)$, where u^n is defined by

$$\begin{aligned} u^n(t,x) = &\int_{\mathbb{R}^d_y} G(t, x-y) u_0(y)\, dy - \int_{s=0}^t \int_{\mathbb{R}^d_y} G(t-s, x-y) \sum_{i=1}^d \partial_{y_i} F_i(u^{n-1}(s,y))\, dy\, ds \\ &+ \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_y} G(t-s, x-y)\, \eta(y, u^{n-1}(s,y); z)\, dy\, \tilde{N}(dz, ds). \end{aligned}$$

As $u^n(t, \cdot) \in \mathcal{S}(\mathbb{R}^d)$, using a property of convolution, for $r = 0, 1, 2, 3, \ldots$, we have

$$\begin{aligned} \Delta^r u^n(t,x) = &\int_{\mathbb{R}^d_y} G(t, x-y) \Delta^r u_0(y)\, dy - \int_{s=0}^t \int_{\mathbb{R}^d_y} G(t-s, x-y) \Delta^r \Big(\sum_{i=1}^d \partial_{y_i} F_i(u^{n-1}(s,y))\Big)\, dy\, ds \\ &+ \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_y} G(t-s, x-y) \Delta^r \eta(y, u^{n-1}(s,y); z)\, dy\, \tilde{N}(dz, ds). \end{aligned}$$

Therefore, $\Delta^r u^n(t,x)$ solves the stochastic differential equation

$$d\Delta^r u^n(t,x) + \nabla \cdot \Delta^r F(u^{n-1}(t,x))dt = \varepsilon \Delta(\Delta^r u^n)(t,x)dt + \int_{|z|>0} \Delta^r \eta(x, u^{n-1}(t,x); z)\, \tilde{N}(dz, dt).$$

Now we apply the Itô-Lévy formula to the function $\phi(u) = u^2$, and integrate with respect to x, returning

$$\begin{aligned} &\int_{\mathbb{R}^d_x} |\Delta^r u^n(t,x)|^2\, dx \\ &= \int_{\mathbb{R}^d_x} |\Delta^r u_0(x)|^2\, dx + 2 \int_{\mathbb{R}^d_x} \int_{s=0}^t \nabla(\Delta^r u^n(s,x)) \cdot \Delta^r F(u^{n-1}(s,x))\, ds\, dx \\ &\qquad - 2\varepsilon \int_{\mathbb{R}^d_x} \int_{s=0}^t \nabla(\Delta^r u^n(s,x)) \cdot \nabla(\Delta^r u^n(s,x))\, ds\, dx \\ &\quad + \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_x} \Big[\Big(\Delta^r u^n(s,x) + \Delta^r \eta(x, u^{n-1}(s,x); z)\Big)^2 - (\Delta^r u^n(s,x))^2\Big]\, dx\, \tilde{N}(dz, ds) \\ &\quad + \int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_x} \Big[\Big(\Delta^r u^n(s,x) + \Delta^r \eta(x, u^{n-1}(s,x); z)\Big)^2 - (\Delta^r u^n(s,x))^2 \\ &\qquad - 2\Delta^r \eta(x, u^{n-1}(s,x); z) \Delta^r u^n(s,x)\Big]\, dx\, m(dz)\, ds. \end{aligned}$$

Taking expectation and using Cauchy's inequality, we obtain

$$\begin{aligned} E\Big[e_{2r}(u^n(t))\Big] \le &E\Big[e_{2r}(u^n(0)) + C_\varepsilon \int_{s=0}^t ||\Delta^r F(u^{n-1}(s))||_2^2\, ds\Big] \\ &+ E\Big[\int_{s=0}^t \int_{|z|>0} \int_{\mathbb{R}^d_x} |\Delta^r_x \eta(x, u^{n-1}(s,x); z)|^2\, dx\, m(dz)\, ds\Big]. \end{aligned}$$

Since F and η are smooth and $|F_k^r(s)| \le C_r$, for $r = 0, 1, 2, \ldots$, and $|D_x^r \eta(x, u; z)| \le K_r(x) \min(|z|, 1)$ for some $K_r(x) \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, there exists a finite constant $\tilde{C}_{\varepsilon,r,T} > 0$, independent of n, such that

$$E\Big[e_{2r}(u^n(t))\Big] \le E\Big[e_{2r}(u^n(0))\Big] + \tilde{C}_{\varepsilon,r,T}\Big(1 + \int_{s=0}^t \sum_{k=1}^r E\Big[e_{2k}(u^{n-1}(s))\Big]\, ds\Big).$$

Denote

$$M_n(t) = \left(1 + \sum_{k=0}^{r} E\Big[e_{2k}(u^n(t))\Big]\right), \quad M(0) = \left(1 + \sum_{k=0}^{r} E\Big[e_{2k}(u(0))\Big]\right).$$

Then, we have $M_n(t) \le CM(0) + C\int_{s=0}^{t} M_{n-1}(s)\,ds$, for some constant $C > 0$, which is independent of n. By induction on n, we conclude that there is a constant $K > 0$ such that $M_n(t) \le CM(0)e^{KT}$ for every $t \in [0,T]$. Therefore, (3.6) follows. □

We now show that u^n converges, in an appropriate sense, to a limiting process. This is done by a classical fixed point argument.

Lemma 3.7. *There exists a $L^2(\mathbb{R}^d)$-valued, $\mathcal{F}_t$-predictable (and cáglád) process u satisfying*

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T} ||u(t) - u^n(t)||_2\Big] = 0, \tag{3.7}$$

and for $\ell = 0, 1, 2, \dots$,

$$E\Big[e_{2\ell}\big(u(t)\big)\Big] \le C_{\varepsilon,\ell,T}\Big(1 + \sum_{0}^{\ell} E\Big[e_{2k}(u_0)\Big]\Big), \qquad t \le T. \tag{3.8}$$

In addition,

$$\sup_{0\le t\le T} E\Big[||u(t)||_p^p\Big] < \infty. \tag{3.9}$$

Furthermore, u is a mild solution of (3.1) in the following sense:

$$\begin{aligned} u(t,x) = \int_{\mathbb{R}^d_y} G(t, x-y)u_0(y)\,dy - \int_{s=0}^{t}\int_{\mathbb{R}^d_y} G(t-s, x-y)\sum_{i=1}^{d}\partial_{y_i}F_i(u(s,y))\,dy\,ds \\ + \int_{s=0}^{t}\int_{|z|>0}\int_{\mathbb{R}^d_y} G(t-s, x-y)\eta(y, u(s,y); z)\,dy\,\tilde{N}(dz, ds), \end{aligned} \tag{3.10}$$

almost surely, for every t.

Proof. Let $\mathbb{L}([0,T] : L^p(\mathbb{R}^d))$ be the space of cáglád and adapted $L^p(\mathbb{R}^d)$-valued processes on $[0,T]$. The distance function between two processes X and Y is defined as

$$|||X - Y|||_p^T = E\big[\sup_{0\le t\le T} ||X(t) - Y(t)||_{L^p}\big] \tag{3.11}$$

It is well-known (see [21, 22]) that the space $\mathbb{L}([0,T] : L^p(\mathbb{R}^d))$ equipped with the metric (3.11) is complete. By Lemma 3.4, it is easily seen that $u^n(t,\cdot) \in \mathbb{L}([0,T] : L^2(\mathbb{R}^d))$ and we want to show that $\{u^n(t,\cdot)\}_n$ converges in this space. At first, by direct integration,

$$\big\|\partial_{x_i}G(t,\cdot)\big\|_1 = \int_{\mathbb{R}^d_x} |\partial_{x_i}G(t,x)|dx = Ct^{\frac{-1}{2}}, \qquad \text{for } t > 0.$$

We denote

$$\mathcal{I}_1(u^n)(t,x) = \int_{s=0}^{t}\int_{\mathbb{R}^d_y} G(t-s, x-y)\sum_{i=1}^{d}\partial_{y_i}F_i(u^n(s,y))\,dy\,ds$$

$$\mathcal{I}_2(u^n)(t,x) = \int_{s=0}^{t}\int_{|z|>0}\int_{\mathbb{R}^d_y} G(t-s, x-y)\eta(y, u^n(s,y); z)\,dy\,\tilde{N}(dz, ds).$$

We define a deterministic measure on $[0,t]$ by

$$\gamma(ds) = \gamma_t(ds) = \big\|\partial_{x_i}G(t-s,.)\big\|_1\,ds = 2Cd(t^{\frac{1}{2}} - (t-s)^{\frac{1}{2}}).$$

Then

$$E\Big[\sup_{0\le s\le t}\big\|\mathcal{I}_1(u^n)(s,.) - \mathcal{I}_1(u^k)(s,.)\big\|_2\Big]$$

$$
\begin{aligned}
&= E\Big[\sup_{0\le s\le t}\Big(\int_{\mathbb{R}^d_x}\Big|\int_{r=0}^s\int_{\mathbb{R}^d_y}\sum_{i=1}^d\big[G(s-r,x-y)F_i'(u^n(r,y))\partial_{y_i}u^n(r,y) \\
&\qquad\qquad - G(s-r,x-y)F_i'(u^k(r,y))\partial_{y_i}u^k(r,y)\big]\,dr\,dy\Big|^2\,dx\Big)^{\frac12}\Big] \\
&\le C_1\sum_{i=1}^d E\left[\sup_{0\le s\le t}\Big(\int_{\mathbb{R}^d_x}\left|\int_{r=0}^s\big(|\partial_{x_i}G(s-r)|*|(u^n(r)-u^k(r)|\big)(x)\,dr\right|^2\,dx\Big)^{\frac12}\right] \\
&\le C_2\sum_{i=1}^d E\left[\sup_{0\le s\le t}\Big(\int_{r=0}^s\big\|\big(|\partial_{x_i}G(s-r)|*|(u^n(r)-u^k(r))|\big)\big\|_2\,dr\Big)\right] \\
&\le C_3E\left[\int_{r=0}^t\big\|u^n(r)-u^k(r)\big\|_2\,\gamma(dr)\right] \\
&\le C_4E\left[\sup_{0\le s\le t}\big\|u^n(s)-u^k(s)\big\|_2\right]\sqrt{t}.
\end{aligned}
$$

The first inequality follows from integration by parts and $|F_k^\ell(r)|\le C_\ell$, for $\ell=0,1,2,\ldots$, the second one follows from Minkowski inequality, while the third inequality follows from Young's inequality for convolutions. Therefore we obtain,

$$
E\Big[\sup_{0\le s\le t}\big\|\mathcal{I}_1(u^n)(s,.)-\mathcal{I}_1(u^k)(s,.)\big\|_2\Big]\le Ct^{\frac12}E\Big[\sup_{0\le s\le t}\big\|u^n(s)-u^k(s)\big\|_2\Big]. \tag{3.12}
$$

We want a similar estimate for $\mathcal{I}_2(u^n)$. This requires maximal inequalities for stochastic convolutions with respect to a compensated Poisson measure, and relevant results are available in [13, 18]:

$$
\begin{aligned}
&E\Big[\sup_{0\le s\le t}\big\|\mathcal{I}_2(u^n)(s,\cdot)-\mathcal{I}_2(u^k)(s,\cdot)\big\|_2\Big] \\
&\big(\text{by maximal inequality for stochastic convolution (see [13, Example 3.1, Prop 1.3])}\big) \\
&\le CE\Big[\Big(\int_{s=0}^t\int_{|z|>0}\int_{\mathbb{R}^d_y}|\eta(y,u^n(s,y);z)-\eta(y,u^k(s,y);z))|^2\,dy\,m(dz)\,ds\Big)^{\frac12}\Big] \\
&\le CE\Big[\Big(\int_{s=0}^t\int_{|z|>0}\int_{\mathbb{R}^d_y}|u^n(s,y)-u^k(s,y)|^2\,dy\min(1,|z|^2)\,m(dz)\,ds\Big)^{\frac12}\Big] \\
&\le C\sqrt{t}\,E\big[\sup_{0\le s\le t}||u^n(s)-u^k(s)||_2\big].
\end{aligned} \tag{3.13}
$$

Combine estimates (3.12) and (3.13), and use (3.3) to conclude that there exist numbers $\alpha\in(0,1)$ and $T_0>0$, which are independent of the initial condition u_0, such that

$$
|||u^n-u^k|||_2^{T_0}\le\alpha|||u^{n-1}-u^{k-1}|||_2^{T_0}.
$$

Hence, by the Banach fixed point argument, we have short time existence in $\mathbb{L}([0,T_0]:L^2(\mathbb{R}^d))$, and pasting the short time existence one can argue for the existence in $\mathbb{L}([0,T]:L^2(\mathbb{R}^d))$. In other words, we have shown the existence of a cáglád and adapted process u such that (3.7) holds. To conclude (3.8), we simply apply Fatou's lemma and let $n\to\infty$ in (3.6). In addition, (3.9) holds as a simple consequence of Sobolev embedding and (3.8). The mild solution property (3.10) is automatic once we note that u is a fixed point of the right-hand side of (3.3).

Remark. While the type of convergence in (3.7) is enough for our existence result, we also point out that in view of (3.9), it is easily seen that $\lim\limits_{n\to\infty}\sup_{0\le t\le T}E\Big[||u(t)-u^n(t)||_p\Big]=0$ for $p=2,3,\ldots$.

In view of Lemma 3.7, we pass to the limit $n\to\infty$ in Lemma 3.5. The result is

Lemma 3.8. *For each* $\varphi\in C_c^\infty(\mathbb{R}^d)$,

$$
\begin{aligned}
&\langle u(t),\varphi\rangle-\langle u(0),\varphi\rangle \\
&\quad=\int_{s=0}^t\langle F(u(s)),\nabla\varphi\rangle ds+\int_{r=0}^t\int_{|z|>0}\int_{\mathbb{R}^d_x}\eta(x,u(r,x);z)\varphi(x)\,dx\,\tilde N(dz,dr)+\varepsilon\int_{r=0}^t\langle\Delta\varphi,u(r)\rangle dr,
\end{aligned}
$$

almost surely, for almost every t.

□

Lemma 3.9. *Suppose that $E\big[e_{2\ell}(u_0)\big] < \infty$ for $2\ell \ge [\frac{d}{2}] + 3$, and let $u = u(t)$ be the limit process given by Lemma 3.7. Then $u = u(t) \in L^\infty_{loc}\big([0,\infty); L^2(\mathbb{R}^d)\big)$ and it is an $\mathcal{F}_t$-predictable (and cádlág) process that satisfies*

(1) *$e_{2\ell}\big(u(t)\big) < \infty$, for all $t > 0$.*
(2) *$\partial_{ij} u = \partial_{x_i\, x_j} u(t,\cdot) \in C(\mathbb{R}^d)$ for all $i,j = 1,\dots,d$.*

In other words, the SPDE (3.1) holds in the classical sense, i.e., (3.1) is satisfied as an one dimensional Lévy driven SDE for every fixed x.

Proof. The proof of (1) is immediate from Lemma 3.7. The proof of (2) is also immediate if we apply the Sobolev embedding [9] along with (1). □

3.2. **A priori estimates for $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$.** We need to approximate the functions $u_0(x), \eta$ and F from (**A**.1)-(**A**.4) by appropriate functions satisfying the assumptions (**B**.1)-(**B**.4) Let $J \in C_c^\infty(\mathbb{R})$ be a one dimensional mollifier and $\varphi \in C_c^\infty(\mathbb{R})$ be a cut-off function such that

$$\varphi(r) = \begin{cases} 0 & \text{for} \quad |r| \ge 2 \\ 1 & \text{for} \quad |r| \le 1. \end{cases}$$

For $\varepsilon > 0$, define the approximations F_ε, $\eta_\varepsilon(x,u;z)$, $u_0^\varepsilon(x)$ as follows:

$$\begin{aligned} F_\varepsilon(r) &= \varphi(\varepsilon|r|^2)F(r) * J_\varepsilon(r), \\ \eta_\varepsilon(x,u;z) &= \int_{\mathbb{R}^d_y}\int_{\mathbb{R}_v}\Big(\prod_{k=1}^d J_\varepsilon\big(x_k - y_k\big)J_\varepsilon(u-v)\Big)\varphi(\varepsilon(|y|^2+|v|^2))\eta(y,v;z)\,dv\,dy, \\ u_0^\varepsilon(x) &= \int_{\mathbb{R}^d_y}\Big(\prod_{k=1}^d J_\varepsilon\big(x_k-y_k\big)u_0(y)\varphi(\varepsilon|y|^2)\Big)\,dy. \end{aligned}$$

It follows from direct computations that

$$\begin{aligned} |F_\varepsilon(r) - F(r)| &\le C\varepsilon(1+|r|^{2p_0}) \quad \text{for some} \quad p_0 \in \mathbb{N}, \\ |\eta_\varepsilon(x,u;z) - \eta(x,u;z)| &\le C\varepsilon(1+|x|+|u|)(1\wedge|z|). \end{aligned} \tag{3.14}$$

Obviously, $u_0^\varepsilon(\cdot) \in C_c^\infty(\mathbb{R}^d)$ and for $p \ge 2$

$$\sup_{\varepsilon>0} E\Big[||u_0^\varepsilon(\cdot)||_p^p\Big] < +\infty.$$

Clearly, the functions F_ε and η_ε depend on ε and satisfy the regularity assumptions (**B**.1)-(**B**.3) Furthermore, we also have following facts:

(C.1) F_ε satisfies same conditions as F.
(C.2) $\sup_{\varepsilon>0}|\eta_\varepsilon(x,u;z)| \le g(x)(1+|u|)(1\wedge|z|)$ where $g \in L^\infty(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$.
(C.3) $\sup_{\varepsilon>0} E\Big[||u_0^\varepsilon||_p^p + ||u_0^\varepsilon||_2^p\Big] < \infty$, for $p = 1,2,\dots$.

We now focus on the equation

$$du_\varepsilon(t,x) + \mathrm{div}_x F_\varepsilon(u_\varepsilon(t,x))\,dt = \int_{|z|>0}\eta_\varepsilon(x,u_\varepsilon(t,x);z)\,\tilde{N}(dz,\,dt) + \varepsilon\Delta_{xx}u_\varepsilon(t,x)\,dt,\ \ t>0,\ \ x\in\mathbb{R}^d, \tag{3.15}$$

with initial condition $u_\varepsilon(0,x) = u_0^\varepsilon(x)$. Clearly, by Lemma 3.9, this problem possesses a unique strong solution $u_\varepsilon(t)$.

Lemma 3.10. *For even positive integers, $p = 2,4,6,\dots$,*

$$\sup_{\varepsilon>0}\sup_{0\le t\le T} E\Big[||u_\varepsilon(t,\cdot)||_p^p\Big] < \infty.$$

Proof. We already know that $\sup_{0\le t\le T} E\Big[||u_\varepsilon(t,\cdot)||_p^p\Big] < \infty$ for every $p \ge 2$ and $T > 0$. Let $\beta(u) = \frac{1}{p}|u|^p$, and apply the Itô-Lévy formula and integrate over the spatial variable x:

$$
\begin{aligned}
&E\Big[||u_\varepsilon(t)||_p^p\Big] - E\Big[||u_\varepsilon(0)||_p^p\Big] \\
&\quad \le p(p-1)\int_{s=0}^t E\Big[\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\lambda=0}^1 |u_\varepsilon(s,x)+\lambda\eta_\varepsilon(x,u_\varepsilon(s,x);z)|^{p-2}\eta_\varepsilon^2(x,u_\varepsilon(s,x);z)\,d\lambda\,m(dz)\,dx\Big]\,ds.
\end{aligned}
$$

We now use (C.1)-(C.3) and apply the Gronwall's inequality to arrive the conclusion

$$
\sup_{0\le t\le T} E\Big[||u_\varepsilon(t,\cdot)||_p^p\Big] \le C_T \sup_{\varepsilon>0} E\Big[||u_\varepsilon(0)||_p^p\Big],
$$

thereby proving the claim. ☐

Lemma 3.11. *For each $p = 1, 2, \ldots$,*

$$
\sup_{\varepsilon>0} E\Big[\big|\varepsilon\int_{s=0}^t\int_{\mathbb{R}^d_x} |\nabla_x u_\varepsilon(s,x)|^2\,dx\,ds\big|^p\Big] < \infty,
$$

for all $t > 0$.

Proof. Taking $p = 2$ in Lemma 3.10 gives

$$
\begin{aligned}
&||u_\varepsilon(t)||_2^2 - ||u_\varepsilon(0)||_2^2 \\
&= \int_{s=0}^t\int_{\mathbb{R}^d_x}\int_{|z|>0} \eta_\varepsilon^2(x,u_\varepsilon(s,x);z)\,m(dz)\,dx\,ds - 2\varepsilon\int_{s=0}^t\int_{\mathbb{R}^d} |\nabla_x u_\varepsilon(s,x)|^2 dx\,ds \\
&\quad + \int_{s=0}^t\int_{|z|>0}\int_{\mathbb{R}^d_x} \eta_\varepsilon(x,u_\varepsilon(s,x);z)(2u_\varepsilon(s,x)+\eta_\varepsilon(x,u_\varepsilon(s,x);z))\,dx\,\tilde{N}(dz,ds).
\end{aligned}
$$

We next apply the Itô-Lévy formula to $||u_\varepsilon(t)||_2^{2p}$ and use the moment estimates from Lemma 3.10 with (C.3) to obtain

$$
E\Big[||u_\varepsilon(t)||_2^{2p}\Big] + E\Big[||u_\varepsilon(0)||_2^{2p}\Big] < \infty.
$$

We apply the moment estimates once again and conclude

$$
\sup_{\varepsilon>0} E\Big[\big|\int_{s=0}^t\int_{\mathbb{R}^d_x}\int_{|z|>0} \eta_\varepsilon^2(x,u_\varepsilon(s,x);z)\,m(dz)\,ds\,dx\big|^p\Big] < \infty.
$$

We can now simply use (C.2) along with uniform moment estimates in Lemma 3.10 and apply the BDG inequality, perhaps more than once, to conclude

$$
\sup_{\varepsilon>0} E\Big[\Big|\int_{s=0}^t\int_{|z|>0}\int_{\mathbb{R}^d_x} \eta_\varepsilon(x,u_\varepsilon(s,x);z)\big(2u_\varepsilon(s,x)+\eta_\varepsilon(x,u_\varepsilon(s,x);z)\big)\,dx\,\tilde{N}(dz,ds)\Big|^p\Big] < \infty,
$$

and hence the proof follows. ☐

There is a generalized version of the above lemma.

Lemma 3.12. *Let $\beta \in C^2(\mathbb{R})$ be a function with β, β', β'' having at most polynomial growth. Then*

$$
\sup_{\varepsilon>0} E\left[\left|\varepsilon\int_{t=0}^T\int_{\mathbb{R}^d_x} \beta''(u_\varepsilon(t,x))|\nabla_x u_\varepsilon(t,x)|^2\,dx\,dt\right|^p\right] < \infty, \qquad p = 1, 2, \ldots, \quad T > 0. \tag{3.16}
$$

Proof. Let (β, ζ) be an entropy-entropy flux pair. Let $\psi_N \in C_c^2(\mathbb{R}^d)$ be such that

$$
\psi_N(x) = \begin{cases} 1, & \text{if } |x| \le N, \\ 0, & \text{if } |x| > N+1. \end{cases}
$$

By the Itô-Lévy formula, we have

$$
\begin{aligned}
&\langle \beta(u_\varepsilon(T,.)), \psi_N\rangle - \langle \beta(u_0^\varepsilon), \psi_N\rangle \\
&= \int_{r=0}^T \langle \zeta(u_\varepsilon(r,.)), \nabla_x\psi_N\rangle\,dr + \varepsilon\int_{r=0}^T \Big(\langle \beta(u_\varepsilon(r,.)), \Delta\psi_N\rangle - \langle \beta''(u_\varepsilon(r,.))|\nabla_x u_\varepsilon(r,.)|^2, \psi_N\rangle\Big)\,dr
\end{aligned}
$$

$$+\int_{r=0}^{T}\int_{|z|>0}\langle\beta(u_\varepsilon(r,.)+\eta_\varepsilon(.,u_\varepsilon(r,.);z))-\beta(u_\varepsilon(r,.)),\psi_N\rangle\,\tilde N(dz,dr)$$
$$+\int_{r=0}^{T}\int_{|z|>0}\langle\beta(u_\varepsilon(r,.)+\eta_\varepsilon(.,u_\varepsilon(r,.);z))-\beta(u_\varepsilon(r,.))-\eta_\varepsilon(.,u_\varepsilon(r,.);z)\beta'(u_\varepsilon(r,.)),\psi_N\rangle\,m(dz)\,dr.$$

Taking expectation and sending $N\to\infty$ result in

$$\begin{aligned}
&E\Big[\Big|\int_{r=0}^{T}\int_{\mathbb{R}^d_x}\varepsilon\beta''(u_\varepsilon(r,x))|\nabla_x u_\varepsilon(r,x)|^2\,dx\,dr\Big|^p\Big]\\
&\quad\le ||\beta(u_\varepsilon(T,\cdot))||_1^p+||\beta(u_0^\varepsilon(\cdot))||_1^p\\
&\qquad+C_1E\Big[\Big|\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\Big(\beta(u_\varepsilon(r,x)+\eta_\varepsilon(x,u_\varepsilon(r,x);z))-\beta(u_\varepsilon(r,x))\Big)\,dx\,\tilde N(dz,dr)\Big|^p\Big] \qquad (3.17)\\
&\qquad+C_2E\Big[\Big|\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\Big(\beta(u_\varepsilon(r,x)+\eta_\varepsilon(x,u_\varepsilon(r,x);z))-\beta(u_\varepsilon(r,x))\\
&\qquad\qquad-\eta_\varepsilon(x,u_\varepsilon(r,x);z)\beta'(u_\varepsilon(r,x))\Big)\,dx\,m(dz)\,dr\Big|^p\Big]. \qquad (3.18)
\end{aligned}$$

Since $|\eta_\varepsilon(x,u;z)|\le g(x)(1+|u|)(|z|\wedge 1)$, for some $g\in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, and β,β',β'' have at most polynomial growth and $\sup_{\varepsilon>0}\sup_{0\le t\le T}E\Big[||u_\varepsilon(t,\cdot)||_p^p\Big]<\infty$, the term (3.18) is finite.

Next, we want to estimate (3.17). Using the BDG inequality we obtain, for any $p\ge 2$,

$$\begin{aligned}
&\mathcal{I}(\varepsilon)\\
&:=E\Big[\Big|\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\Big(\beta(u_\varepsilon(r,x)+\eta_\varepsilon(x,u_\varepsilon(r,x);z))-\beta(u_\varepsilon(r,x))\Big)\,dx\,\tilde N(dz,dr)\Big|^p\Big]\\
&=E\Big[\Big|\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\lambda=0}^{1}\beta'(u_\varepsilon(r,x)+\lambda\eta_\varepsilon(x,u_\varepsilon(r,x);z))\eta_\varepsilon(x,u_\varepsilon(r,x);z)\,d\lambda\,dx\,\tilde N(dz,dr)\Big|^p\Big]\\
&\le CE\Big[\Big|\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\lambda=0}^{1}\beta'^2(u_\varepsilon(r,x)+\lambda\eta_\varepsilon(x,u_\varepsilon(r,x);z))\eta_\varepsilon^2(x,u_\varepsilon(r,x);z)\,d\lambda\,dx\,N(dz,dr)\Big|^{\frac{p}{2}}\Big].
\end{aligned}$$

Since $\tilde N(dz,dr)=N(dz,dr)-m(dz)\,dr$,

$$\begin{aligned}
\mathcal{I}(\varepsilon)\le&E\Big[\Big|\int_{r=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\lambda=0}^{1}\beta'^2(u_\varepsilon(r,x)+\lambda\eta_\varepsilon(x,u_\varepsilon(r,x);z))\\
&\qquad\times\eta_\varepsilon^2(x,u_\varepsilon(r,x);z)\,d\lambda\,dx\tilde N(dz,dr)\Big|^{\frac{p}{2}}\Big]+C.
\end{aligned}$$

Using the BDG inequality, perhaps repeatedly, we see that $\sup_{\varepsilon>0}\mathcal{I}(\varepsilon)<\infty$.

Finally, assuming $\beta(u)=C|u|^{2(\ell+1)}$ for some $\ell\in\mathbb{N}$, the above estimates imply

$$\sup_{\varepsilon>0}E\left[\left|\varepsilon\int_{t=0}^{T}\int_{\mathbb{R}^d_x}(u_\varepsilon(t,x))^{2\ell}|\nabla_x u_\varepsilon(t,x)|^2\,dx\,dt\right|^p\right]<\infty,\qquad p=1,2,\dots,\qquad T>0. \tag{3.19}$$

For general β with polynomially growing derivatives, there is $\ell\in\mathbb{N}$ such that $|\beta''(u)|\le C(1+u^{2\ell})$. We use this information along with (3.19) and Lemma 3.11 to conclude (3.16). □

The achieved results can be summarized into the following proposition:

Proposition 3.13. *Suppose assumptions* (**A**.1)-(**A**.4) *hold and fix any* $\varepsilon>0$. *Then there exists a unique* $C^2(\mathbb{R}^d)$*-valued predictable process* $u_\varepsilon(t,\cdot)$ *which solves* (3.1) *with initial data* $u_\varepsilon(0,x)=u_0^\varepsilon(x)$. *Moreover,*

1) *For even positive integers* $p=2,4,6,\dots,$

$$\sup_{\varepsilon>0}\sup_{0\le t\le T}E\Big[||u_\varepsilon(t,\cdot)||_p^p\Big]<+\infty. \tag{3.20}$$

2) *For $\phi \in C^2(\mathbb{R})$ with ϕ, ϕ', ϕ'' having at most polynomial growth,*

$$\sup_{\varepsilon>0} E\Big[\Big|\varepsilon \int_{t=0}^{T} \int_{\mathbb{R}^d_x} \phi''(u_\varepsilon(t,x)) |\nabla_x u_\varepsilon(t,x)|^2 \,dx\,dt\Big|^p\Big] < \infty, \quad p = 1, 2, \ldots, \quad T > 0. \tag{3.21}$$

4. Existence of generalized entropy solution

The proof of existence depends largely on the appropriate compactness of the family $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$. The moment estimates (3.20) only guarantee weak compactness, which is inadequate in view of the nonlinearities in the equation. Drawing inspiration from deterministic conservation laws, we look for compactness in the space of Young measures. We also mention here that similar strategies have been adopted by Bauzet, Vallet, and Wittbold [2] in the context of pure diffusion driven conservation laws. Before we proceed further, let us define the Young measures and the notion of narrow convergence. We refer to [5, 9] for more on the topic of Young measures in deterministic settings and to [1] for the stochastic version of the theory.

4.1. A few facts about Young measures.

Let (Θ, Σ, μ) be a σ-finite measure space and $\mathcal{P}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$.

Definition 4.1 (Young measure)**.** A Young measure from Θ into $\mathbb{R}$ is a map $\nu \mapsto \mathcal{P}(\mathbb{R})$ such that $\nu(\cdot) : \theta \mapsto \nu(\theta)(B)$ is Σ-measurable for every Borel subset B of $\mathbb{R}$. The set of all Young measures from Θ into $\mathbb{R}$ is denoted by $\mathcal{R}(\Theta, \Sigma, \mu)$ or simply by $\mathcal{R}$.

Definition 4.2 (narrow convergence)**.** A sequence of Young measures $\{\nu_n\}_n$ in $\mathcal{R}$ is said to converge *narrowly* to ν_0 iff for every $A \in \Sigma$ and $h \in C_b(\mathbb{R})$,

$$\lim_{n\to\infty} \int_A \Big[\int_{\mathbb{R}_\xi} h(\xi)\nu_n(\theta)(d\xi)\Big]\mu(d\theta) = \int_A \Big[\int_{\mathbb{R}_\xi} h(\xi)\nu_0(\theta)(d\xi)\Big]\,\mu(d\theta).$$

Remark. Young measures can be viewed as a parametrized family of probability measures where the parametrization is measurable. Clearly, if $u(\theta)$ is a real-valued measurable function on (Θ, Σ, μ), then $\nu(\theta) = \delta(\xi - u(\theta))$ defines a Young measure on Θ. In other words, with an appropriate choice of (Θ, Σ, μ), the family $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$ can be thought of as a family of Young measures and we are interested in extracting a subsequence which converges *narrowly* in $\mathcal{R}$. This requires setting up suitable tightness criterion.

Definition 4.3 (tightness)**.** A family of Young measures $\{\nu_n\}_n$ in $\mathcal{R}$ is called tight if there exists an inf-compact integrand h on $\Theta \times \mathbb{R}$ such that

$$\sup_n \int_\Theta \Big[\int_{\mathbb{R}_\xi} h(\theta,\xi)\nu_n(\theta)(d\xi)\Big]\,\mu(d\theta) < \infty.$$

Remark. Without getting into details about the whole class of inf-compact functions, we point out that $h(\theta, \xi) = \xi^2$ is one such example. With this choice of h and an appropriate choice of (Θ, Σ, μ), by (3.20) the family $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$ is tight when viewed as a family of Young measures.

The tightness condition enables us to extract a subsequence from a tight family and we have the following version of Prohorov's theorem to this end, a detailed proof which could be found in [1].

Theorem 4.1 (Prohorov's theorem)**.** *(1) Let (Θ, Σ, μ) be a finite measure space and $\{\nu_n\}_n$ be a tight family of Young measures in $\mathcal{R}$. Then there exists a subsequence $\{\nu_{n'}\}$ of $\{\nu_n\}_n$ and $\nu_0 \in \mathcal{R}$ such that $\{\nu_{n'}\}$ converges narrowly to ν_0.*

(2) Moreover, if $\nu_n = \delta_{f_n(\theta)}(\xi)$ and given a Caratheodory function $h(\theta, \xi)$ on $\Theta \times \mathbb{R}$, if $h(\theta, f_{n'}(\theta))$ is uniformly integrable, then

$$\lim_{n'\to\infty} \int_\Theta h(\theta, f_{n'}(\theta))\,\mu(d\theta) = \int_\Theta \Big[\int_{\mathbb{R}_\xi} h(\theta,\xi)\,\nu_0(\theta)(d\xi)\Big]\,\mu(d\theta).$$

4.2. **Extraction of an inviscid Young measure limit.** The predictable σ-field of $\Omega \times (0,T)$ with respect to $\{\mathcal{F}_t\}$ is denoted by $\mathcal{P}_T$, and we set

$$\Theta = \Omega \times (0,T) \times \mathbb{R}^d, \quad \Sigma = \mathcal{P}_T \times \mathcal{L}(\mathbb{R}^d) \quad \text{and} \quad \mu = P \otimes \lambda_t \otimes \lambda_x,$$

where λ_t and λ_x are respectively the Lebesgue measures on $(0,T)$ and $\mathbb{R}^d$. Moreover, for $M \in \mathbb{N}$, let

$$\Theta_M = \Omega \times (0,T) \times B_M, \quad \Sigma_M = \mathcal{P}_T \times \mathcal{L}(B_M) \quad \text{and} \quad \mu_M = \mu\big|_{\Theta_M},$$

where B_M is the ball of radius M around zero in $\mathbb{R}^d$ and $\mathcal{L}(B_M)$ is the Lebesgue sigma algebra on B_M. Clearly $(\Theta_M, \Sigma_M, \mu_M)$ is a finite measure space and $\{u_\varepsilon(\omega;t,x)\}_{\varepsilon>0}$ is a tight family of Young measures in $\mathcal{R}(\Theta_M, \Sigma_M, \mu_M)$. Therefore by Theorem 4.1 there exists a subsequence $\varepsilon_n \to 0$ and $\nu^M \in \mathcal{R}(\Theta_M, \Sigma_M, \mu_M)$ such that $\{u_{\varepsilon_n}(\omega;t,x)\}$ converges narrowly to ν^M.

Furthermore, for $\bar{M} > M$, the sequence $\{u_{\varepsilon_n}(\omega;t,x)\}$ is tight in $\mathcal{R}(\Theta_{\bar{M}}, \Sigma_{\bar{M}}, \mu_{\bar{M}})$, and hence admits a further subsequence, say $\{u_{\varepsilon_{n'}}(\omega;t,x)\}$, and $\nu^{\bar{M}} \in \mathcal{R}(\Theta_{\bar{M}}, \Sigma_{\bar{M}}, \mu_{\bar{M}})$ such that $\{u_{\varepsilon_{n'}}(\omega;t,x)\}$ converges narrowly to $\nu^{\bar{M}}$. We now invoke diagonalization and conclude that there exist a subsequence $\{u_{\varepsilon_{n'}}(\omega;t,x)\}$ with $\varepsilon_n \to 0$ and a Young mesures $\nu^M \in \mathcal{R}(\Theta_M, \Sigma_M, \mu_M)$, $M = 1,2,3,\ldots$, such that $\{u_{\varepsilon_n}(\omega;t,x)\}$ converges narrowly to ν^M in $\mathcal{R}(\Theta_M, \Sigma_M, \mu_M)$ for every $M = 1,2,\ldots$. It is also trivial to prove that

$$\text{if } \bar{M} > M \text{ then } \nu^M = \nu^{\bar{M}} \text{ on } (\Theta_M, \Sigma_M, \mu).$$

Now we define

$$\nu(\theta) = \nu^M(\theta) \text{ if } \theta \in \Theta_M. \tag{4.1}$$

Clearly, ν is well defined and ν is a Young measure belonging to $\mathcal{R}(\Theta, \Sigma, \mu)$.

We summarize the findings in a next lemma.

Lemma 4.2. *Let $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$ be a sequence of $L^p(\mathbb{R}^d)$-valued predictable processes such that* (3.20) *holds. Then there exists a subsequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ and a Young measure $\nu \in \mathcal{R}(\Theta, \Sigma, \mu)$ such that if $h(\theta,\xi)$ is a Caratheodory function on $\Theta \times \mathbb{R}$ such that supp$(h) \subset \Theta_M \times \mathbb{R}$ for some $M \in \mathbb{N}$ and $\{h(\theta, u_{\varepsilon_n}(\theta)\}_n$ (where $\theta \equiv (\omega;t,x)$) is uniformly integrable, then*

$$\lim_{\varepsilon_n \to 0} \int_\Theta h(\theta, u_{\varepsilon_n}(\theta))\,\mu(d\theta) = \int_\Theta \Big[\int_{\mathbb{R}_\xi} h(\theta,\xi)\,\nu(\theta)(d\xi)\Big]\,\mu(d\theta).$$

Proof. The extraction of a subsequence is done as described above and ν is defined in (4.1). Note that if $M \in \mathbb{N}$ such that supp$(h) \subset \Theta_M \times \mathbb{R}$, then

$$\int_\Theta h(\theta, u_{\varepsilon_n}(\theta))\,\mu(d\theta) = \int_{\Theta_M} h(\theta, u_{\varepsilon_n}(\theta))\,\mu_M(d\theta)$$

and

$$\int_\Theta \Big[\int_{\mathbb{R}_\xi} h(\theta,\xi)\,\nu(\theta)(d\xi)\Big]\,\mu(d\theta) = \int_{\Theta_M} \Big[\int_{\mathbb{R}_\xi} h(\theta,\xi)\,\nu^M(\theta)(d\xi)\Big]\,\mu_M(d\theta),$$

and the convergence follows from Theorem 4.1. □

4.3. **Construction of a generalized entropy solution.** With the Young measure valued limit ν of $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$ (upto a subsequence) at hand, we follow the standard recipe of Panov [20] (and its adaptation to a stochastic case [2]) to turn it into a generalized entropy solution. Define the real valued function $u(\theta,\lambda)$ by

$$u(\theta,\lambda) = \inf\Big\{c \in \mathbb{R} : \nu(\theta)\Big((-\infty,c)\Big) > \lambda\Big\}, \qquad \text{for } \lambda \in (0,1) \text{ and } \theta \in \Theta.$$

Lemma 4.3. *For fixed $\theta \in \Theta$, the function $u(\theta,\cdot)$ is non-decreasing and right-continuous on* $(0,1)$*. Moreover, if $h(\theta,\xi)$ is a nonnegative Caratheodory function on $\Theta \times \mathbb{R}$, then*

$$\int_\Theta \Big[\int_{\mathbb{R}_\xi} h(\theta,\xi)\,\nu(\theta)(d\xi)\Big]\,\mu(\,d\theta) = \int_\Theta \int_{\lambda=0}^1 h(\theta, u(\theta,\lambda))\,d\lambda\,\mu(\,d\theta).$$

Proof. The proof is classical, and we refer to [[20] Lemma 3.1] for the details. □

Any prospective generalized entropy solution has to be predictable. The presence of Lévy noise makes this condition indispensable. The next lemma affirms that condition for $u(\omega;t,x,\lambda)$.

Lemma 4.4. *u is $\mathcal{P}_T \times \mathcal{L}(\mathbb{R}^d \times (0,1))$ measurable.*

Proof. We establish that u satisfies the basic condition of measurability. Let $\sigma \in \mathbb{R}$ and $E_\sigma = \{(\theta,\lambda) : u(\theta,\lambda) < \sigma\}$. We want to show that $E_\sigma \in \mathcal{P}_T \times \mathcal{L}(\mathbb{R}^d \times (0,1))$. Let $H_\sigma = \{(\theta,\lambda) : \nu(\theta)\big((-\infty,\sigma)\big) > \lambda\}$. For $(\theta,\lambda) \in E_\sigma$, it holds that $u(\theta,\lambda) < \sigma$ i.e., there exits c with $u(\theta,\lambda) < c < \sigma$ such that $\nu(\theta)\big((-\infty,c)\big) > \lambda$ and hence $\nu(\theta)\big((-\infty,\sigma)\big) > \lambda$, implying $E_\sigma \subset H_\sigma$. For the converse, let $(\theta,\lambda) \in H_\sigma$.

Note that the map $\sigma \mapsto \nu(\theta)\big((-\infty,\sigma)\big)$ is left continuous and therefore $\nu(\theta)\big((-\infty,\sigma)\big) > \lambda$ implies that there exists $c < \sigma$ such that $\nu(\theta)\big((-\infty,c)\big) > \lambda$. Thus, by the definition of u, $u(\theta,\lambda) < \sigma$ and hence $H_\sigma \subset E_\sigma$, implying $H_\sigma = E_\sigma$. Note that $\theta \mapsto \nu(\theta)\big((-\infty,\sigma)\big)$ is Σ-measurable, implying $H_\sigma \in \mathcal{P}_T \times \mathcal{L}(\mathbb{R}^d \times (0,1))$ for all $\sigma \in \mathbb{R}$. □

Let $\Gamma = \Omega \times [0,T] \times \mathbb{R}$, $\mathcal{G} = \mathcal{P}_T \times \mathcal{L}(\mathbb{R})$ and $\varsigma = P \otimes \lambda_t \otimes m(dz)$. Then $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ consists of all square integrable predictable processes which are Lebesgue measurable functions of z-variable. In other words, if $\psi(t,z) \in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$, then

$$E\Big[\int_{t=0}^T \int_{\mathbb{R}_z} \psi^2(t,z)\, m(dz)\, dt\Big] < +\infty.$$

The space $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ represents the square integrable predictable integrands for Itô-Lévy integrals with respect to the compensated Poisson random measure $\tilde{N}(dz,dt)$. Moreover, an Itô-Lévy integral defines a linear operator from $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ to $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$ and it preserves the norm, thanks to the Itô-Lévy isometry. Furthermore, for any random variable $Y \in L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$ we can invoke the *martingale representation theorem* for marked point processes and conclude that there exists $\psi \in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ such that

$$Y = \int_{t=0}^T \int_{\mathbb{R}_z} \psi(t,z)\, \tilde{N}(dz,dt).$$

Hence, the Itô-Lévy integral operator is an isometry from $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ onto $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. To this end, note that any isometry between two Hilbert spaces preserves weak convergence. Therefore for any weakly converging sequence of integrands $\{\psi_n(t,z)\} \in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$, the corresponding sequence of Itô-Lêvy integrals with respect to $\tilde{N}(dz,dt)$ will also converge weakly in $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. Moreover, the weak limits are preserved under Itô-Lévy integral operators. To see this, define

$$\chi_n(t,z) = \int_{\mathbb{R}_x^d} \Big(\beta\big(u_{\varepsilon_n}(t,x) + \eta(x,u_{\varepsilon_n}(t,x);z) - \beta\big(u_{\varepsilon_n}(t,x)\big)\Big)\phi(t,x)\, dx,$$

where β is a smooth function with bounded derivatives and ϕ is a compactly supported smooth function on Π_T. Then clearly $\chi_n(t,z) \in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ and the sequence $\{\chi_n(t,z)\}_n$ is bounded in $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$.

We have the following lemma:

Lemma 4.5. *The sequence $\{\chi_n(t,z)\}_n$ is weakly convergent in $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ and the weak limit $\chi(t,z)$ is given by*

$$\chi(t,z) = \int_{\mathbb{R}_x^d} \int_{\alpha=0}^1 \Big(\beta\big(u(t,x,\alpha) + \eta(x,u(t,x,\alpha);z) - \beta\big(u(t,x,\alpha)\big)\Big)\phi(t,x)\, d\alpha\, dx.$$

Proof. Fix $h(t,z) \in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$. Then

$$\begin{aligned} &E\Big[\int_{t=0}^T \int_{\mathbb{R}_z} h(t,z)\chi_n(t,z)\, m(dz)\, dt\Big] \\ &\quad = \int_{\mathbb{R}_z} E\Big[\int_{t=0}^T \int_{\mathbb{R}_x^d} \Big(\beta\big(u_{\varepsilon_n}(t,x) + \eta(x,u_{\varepsilon_n}(t,x);z) - \beta\big(u_{\varepsilon_n}(t,x)\big)\Big)\phi(t,x)h(t,z)\, dx\, dt\Big]\, m(dz). \end{aligned}$$

For $m(dz)$-almost every $z \in \mathbb{R}$, we apply Lemmas 4.2 and 4.3 and conclude that

$$\begin{aligned} &\lim_{n\to\infty} E\Big[\int_{t=0}^T \int_{\mathbb{R}_x^d} \Big(\beta\big(u_{\varepsilon_n}(t,x) + \eta(x,u_{\varepsilon_n}(t,x);z) - \beta\big(u_{\varepsilon_n}(t,x)\big)\Big)\phi(t,x)h(t,z)\, dx\, dt\Big] \\ &\quad = E\Big[\int_{t=0}^T \int_{\mathbb{R}_x^d} \int_{\alpha=0}^1 \Big(\beta\big(u(t,x,\alpha) + \eta(x,u(t,x,\alpha);z) - \beta\big(u(t,x,\alpha)\big)\Big)\phi(t,x)h(t,z)\, d\alpha\, dx\, dt\Big]. \end{aligned}$$

We now invoke (**A**.4) and uniform moment estimates in order to apply the bounded convergence theorem, with the result that

$$\begin{aligned}
&\lim_{n\to\infty} E\Big[\int_{t=0}^{T}\int_{\mathbb{R}_z} h(t,z)\chi_n(t,z)\, m(dz)\, dt\Big] \\
&\quad = E\Big[\int_{t=0}^{T}\int_{\mathbb{R}_z}\Big[\int_{\mathbb{R}_x^d}\int_{\alpha=0}^{1}\Big(\beta\big(u(t,x,\alpha)+\eta(x,u(t,x,\alpha);z)\big) \\
&\qquad\qquad - \beta\big(u(t,x,\alpha)\big)\Big)\phi(t,x)h(t,z)\, d\alpha\, dx\Big]\, m(dz)\, dt\Big].
\end{aligned}$$

This completes the proof. □

As a consequence of the discussion prior to Lemma 4.5, the Itô-Lévy integrals $\int_{t=0}^{T}\int_{\mathbb{R}_z}\chi_n(t,z)\,\tilde N(dz,dt)$ converges weakly to $\int_{t=0}^{T}\int_{\mathbb{R}_z}\chi(t,z)\,\tilde N(dz,dt)$ in $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. Hence, we have the following

Corollary 4.6. *For every* $B\in\mathcal{F}_T$,

$$\begin{aligned}
&\lim_{n\to\infty} E\Big[\mathbf{1}_B\int_{t=0}^{T}\int_{\mathbb{R}_z}\int_{\mathbb{R}_x^d}\Big(\beta\big(u_{\varepsilon_n}(t,x)+\eta(x,u_{\varepsilon_n}(t,x);z)-\beta\big(u_{\varepsilon_n}(t,x)\big)\Big)\phi(t,x)\, dx\,\tilde N(dz,dt)\Big] \\
&= E\Big[\mathbf{1}_B\int_{t=0}^{T}\int_{\mathbb{R}_z}\int_{\mathbb{R}_x^d}\int_{\alpha=0}^{1}\Big(\beta\big(u(t,x,\alpha)+\eta(x,u(t,x,\alpha);z)-\beta\big(u(t,x,\alpha)\big)\Big)\phi(t,x)\, d\alpha\, dx\,\tilde N(dz,dt)\Big].
\end{aligned}$$

Proof. The proof is obvious in view of the above discussion as $\mathbf{1}_B\in L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. □

At this point we fix a nonnegative test function $\psi\in C_c^\infty([0,\infty)\times\mathbb{R}^d)$, $B\in\mathcal{F}_T$, and a convex entropy pair (β,ζ). Let ζ_ε be the entropy flux based on F_ε, and thus ζ_ε is approximating ζ. We use the Itô-Lévy formula to compute $\beta(u_\varepsilon(t,x))$, apply the product rule to $\psi(t,x)\beta(u_\varepsilon(t,x))$, and then integrate. The result is

$$\begin{aligned}
0 \le\ & E\Big[\mathbf{1}_B\int_{\mathbb{R}_x^d}\beta(u_0^{\varepsilon_n}(x))\psi(0,x)\, dx\Big] - \varepsilon_n E\Big[\mathbf{1}_B\int_{\Pi_T}\beta'(u_{\varepsilon_n}(t,x))\nabla u_{\varepsilon_n}(t,x).\nabla\psi(t,x)\, dx\, dt\Big] \\
&+ E\Big[\mathbf{1}_B\int_{\Pi_T}\Big(\beta(u_{\varepsilon_n}(t,x))\partial_t\psi(t,x)+\zeta_{\varepsilon_n}(u_{\varepsilon_n}(t,x))\cdot\nabla\psi(t,x)\Big)\, dt\, dx\Big] \\
&+ E\Big[\mathbf{1}_B\int_{|z|>0}\int_{\Pi_T}\Big(\beta(u_{\varepsilon_n}(t,x)+\eta_{\varepsilon_n}(x,u_{\varepsilon_n}(t,x);z))-\beta(u_{\varepsilon_n}(t,x))\Big)\psi(t,x)\, dx\,\tilde N(dz,dt)\Big] \\
&+ E\Big[\mathbf{1}_B\int_{|z|>0}\int_{\Pi_T}\Big(\beta(u_{\varepsilon_n}(t,x)+\eta_{\varepsilon_n}(x,u_{\varepsilon_n}(t,x);z))-\beta(u_{\varepsilon_n}(t,x)) \\
&\qquad - \eta_{\varepsilon_n}(x,u_{\varepsilon_n}(t,x);z)\beta'(u_{\varepsilon_n}(t,x))\Big)\psi(t,x)\, dx\, m(dz)\, dt\Big]. \qquad (4.2)
\end{aligned}$$

With the help of uniform moment estimates and (3.14), it follows from (4.2) that

$$\begin{aligned}
0 \le\ & E\Big[\mathbf{1}_B\int_{\mathbb{R}_x^d}\beta(u_0^{\varepsilon_n}(x))\psi(0,x)\, dx\Big] - \varepsilon_n E\Big[\mathbf{1}_B\int_{\Pi_T}\beta'(u_{\varepsilon_n}(t,x))\nabla u_{\varepsilon_n}(t,x).\nabla\psi(t,x)\, dx\, dt\Big] \\
&+ E\Big[\mathbf{1}_B\int_{\Pi_T}\Big(\beta(u_{\varepsilon_n}(t,x))\partial_t\psi(t,x)+\zeta(u_{\varepsilon_n}(t,x))\cdot\nabla\psi(t,x)\Big)\, dt\, dx\Big] \\
&+ E\Big[\mathbf{1}_B\int_{t=0}^{T}\int_{|z|>0}\int_{\mathbb{R}_x^d}\Big(\beta(u_{\varepsilon_n}(t,x)+\eta(x,u_{\varepsilon_n}(t,x);z))-\beta(u_{\varepsilon_n}(t,x))\Big)\psi(t,x)\, dx\,\tilde N(dz,dt)\Big] \\
&+ E\Big[\mathbf{1}_B\int_{t=0}^{T}\int_{|z|>0}\int_{\mathbb{R}_x^d}\Big(\beta(u_{\varepsilon_n}(t,x)+\eta(x,u_{\varepsilon_n}(t,x);z)) \\
&\qquad - \beta(u_{\varepsilon_n}(t,x))-\eta(x,u_{\varepsilon_n}(t,x);z)\beta'(u_{\varepsilon_n}(t,x))\Big)\psi(t,x)\, dx\, dt\, m(dz)\Big] + o(\varepsilon_n). \qquad (4.3)
\end{aligned}$$

We now pass to the limit $\varepsilon_n\to 0$ in (4.3). Thanks to (3.21),

$$\lim_{\varepsilon_n\to 0}\varepsilon_n E\Big[\mathbf{1}_B\int_{\Pi_T}\beta'(u_{\varepsilon_n}(t,x))\nabla u_{\varepsilon_n}(t,x)\cdot\nabla\psi(t,x)\, dx\, dt\Big] = 0. \qquad (4.4)$$

Moreover, it is straightforward to see that

$$\lim_{\varepsilon_n \to 0} E\Big[\mathbf{1}_B \int_{\mathbb{R}^d_x} \beta(u_0^{\varepsilon_n}(x))\psi(0,x)\,dx\Big] = E\Big[\mathbf{1}_B \int_{\mathbb{R}^d_x} \beta(u_0(x))\psi(0,x)\,dx\Big]. \tag{4.5}$$

We now recall that $L^2\big(\Theta,\Sigma,\mu\big)$ is closed subspace of the larger space $L^2\big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\big)$, and hence weak convergence in $L^2\big(\Theta,\Sigma,\mu\big)$ would imply weak convergence in $L^2\big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\big)$.

In addition, for $B \in \mathcal{F}_T$, the functions $\mathbf{1}_B\partial_t\psi(t,x), \mathbf{1}_B\partial_{x_i}\psi(t,x), \mathbf{1}_B\psi(t,x)$ are all members of $L^2\big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\big)$, and hence by Lemmas 4.2 and 4.3, we have

$$\begin{aligned}
&\lim_{\varepsilon_n \to 0} E\Big[\mathbf{1}_B \int_{\Pi_T} \Big(\beta(u_{\varepsilon_n}(t,x))\partial_t\psi(t,x) + \zeta(u_{\varepsilon_n}(t,x))\cdot\nabla\psi(t,x)\Big)\,dt\,dx\Big] \\
&\quad = E\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\alpha=0}^1 \Big(\beta(u(t,x,\alpha))\partial_t\psi(t,x) + \zeta(u(t,x,\alpha)).\nabla\psi(t,x)\Big)\,d\alpha\,dt\,dx\Big],
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\lim_{\varepsilon_n \to 0} E\Big[\mathbf{1}_B \int_{\Pi_T}\int_{|z|>0} \Big(\beta(u_{\varepsilon_n}(t,x) + \eta(x,u_{\varepsilon_n}(t,x);z)) - \beta(u_{\varepsilon_n}(t,x)) \\
&\qquad\qquad - \eta(x,u_{\varepsilon_n}(t,x);z)\beta'(u_{\varepsilon_n}(t,x))\Big)\psi(t,x)\,m(dz)\,dt\,dx\Big] \\
&= E\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\alpha=0}^1\int_{|z|>0} \Big(\beta(u(t,x,\alpha) + \eta(x,u(t,x,\alpha);z)) - \beta(u(t,x,\alpha)) \\
&\qquad\qquad - \eta(x,u(t,x,\alpha);z)\beta'(u(t,x,\alpha))\Big)\psi(t,x)\,m(dz)\,d\alpha\,dt\,dx\Big].
\end{aligned} \tag{4.7}$$

Now, combining (4.4)-(4.7) along with Corollary 4.6, passing to the limit $\varepsilon_n \downarrow 0$ in (4.3) gives

$$\begin{aligned}
0 &\le E\Big[\mathbf{1}_B \int_{\mathbb{R}^d_x} \beta(u_0(x))\psi(0,x)\,dx\Big] \\
&+ E\Big[\mathbf{1}_B \int_{\Pi_T}\int_{\alpha=0}^1 \Big(\beta(u(t,x,\alpha))\partial_t\psi(t,x) + \zeta(u(t,x,\alpha))\cdot\nabla\psi(t,x)\Big)\,d\alpha\,dt\,dx\Big] \\
&+ E\Big[\mathbf{1}_B \int_{\Pi_T}\int_{|z|>0}\int_{\alpha=0}^1 \Big(\beta(u(t,x,\alpha) + \eta(x,u(t,x,\alpha);z)) - \beta(u(t,x,\alpha))\Big)\psi(t,x)\,d\alpha\,\tilde{N}(dz,dt)\,dx\Big] \\
&+ E\Big[\mathbf{1}_B \int_{\Pi_T}\int_{|z|>0}\int_{\alpha=0}^1 \Big(\beta(u(t,x,\alpha) + \eta(x,u(t,x,\alpha);z)) - \beta(u(t,x,\alpha)) \\
&\qquad\qquad - \eta(x,u(t,x,\alpha);z)\beta'(u(t,x,\alpha))\Big)\psi(t,x)d\alpha\,m(dz)\,dt\,dx\Big].
\end{aligned} \tag{4.8}$$

Proof of the Theorem 2.1. The predictability of $u(t,\cdot,\cdot)$ follows from Lemma 4.4, and the uniform moment estimate together with the classical Fatou lemma, which gives

$$\sup_{0\le t\le T} E\Big[||u(t,\cdot,\cdot)||_p^p\Big] < \infty \qquad \text{for } p = 2,3,4,\ldots.$$

For any $0 \le \psi \in C_c^\infty([0,\infty)\times\mathbb{R}^d)$ and any convex entropy flux-pair (β,ζ), (4.8) holds for each set $B \in \mathcal{F}_T$. Hence

$$\begin{aligned}
&\int_{\mathbb{R}^d_x} \beta(u_0(x))\psi(0,x)\,dx + \int_{\Pi_T}\int_{\lambda=0}^1 \Big(\beta(u(t,x,\lambda))\partial_t\psi(t,x) + \zeta(u(t,x,\lambda))\cdot\nabla\psi(t,x)\Big)\,d\lambda\,dt\,dx \\
&\quad + \int_{\Pi_T}\int_{|z|>0}\int_{\lambda=0}^1 \Big(\beta\big(u(t,x,\lambda) + \eta(x,u(t,x,\lambda);z)\big) - \beta(u(t,x,\lambda)) \\
&\qquad\qquad - \eta(x,u(t,x,\lambda);z)\beta'(u(t,x,\lambda))\Big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx \\
&\quad + \int_{\Pi_T}\int_{|z|>0}\int_{\lambda=0}^1 \Big(\beta\big(u(t,x,\lambda) + \eta(x,u(t,x,\lambda);z)\big) - \beta(u(t,x,\lambda))\Big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx \\
&\ge 0 \quad P\text{-a.s.},
\end{aligned}$$

which completes the proof. □

5. Uniqueness and existence of entropy solutions

A natural strategy for proving uniqueness in the presence of noise is to adapt the Kružkov approach for deterministic equations. The main difficulty lies in "doubling" the time variable, which gives rise to stochastic integrands that are anticipative and hence cannot be interpreted in the usual Itô sense. One way to get around this problem seems to be through the vanishing viscosity regularization. For conservation laws with Brownian white noise, there are two routes based on this strategy. The first one is by introducing the so called *strong entropy condition* (see [3, 4, 11]) and then showing that the vanishing viscosity limit obeys this condition. The other one uses a more direct approach (see [2]) by comparing the entropy solution against the solution of the viscous problem and subsequently sending the viscosity parameter to zero, relying on "weak compactness" of the viscous approximations. In the presence of Lévy noise, the paths of the solution are discontinuous and the Feng-Nualart strategy of introducing a "strong entropy condition" has proven difficult to implement. However, as it will be detailed in the sequel, the approach of directly comparing an entropy solution against that of a weakly converging sequence of viscous approximations is successful.

Let ρ and ϱ be the standard nonnegative mollifiers on $\mathbb{R}$ and $\mathbb{R}^d$ respectively such that supp $(\rho) \subset [-1,0]$ and supp $(\varrho) = B_1(0)$. We define $\rho_{\delta_0}(r) = \frac{1}{\delta_0}\rho(\frac{r}{\delta_0})$ and $\varrho_\delta(x) = \frac{1}{\delta^d}\varrho(\frac{x}{\delta})$, where δ and δ_0 are two positive constants. Given a nonnegative test function $\psi \in C_c^{1,2}([0,\infty)\times\mathbb{R}^d)$ and two positive constants δ and δ_0, we define

$$\phi_{\delta,\delta_0}(t,x,s,y) = \rho_{\delta_0}(t-s)\varrho_\delta(x-y)\psi(s,y).$$

Clearly $\rho_{\delta_0}(t-s) \neq 0$ only if $s-\delta_0 \le t \le s$ and hence $\phi_{\delta,\delta_0}(t,x;s,y) = 0$ outside $s-\delta_0 \le t \le s$.

Let $v(t,x,\alpha)$ be a generalized entropy solution of (1.2). Moreover, let ς be the standard symmetric nonnegative mollifier on $\mathbb{R}$ with support in $[-1,1]$ and $\varsigma_l(r) = \frac{1}{l}\varsigma(\frac{r}{l})$ for $l>0$. We use the generic β for the functions β_ϑ introduced in Section 2. Given $k\in\mathbb{R}$, the function $\beta(\cdot-k)$ is a smooth convex function and $(\beta(\cdot-k), F^\beta(\cdot,k))$ is a convex entropy pair.

We now write the entropy inequality for $v(t,x,\alpha)$, based on the entropy pair $(\beta(\cdot-k), F^\beta(\cdot,k))$, and then multiply by $\varsigma_l(u_\varepsilon(s,y)-k)$, integrate with respect to s,y,k and take the expectation. The result is

$$\begin{aligned}
0 \le\, & E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k} \beta(v(0,x)-k)\phi_{\delta,\delta_0}(0,x,s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,dx\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^1\int_{\mathbb{R}_k} \beta(v(t,x,\alpha)-k)\partial_t\phi_{\delta,\delta_0}\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\mathbb{R}_k}\int_{t=0}^T\int_{|z|>0}\int_{\alpha=0}^1\int_{\mathbb{R}^d_x} \Big(\beta\big(v(t,x,\alpha)+\eta(x,v(t,x,\alpha);z)-k\big) - \beta(v(t,x,\alpha)-k)\Big) \\
&\qquad\qquad \times \phi_{\delta,\delta_0}\,dx\,d\alpha\,\tilde N(dz,dt)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{t=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \Big(\beta\big(v(t,x,\alpha)+\eta(x,v(t,x,\alpha);z)-k\big) - \beta(v(t,x,\alpha)-k) \\
&\qquad\qquad - \eta(x,v(t,x,\alpha);z)\beta'(v(t,x,\alpha)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y) \\
&\qquad\qquad \times \varsigma_l(u_\varepsilon(s,y)-k)\,d\alpha\,dk\,dx\,m(dz)\,dt\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^1\int_{\mathbb{R}_k} F^\beta(v(t,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\,\rho_{\delta_0}(t-s) \\
&\qquad\qquad \times \varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{5.1}$$

We now apply the Itô-Lévy formula to (3.15), giving

$$0 \le E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k} \beta(u_\varepsilon(0,y)-k)\phi_{\delta,\delta_0}(t,x,0,y)\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,dt\Big]$$

$$
\begin{aligned}
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} \beta(u_\varepsilon(s,y)-k)\partial_s\phi_{\delta,\delta_0}\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dy\,ds\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{s=0}^{T}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^{1}\int_{\mathbb{R}^d_y} \Big(\beta\big(u_\varepsilon(s,y)+\eta_\varepsilon(y,u_\varepsilon(s,y);z)-k\big)-\beta(u_\varepsilon(s,y)-k)\Big) \\
&\qquad\qquad \times \phi_{\delta,\delta_0}\varsigma_l(v(t,x,\alpha)-k)\,dy\,d\alpha\,dk\,\tilde{N}(dz,ds)\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{s=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\int_{\alpha=0}^{1} \Big(\beta\big(u_\varepsilon(s,y)+\eta_\varepsilon(y,(u_\varepsilon(s,y);z)-k\big)-\beta(u_\varepsilon(s,y)-k) \\
&\qquad\qquad - \eta_\varepsilon(y,u_\varepsilon(s,y);z)\beta'(u_\varepsilon(s,y)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y) \\
&\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,m(dz)\,ds\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} F_\varepsilon^\beta(u_\varepsilon(s,y),k)\cdot\nabla_y\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s) \\
&\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} F_\varepsilon^\beta(u_\varepsilon(s,y),k)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\rho_{\delta_0}(t-s) \\
&\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&- \varepsilon E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} \beta'(u_\varepsilon(s,y)-k)\nabla_y u_\varepsilon(s,y)\cdot\nabla_y\phi_{\delta,\delta_0}\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dy\,ds\,dx\,dt\Big],
\end{aligned}
\tag{5.2}
$$

where $F_\varepsilon^\beta(a,b)=\int_a^b\beta'(\sigma-b)F_\varepsilon'(\sigma)\,d\sigma$. It follows by direct computations that there is $p\in\mathbb{N}$ such that

$$\big|F_\varepsilon^\beta(a,b)-F^\beta(a,b)\big|\le C\varepsilon\big(1+|a|^{2p}+|b|^{2p}\big).$$

In view of the uniform moment estimates (3.20), it follows from (5.2) that

$$
\begin{aligned}
0 \le\, & E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} \beta(u_\varepsilon(0,y)-k)\phi_{\delta,\delta_0}(t,x,0,y)\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} \beta(u_\varepsilon(s,y)-k)\partial_s\phi_{\delta,\delta_0}\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dy\,ds\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{s=0}^{T}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^{1}\int_{\mathbb{R}^d_y} \Big(\beta\big(u_\varepsilon(s,y)+\eta_\varepsilon(y,u_\varepsilon(s,y);z)-k\big)-\beta(u_\varepsilon(s,y)-k)\Big) \\
&\qquad\qquad \times \phi_{\delta,\delta_0}\varsigma_l(v(t,x,\alpha)-k)\,dy\,d\alpha\,dk\,\tilde{N}(dz,ds)\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{s=0}^{T}\int_{|z|>0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\int_{\alpha=0}^{1} \Big(\beta\big(u_\varepsilon(s,y)+\eta_\varepsilon(y,(u_\varepsilon(s,y);z)-k\big)-\beta(u_\varepsilon(s,y)-k) \\
&\qquad\qquad - \eta_\varepsilon(y,u_\varepsilon(s,y);z)\beta'(u_\varepsilon(s,y)-k)\Big)\phi_{\delta,\delta_0}(t,x;s,y) \\
&\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,m(dz)\,ds\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} F^\beta(u_\varepsilon(s,y),k)\cdot\nabla_y\varrho_\delta(x-y)\psi(s,y)\rho_{\delta_0}(t-s) \\
&\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k} F^\beta(u_\varepsilon(s,y),k)\cdot\nabla_y\psi(s,y)\varrho_\delta(x-y)\rho_{\delta_0}(t-s) \\
&\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dx\,dt\,dy\,ds\Big]
\end{aligned}
$$

$$-\varepsilon E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k}\beta'(u_\varepsilon(s,y)-k)\nabla_y u_\varepsilon(s,y)\cdot\nabla_y\phi_{\delta,\delta_0}\varsigma_l(v(t,x,\alpha)-k)\,dk\,d\alpha\,dy\,ds\,dx\,dt\Big]$$
$$+\,C(\delta,\beta,\psi)o(\varepsilon)$$
$$=: J_1+J_2+J_3+J_4+J_5+J_6+J_7+C(\delta,\beta,\psi)o(\varepsilon) \tag{5.3}$$

where $C(\delta,\beta,\psi)$ is a constant depending only the quantities in the parentheses.

We now add (5.1) and (5.3), and compute limits with respect to the various parameters involved.

Lemma 5.1. *It holds that*

$$I_1+J_1 \underset{\delta_0\to 0}{\longrightarrow} E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\beta(v(0,x)-k)\psi(0,y)\varrho_\delta(x-y)\varsigma_l(u_\varepsilon(0,y)-k)\,dk\,dx\,dy\Big]$$
$$\underset{l\to 0}{\longrightarrow} E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\beta(v(0,x)-u_\varepsilon(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big]$$
$$\underset{\varepsilon\to 0}{\longrightarrow} E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\beta(v(0,x)-u(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big],$$

and

$$\lim_{(\vartheta,\delta)\to(0,0)} E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\beta_\vartheta(v(0,x)-u(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big]$$
$$= E\Big[\int_{\mathbb{R}^d_x}|v(0,x)-u(0,x)|\psi(0,x)\,dx\Big].$$

Proof. The first part of the proof is divided into three steps, and we note that $J_1=0$.

Step 1: In this step, we want to let $\delta_0\to 0$. For this, let

$$\begin{aligned}
\mathcal{A}_1 :=& E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\beta(v(0,x)-k)\psi(s,y)\varrho_\delta(x-y)\,\rho_{\delta_0}(-s)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,dx\,dy\,ds\Big]\\
&- E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\beta(v(0,x)-k)\psi(0,y)\varrho_\delta(x-y)\varsigma_l(u_\varepsilon(0,y)-k)\,dk\,dx\,dy\Big]\\
=& E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\beta(v(0,x)-u_\varepsilon(s,y)+k)\Big(\psi(s,y)-\psi(0,y)\Big)\varrho_\delta(x-y)\\
&\qquad\qquad\times\rho_{\delta_0}(-s)\varsigma_l(k)\,dk\,dx\,dy\,ds\Big]\\
&+ E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\Big(\beta(v(0,x)-u_\varepsilon(s,y)+k)-\beta(v(0,x)-u_\varepsilon(0,y)+k)\Big)\psi(0,y)\\
&\qquad\qquad\times\varrho_\delta(x-y)\rho_{\delta_0}(-s)\varsigma_l(k)\,dk\,dx\,dy\,ds\Big].
\end{aligned}$$

Since support $\psi(s,\cdot)\subset K$, we have

$$\begin{aligned}
\big|\mathcal{A}_1\big| \le& ||\partial_t\psi||_\infty E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\chi_K(y)\beta(v(0,x)-u_\varepsilon(s,y)+k)\varrho_\delta(x-y)\\
&\qquad\qquad\times s\rho_{\delta_0}(-s)\varsigma_l(k)\,dk\,dx\,dy\,ds\Big]\\
&+||\beta'||_\infty E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}|u_\varepsilon(s,y)-u_\varepsilon(0,y)|\psi(0,y)\varrho_\delta(x-y)\rho_{\delta_0}(-s)\,dx\,dy\,ds\Big]\\
\le& ||\partial_t\psi||_\infty\delta_0\,||\beta'||_\infty E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\chi_K(y)|v(0,x)-u_\varepsilon(s,y)|\rho_{\delta_0}(-s)\varrho_\delta(x-y)\,dx\,dy\,ds\Big]\\
&+||\partial_t\psi||_\infty\delta_0\,||\beta'||_\infty\,l\,E\Big[\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\chi_K(y)\varrho_\delta(x-y)\,dx\,dy\Big]\\
&+||\beta'||_\infty E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}|u_\varepsilon(s,y)-u_\varepsilon(0,y)|\psi(0,y)\varrho_\delta(x-y)\rho_{\delta_0}(-s)\,dx\,dy\,ds\Big]\\
\le& ||\partial_t\psi||_\infty\delta_0\,||\beta'||_\infty C(\psi)\Big(E\big[||v(0,\cdot,\cdot)||_2\big]+\sup_{\varepsilon>0}\sup_{0\le s\le T}E\big[||u_\varepsilon(s,\cdot)||_2\big]\Big)
\end{aligned}$$

$$+ ||\partial_t \psi||_\infty \delta_0 \, ||\beta'||_\infty \, l \, C(\psi) + C||\beta'||_\infty E\Big[\frac{1}{\delta_0}\int_{s=0}^{\delta_0}\int_{\mathbb{R}^d} |u_\varepsilon(s,y) - u_\varepsilon(0,y)|\psi(0,y)\,dy\,ds\Big].$$

Clearly, the results of Lemma 2.3 continue to hold if we replace u by u_ε. Hence the last term vanishes as $\delta_0 \to 0$. Therefore $\mathcal{A}_1 \to 0$ as $\delta_0 \to 0$.

Step 2: In this step, we verify the passage to the limit as $l \to 0$. To this end, let

$$\begin{aligned}
&\mathcal{A}_2 \\
&:= E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d}\int_{\mathbb{R}_k} \beta(v(0,x) - k)\psi(0,y)\varrho_\delta(x-y)\varsigma_l(u_\varepsilon(0,y) - k)\,dk\,dx\,dy\Big] \\
&\quad - E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} \beta(v(0,x) - u_\varepsilon(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big] \\
&= E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d}\int_{\mathbb{R}_k} \Big(\beta(v(0,x) + k - u_\varepsilon(0,y)) - \beta(v(0,x) - u_\varepsilon(0,y))\Big)\psi(0,y)\varrho_\delta(x-y)\varsigma_l(k)\,dk\,dx\,dy\Big].
\end{aligned}$$

Hence

$$\begin{aligned}
\big|\mathcal{A}_2\big| &\le ||\beta'||_\infty E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d}\int_{\mathbb{R}_k} k\psi(0,y)\varrho_\delta(x-y)\varsigma_l(k)\,dk\,dx\,dy\Big] \le C(\psi)||\beta'||_\infty\, l \\
&\to 0 \quad \text{as} \quad l \to 0.
\end{aligned}$$

Step 3: In this step, we verify the passage to the limit as $\varepsilon \to 0$. Let

$$\begin{aligned}
\mathcal{A}_3 &:= E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} \beta(v(0,x) - u_\varepsilon(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big] \\
&\quad - E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} \beta(v(0,x) - u(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big] \\
&= E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} \Big(\beta(v(0,x) - u_\varepsilon(0,y)) - \beta(v(0,x) - u(0,y))\Big)\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big].
\end{aligned}$$

Therefore,

$$\big|\mathcal{A}_3\big| \le ||\beta'||_\infty \int_{\mathbb{R}_y^d} |u_\varepsilon(0,y) - u(0,y)|\psi(0,y)\,dy \to 0 \quad \text{as } \varepsilon \to 0.$$

For the second part of the lemma, consider

$$\mathcal{A}_4(\vartheta,\delta) := E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} \Big(\beta_\vartheta(v(0,x) - u(0,y)) - |v(0,x) - u(0,y)|\Big)\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big].$$

Note that $(\beta_\vartheta)_{\vartheta>0}$ is a sequence of functions satisfying $\big|\beta_\vartheta(r) - |r|\big| \le C\vartheta$ for any $r \in \mathbb{R}$. Therefore,

$$\big|\mathcal{A}_4(\vartheta,\delta)\big| \le \text{Const}(\psi)\vartheta \to 0 \quad \text{as } \vartheta \to 0.$$

Furthermore, let

$$\begin{aligned}
\mathcal{A}_5(\delta) &:= \Big|E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} |v(0,x) - u(0,y)|\psi(0,y)\varrho_\delta(x-y)\,dx\,dy - \int_{\mathbb{R}_y^d} |v(0,y) - u(0,y)|\psi(0,y)\,dy\Big]\Big| \\
&\le E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} |v(0,y) - v(0,x)|\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big] \\
&\le ||\psi(0,\cdot)||_\infty E\Big[\int_{|z|\le 1}\int_{\mathbb{R}_y^d} |v(0,y) - v(0,y+\delta z)|\varrho(z)\,dy\,dz\Big].
\end{aligned}$$

Note that $\lim_{\delta\downarrow 0} \int_{\mathbb{R}_x^d} |v(0,x) - v(0,x+\delta z)|\,dx \to 0$ for $||z|| \le 1$, and therefore by the bounded convergence theorem we have $\lim_{\delta\downarrow 0} E\Big[\int_{\mathbb{R}_z^d}\int_{\mathbb{R}_x^d} |v(0,x) - v(0,x+\delta z)|\varrho(z)\,dx\,dz\Big] = 0$ and hence $\mathcal{A}_5(\delta) \to 0$ as $\delta \to 0$. Finally, since

$$\Big|E\Big[\int_{\mathbb{R}_y^d}\int_{\mathbb{R}_x^d} \beta_\vartheta(v(0,x) - u(0,y))\psi(0,y)\varrho_\delta(x-y)\,dx\,dy\Big] - E\Big[\int_{\mathbb{R}^d} |v(0,x) - u(0,x)|\psi(0,x)\,dx\Big]\Big|$$

$$\le \mathcal{A}_4(\vartheta,\delta) + \mathcal{A}_5(\delta),$$

we can conclude the proof of the second part of the lemma. □

We now turn our attention to $(I_2 + J_2)$:

$$\begin{aligned}
I_2 + J_2 =& E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(v(t,x,\alpha)-k)\psi(s,y)\partial_t\rho_{\delta_0}(t-s) \\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\varsigma_l(u_\varepsilon(s,y)-k)\,d\alpha\,dk\,dx\,dt\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\rho_{\delta_0}(t-s) \\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,ds\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(u_\varepsilon(s,y)-k)\psi(s,y)\partial_s\rho_{\delta_0}(t-s) \\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,ds\,dx\,dt\Big] \\
=& - E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(v(t,x,\alpha)-u_\varepsilon(s,y)+k)\psi(s,y)\partial_s\rho_{\delta_0}(t-s) \\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\varsigma_l(k)\,d\alpha\,dk\,dx\,dt\,dy\,ds\Big] \\
&+ E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta((v(t,x,\alpha)-u_\varepsilon(s,y)+k))\psi(s,y)\partial_s\rho_{\delta_0}(t-s) \\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\varsigma_l(k)\,d\alpha\,dk\,dy\,ds\,dx\,dt\Big] \\
&+ E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\,\rho_{\delta_0}(t-s) \\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,ds\,dx\,dt\Big],
\end{aligned}$$

as β, ς_l are even functions. Hence, we are left with

$$\begin{aligned}
I_2 + J_2 =& E\Big[\int_{\Pi_T\times\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\,\rho_{\delta_0}(t-s)\varrho_\delta(x-y) \\
&\qquad\qquad\qquad\qquad \times \varsigma_l(v(t,x,\alpha)-k)\,d\alpha\,dk\,dy\,ds\,dxdt\Big].
\end{aligned}$$

Lemma 5.2. *It holds that*

$$\begin{aligned}
I_2 + J_2 \underset{\delta_0\to 0}{\longrightarrow}& E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\,\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,d\alpha\,dk\,dy\,dx\,ds\Big] \\
\underset{l\to 0}{\longrightarrow}& E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1 \beta(u_\varepsilon(s,y)-v(s,x,\alpha))\partial_s\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dy\,dx\,ds\Big] \\
\underset{\varepsilon\to 0}{\longrightarrow}& E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\gamma=0}^1\int_{\alpha=0}^1 \beta(u(s,y,\gamma)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dy\,dx\,ds\Big],
\end{aligned}$$

and

$$\begin{aligned}
\lim_{(\vartheta,\delta)\to(0,0)} & E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\gamma=0}^1\int_{\alpha=0}^1 \beta_\vartheta(u(s,y,\gamma)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dy\,dx\,ds\Big] \\
&= E\Big[\int_{s=0}^T\int_{\mathbb{R}^d}\int_{\alpha=0}^1\int_{\gamma=0}^1 |u(s,y,\gamma)-v(s,y,\alpha)|\partial_s\psi(s,y)d\gamma\;d\alpha\,dy\,ds\Big].
\end{aligned}$$

Proof. The proof of the first part of the lemma is divided into three steps.

Step 1: In this step we justify passing to the limit $\delta_0 \to 0$. Let

$$
\begin{aligned}
\mathcal{B}_1 := \Big| E\Big[\int_{\Pi_T} \int_{\Pi_T} \int_{\mathbb{R}_k} \int_{\alpha=0}^{1} & \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\, \rho_{\delta_0}(t-s)\varrho_\delta(x-y) \\
& \times \varsigma_l(v(t,x,\alpha)-k)\, d\alpha\, dk\, dx\, dt\, dy\, ds\Big] \\
- E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_x} \int_{\mathbb{R}_k} \int_{\alpha=0}^{1} & \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\, \varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\, d\alpha\, dk\, dx\, dy\, ds\Big]\Big| \\
= \Big| E\Big[\int_{s=\delta_0}^{T} \int_{\mathbb{R}^d_y} \int_{\Pi_T} \int_{\mathbb{R}_k} \int_{\alpha=0}^{1} & \Big(\beta(u_\varepsilon(s,y)-v(t,x,\alpha)+k) - \beta(u_\varepsilon(s,y)-v(s,x,\alpha)+k)\Big) \\
& \times \partial_s\psi(s,y)\, \rho_{\delta_0}(t-s)\varrho_\delta(x-y)\varsigma_l(k)\, d\alpha\, dk\, dx\, dt\, dy\, ds\Big]\Big| + o(\delta_0).
\end{aligned}
$$

Observe that

$$
\begin{aligned}
& \mathcal{B}_1 \\
& \le C(\beta')E\Big[\int_{s=\delta_0}^{T} \int_{\mathbb{R}^d_y} \int_{\Pi_T} \int_{\alpha=0}^{1} |v(t,x,\alpha)-v(s,x,\alpha)| \varrho_\delta(x-y) |\partial_s\psi(s,y)| \rho_{\delta_0}(t-s)\, d\alpha\, dt\, dx\, dy\, ds\Big] \\
& \qquad + o(\delta_0) \\
& \le C(\beta', \partial_s\psi)\Big(E\Big[\int_{s=\delta_0}^{T} \int_{\mathbb{R}^d_y} \int_{\Pi_T} \int_{\alpha=0}^{1} |v(t,x,\alpha)-v(s,x,\alpha)|^2 \varrho_\delta(x-y)\rho_{\delta_0}(t-s)\, d\alpha\, dt\, dx\, dy\, ds\Big]\Big)^{\frac{1}{2}} \\
& \qquad + o(\delta_0) \\
& \big(\text{used Cauchy-Schwartz's inequality w.r.t.} \quad \varrho_\delta(x-y)\rho_{\delta_0}(t-s)\, d\alpha\, dt\, dx\, dy\, ds\, dP(\omega)\big) \\
& \le C(\beta') \|\partial_s\psi\|_\infty \Big(E\Big[\int_{r=0}^{1} \int_{\Pi_T} \int_{\alpha=0}^{1} |v(t+\delta_0 r, x,\alpha) - v(t,x,\alpha)|^2 \rho(-r)\, d\alpha\, dt\, dx\, dr\Big]\Big)^{\frac{1}{2}} + o(\delta_0).
\end{aligned}
$$

Note that $\lim_{\delta_0 \downarrow 0} \int_{t=0}^{T} \int_{\mathbb{R}^d_x} \int_{\alpha=0}^{1} |v(t+\delta_0 r, x, \alpha) - v(t,x,\alpha)|^2 \, d\alpha\, dx\, dt = 0$, almost surely, for every fixed $r \in [0,1]$. Therefore, by the bounded convergence theorem,

$$
\lim_{\delta_0 \downarrow 0} E\Big[\int_{t=0}^{T} \int_{r=0}^{1} \int_{\mathbb{R}^d_x} \int_{\alpha=0}^{1} |v(t+\delta_0 r, x,\alpha) - v(t,x,\alpha)|^2 \rho(-r)\, d\alpha\, dx\, dr\, dt\Big] = 0.
$$

This concludes the first step.

Step 2: Let

$$
\begin{aligned}
\mathcal{B}_2 := \Big| E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_y} \int_{\mathbb{R}_k} \int_{\alpha=0}^{1} & \beta(u_\varepsilon(s,y)-k)\partial_s\psi(s,y)\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\, d\alpha\, dk\, dy\, dx\, ds\Big] \\
- E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_y} \int_{\alpha=0}^{1} & \beta(u_\varepsilon(s,y)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\, d\alpha\, dy\, dx\, ds\Big]\Big| \\
= \Big| E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_x} \int_{\mathbb{R}_k} \int_{\alpha=0}^{1} & \Big(\beta(u_\varepsilon(s,y)+k-v(s,x,\alpha)) - \beta(u_\varepsilon(s,y)-v(s,x,\alpha))\Big)\partial_s\psi(s,y) \\
& \times \varrho_\delta(x-y)\varsigma_l(k)\, d\alpha\, dk\, dx\, dy\, ds\Big]\Big|,
\end{aligned}
$$

and note that

$$
\big|\mathcal{B}_2\big| \le \|\beta'\|_\infty E\Big[\int_{\Pi_T} \int_{\mathbb{R}_k} |k| |\partial_s\psi(s,y)|\, \varsigma_l(k)\, dk\, dy\, ds\Big] \le \|\partial_s\psi\|_\infty\, l\, \|\beta'\|_\infty\, C(\psi) \to 0, \quad \text{as } l \to 0.
$$

Step 3: Note that $u(s,y,\gamma)$ is the $L^2(\mathbb{R}^d \times (0,1))$-valued process that was recovered from the Young measure valued narrow limit of the sequence $\{u_\varepsilon(s,y)\}_{\varepsilon>0}$ and it satisfies Lemmas 4.2 and 4.3. Let

$$
\Gamma_{(x,\alpha)}(s,y,\omega;\xi) = \beta(\xi - v(s,x,\alpha))\partial_s\psi(s,y)\, \varrho_\delta(x-y).
$$

Clearly, for every fixed $(x,\alpha)\in\mathbb{R}^d\times(0,1)$, $\Gamma_{(x,\alpha)}$ is a Caratheodory function and $\big\{\Gamma_{(x,\alpha)}(s,y,\omega;u_{\varepsilon_n}(s,y))\big\}_n$ is uniformly integrable in $L^1\big((\Theta,\Sigma,\mu),\mathbb{R}\big)$ and satisfies the conditions of Lemma 4.2. Hence, for every $(x,\alpha)\in\mathbb{R}^d\times(0,1)$,

$$\begin{aligned}&\lim_{\varepsilon_n\to 0}\int_\Omega\int_{\Pi_T}\Gamma_{(x,\alpha)}(s,y,\omega;u_{\varepsilon_n}(s,y))\,ds\,dy\,dP(\omega)\\&=\int_\Omega\int_{\Pi_T}\int_{\gamma=0}^1\Gamma_{(x,\alpha)}(s,y,\omega;u(s,y,\gamma))\,d\gamma\,ds\,dy\,dP(\omega).\end{aligned}\tag{5.4}$$

In view of (5.4), we now invoke the bounded convergence and Fubini theorems to conclude

$$\begin{aligned}&\lim_{\varepsilon_n\to 0}E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\beta(u_{\varepsilon_n}(s,y)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,dy\,dx\,ds\Big]\\&\quad=E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\gamma=0}^1\beta(u(s,y,\gamma)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dy\,dx\,ds\Big].\end{aligned}$$

This concludes the proof of the first part of the lemma.

For the second part we proceed as follows:

$$\begin{aligned}\mathcal{B}_3(\vartheta,\delta):=\Big|&E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\gamma=0}^1\int_{\alpha=0}^1\beta_\vartheta(u(s,y,\gamma)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dx\,dy\,ds\Big]\\&-E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\gamma=0}^1\int_{\alpha=0}^1|u(s,y,\gamma)-v(s,x,\alpha)|\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dx\,dy\,ds\Big]\Big|\\\le||\partial_s\psi||_\infty\,&E\Big[\int_{|y|\le C_\psi}\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1\big|\beta_\vartheta(u(s,y,\gamma)-v(s,x,\alpha))-|u(s,y,\gamma)-v(s,x,\alpha)|\big|\\&\qquad\times\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dx\,ds\,dy\Big].\end{aligned}$$

Since $\big|\beta_\vartheta(r)-|r|\big|\le C\vartheta$, for any $r\in\mathbb{R}$, it follows that $\mathcal{B}_3\le||\partial_s\psi||_\infty\,\vartheta\,C(\psi,T)$. Observe that

$$\begin{aligned}\mathcal{B}_4(\delta):=\Big|&E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\gamma=0}^1|u(s,y,\gamma)-v(s,x,\alpha)|\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dy\,dx\,ds\Big]\\&-E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\gamma=0}^1|u(s,y,\gamma)-v(s,y,\alpha)|\partial_s\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big]\Big|\\\le\ &E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_0^1|v(s,y,\alpha)-v(s,x,\alpha)|\,|\partial_s\psi(s,y)|\varrho_\delta(x-y)\,d\alpha\,dy\,dx\,ds\Big]\\\le\ &C(\psi)\Big(E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_0^1|v(s,y,\alpha)-v(s,x,\alpha)|^2\varrho_\delta(x-y)\,d\alpha\,dy\,dx\,ds\Big]\Big)^{\frac12}\\&\quad\text{(here we used Cauchy-Schwartz's inequality)}\\\to\ &0\text{ as }\delta\to 0,\end{aligned}$$

where the $\delta\to0$ limit follows by arguments similar to those used to prove the last part of Lemma 5.1.

Since

$$\begin{aligned}\Big|&E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\gamma=0}^1\int_{\alpha=0}^1\beta_\vartheta(u(s,y,\gamma)-v(s,x,\alpha))\partial_s\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dx\,dy\,ds\Big]\\&-E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\gamma=0}^1|u(s,y,\gamma)-v(s,y,\alpha)|\partial_s\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big]\Big|\\&\le\mathcal{B}_3(\vartheta,\delta)+\mathcal{B}_4(\delta)\to0,\quad\text{as }(\vartheta,\delta)\to(0,0),\end{aligned}$$

the second part of the lemma follows. □

Next we consider the stochastic term I_3+J_3; we begin with the following assertion:

Lemma 5.3. *For any two constants $T_1, T_2 \geq 0$ with $T_1 < T_2$,*

$$E\Big[X_{T_1} \int_{T_1}^{T_2} \int_{|z|>0} \zeta(t,z)\, \tilde{N}(dz,dt)\Big] = 0, \tag{5.5}$$

where ζ is a predictable integrand with $E\Big[\int_0^T \int_{|z|>0} \zeta^2(t,z)\, m(dz)\, dt\Big] < \infty$ and X is an adapted process.

Proof. Let $\mathcal{M}(t) = \int_0^t \int_{|z|>0} \zeta(s,z)\, \tilde{N}(dz,ds)$. Clearly, $\mathcal{M}(t)$ is a martingale, and thus

$$\begin{aligned}
& E\Big[X_{T_1} \int_{T_1}^{T_2} \int_{|z|>0} \zeta(t,z)\, \tilde{N}(dz,dt)\Big] \\
&= E\Big[X_{T_1}\Big(\mathcal{M}(T_2) - \mathcal{M}(T_1)\Big)\Big] \\
&= E\Big[E\big(X_{T_1} M(T_2) | \mathcal{F}_{T_1}\big)\Big] - E\Big[X_{T_1}\mathcal{M}(T_1)\Big] \\
&= E\Big[X_{T_1} E\big(M(T_2) | \mathcal{F}_{T_1}\big)\Big] - E\Big[X_{T_1}\mathcal{M}(T_1)\Big] \\
&= E\Big[X_{T_1}\mathcal{M}(T_1)\Big] - E\Big[X_{T_1}\mathcal{M}(T_1)\Big] = 0.
\end{aligned}$$

□

For any $\beta \in C^\infty(\mathbb{R})$ with $\beta', \beta'' \in C_b(\mathbb{R})$ and any nonnegative $\phi \in C_c^\infty(\Pi_\infty \times \Pi_\infty)$, we define

$$\begin{aligned}
J[\beta,\phi](s;y,v) := \int_{r=0}^T \int_{|z|>0} \int_{\mathbb{R}_x^d} \int_0^1 & \Big(\beta\big(v(r,x,\alpha) + \eta(x, v(r,x,\alpha); z) - v\big) - \beta\big(v(r,x,\alpha) - v\big)\Big) \\
& \times \phi(r,x,s,y)\, d\alpha\, dx\, \tilde{N}(dz,dr),
\end{aligned}$$

where $0 \leq s \leq T$ and $(y,v) \in \mathbb{R}^d \times \mathbb{R}$. Note that ϕ has compact support, i.e., there exists a constant $c_\phi > 0$ such that $\phi(\cdot,\cdot,\cdot,y) = 0$ for $|y| \geq c_\phi$. As a result $J[\beta,\phi](s;y,v) = 0$ if $|y| > c_\phi$ and $0 \leq s \leq T$. Furthermore, we extend the process $u_\varepsilon(\cdot,y)$ for negative times by setting $u_\varepsilon(s,y) = u_\varepsilon(0,y)$ if $s < 0$. With this convention, by Itô-Lévy product rule

$$\begin{aligned}
J[\beta,\phi_{\delta,\delta_0}](s;y,v) &= \int_{s-\delta_0}^s \int_{|z|>0} \int_{\mathbb{R}_x^d} \int_0^1 \Big(\beta\big(v(r,x,\alpha) + \eta(x, v(r,x,\alpha); z) - v\big) - \beta\big(v(r,x,\alpha) - v\big)\Big) \\
&\qquad\qquad \times \phi_{\delta,\delta_0}(r,x,s,y)\, d\alpha\, dx\, \tilde{N}(dz,dr) \\
&= -\, \psi(s,y) \int_{s-\delta_0}^s \Big[\int_{s-\delta_0}^r \int_{|z|>0} \zeta_{(y,v)}(\sigma,z) \tilde{N}(\,dz,\, d\sigma)\Big] \rho'_{\delta_0}(r-s)\, dr
\end{aligned}$$

where

$$\zeta_{(y,v)}(r,z) = \int_{\mathbb{R}_x^d} \int_0^1 \Big(\beta\big(v(r,x,\alpha) + \eta(x, v(r,x,\alpha); z) - v\big) - \beta\big(v(r,x,\alpha) - v\big)\Big) \varrho_\delta(x-y)\, d\alpha\, dx.$$

Therefore, by Fubini's theorem and (5.5),

$$\begin{aligned}
& E\Big[J[\beta,\phi_{\delta,\delta_0}](s;y,v)\, \varsigma_l(u_\varepsilon(s-\delta_0,y) - v)\Big] \\
&= -\psi(s,y) \int_{s-\delta_0}^s E\Big[\varsigma_l(u_\varepsilon(s-\delta_0,y) - v) \int_{s-\delta_0}^r \int_{|z|>0} \zeta_{(y,v)}(\sigma,z) \tilde{N}(\,dz,\, d\sigma)\Big] \rho'_{\delta_0}(r-s)\, dr \\
&= 0
\end{aligned} \tag{5.6}$$

for all (s,y,v). Finally, we apply Fubini's theorem along with (5.6) and obtain

$$E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta,\phi_{\delta,\delta_0}](s;y,v)\, \varsigma_l(u_\varepsilon(s-\delta_0,y) - v)\, dy\, ds\, dv\Big] = 0.$$

Therefore

$$I_3 = E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta,\phi_{\delta,\delta_0}](s;y,v) \Big(\varsigma_l(u_\varepsilon(s,y) - v) - \varsigma_l(u_\varepsilon(s-\delta_0,y) - v)\Big)\, dy\, ds\, dv\Big]. \tag{5.7}$$

Lemma 5.4. *The following identities hold:*

$$\partial_v J[\beta,\phi](s;y,v) = J[-\beta',\phi](s;y,v)$$
$$\partial_{y_k} J[\beta,\phi](s;y,v) = J[\beta,\partial_{y_k}\phi](s;y,v).$$

Proof. The proof follows by a classical argument validating differentiation under the integral sign. □

Lemma 5.5. *Let $\beta\in C^\infty(\mathbb{R})$ be function such that $\beta',\beta''\in C_c^\infty(\mathbb{R})$ and p be a positive integer of the form $p=2^k$ for some $k\in\mathbb{N}$. If $p\ge d+3$, then there exists a constant $C=C(\beta',\psi,\delta)$ such that*

$$\sup_{0\le s\le T}\Big(E\Big[||J[\beta,\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^2_{L^\infty(\mathbb{R}^d\times\mathbb{R})}\Big]\Big)\le \frac{C(\beta',\psi,\delta)}{\delta_0^{\frac{2(p-1)}{p}}}. \tag{5.8}$$

Proof. We estimate as follows:

$$\begin{aligned}
&E\Big[||J[\beta,\phi_{\delta,\delta_0}](s;\cdot,\cdot)||_p^p\Big]\\
&\quad= E\Big[\int_{\mathbb{R}_v}\int_{\mathbb{R}^d_y}\Big|J[\beta,\phi_{\delta,\delta_0}](s;y,v)\Big|^p\,dy\,dv\Big]\\
&\quad= E\Big[\int_{\mathbb{R}_v}\int_{\mathbb{R}^d_y}\Big|\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\lambda=0}^1\int_{\alpha=0}^1 \eta(x,v(r,x,\alpha);z)\beta'(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad \times\rho_{\delta_0}(r-s)\varrho_\delta(x-y)\psi(s,y)\,d\alpha\,d\lambda\,dx\,\tilde N(dz,dr)\Big|^p\,dy\,dv\Big]\\
&\quad\text{(by the BDG inequality)}\\
&\quad\le C\int_{\mathbb{R}_v}\int_{\mathbb{R}^d_y}E\Big[\Big(\int_{r=0}^T\int_{|z|>0}\Big|\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1\beta'(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad\times\eta(x,v(r,x,\alpha);z)\rho_{\delta_0}(r-s)\psi(s,y)\,\varrho_\delta(x-y)\,d\lambda\,d\alpha\,dx\Big|^2 N(dz,dr)\Big)^{\frac{p}{2}}\Big]\,dy\,dv\\
&\quad\text{(by Schwartz's inequality w.r.t. the measure } \varrho_\delta(x-y)\,d\lambda\,d\alpha\,dx)\\
&\quad\le C\int_{\mathbb{R}_v}\int_{|y|<C_\psi}E\Big[\Big(\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1\beta'^2(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad\times\eta^2(x,v(r,x,\alpha);z)\rho^2_{\delta_0}(r-s)\psi^2(s,y)\varrho_\delta(x-y)\,d\lambda\,d\alpha\,dx\,N(dz,dr)\Big)^{\frac{p}{2}}\Big]\,dy\,dv\\
&\quad\le C\int_{\mathbb{R}_v}\int_{|y|<C_\psi}E\Big[\Big(\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1\beta'^2(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad\times\eta^2(x,v(r,x,\alpha);z)\rho^2_{\delta_0}(r-s)\psi^2(s,y)\,\varrho_\delta(x-y)\,d\lambda\,d\alpha\,dx\,\tilde N(dz,dr)\Big)^{\frac{p}{2}}\Big]\,dy\,dv\\
&\qquad+C\int_{\mathbb{R}_v}\int_{|y|<C_\psi}E\Big[\Big(\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1\beta'^2(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad\times\eta^2(x,v(r,x,\alpha);z)\rho^2_{\delta_0}(r-s)\varrho_\delta(x-y)\psi^2(s,y)\,d\lambda\,d\alpha\,dx\,m(dz)\,dr\Big)^{\frac{p}{2}}\Big]\,dy\,dv\\
&\quad\Big(\text{noting that } p=2^k,\text{ and applying the BDG inequality followed by the Cauchy-Schwartz's}\\
&\quad\text{inequality another } (k-1)\text{ times, we obtain}\Big)\\
&\quad\le\sum_{j=0}^{k-1}C_j\int_{\mathbb{R}_v}\int_{|y|<C_\psi}E\Big[\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1\big|\beta'(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad\times\eta(x,v(r,x,\alpha);z)\rho_{\delta_0}(r-s)\psi(s,y)\big|^{\frac{p}{2^j}}\,\varrho_\delta(x-y)\,d\lambda\,d\alpha\,dx\,m(dz)\,dr\Big]^{2^j}\,dy\,dv\\
&\quad\le\sum_{j=0}^{k-1}C_j\int_{\mathbb{R}_v}\int_{|y|<C_\psi}E\Big[\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1\big|\beta'(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))g(x)
\end{aligned}$$

$$\times (1+|v(r,x,\alpha)|)\rho_{\delta_0}(r-s)\psi(s,y)\Big|^{\frac{p}{2^j}}\,\varrho_\delta(x-y)\min(1,|z|^2)\,d\lambda\,d\alpha\,dx\,m(dz)\,dr\Big]^{2^j}\,dy\,dv$$

$\big($applying Hölder's inequality w.r.t. the measure $\varrho_\delta(x-y)\min(1,|z|^2)\,d\lambda\,d\alpha\,dx\,m(dz)\,dr\big)$

$$\begin{aligned}
&\le \sum_{j=0}^{k-1} C_j \int_{\mathbb{R}_v}\int_{|y|<C_\psi} E\Big[\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\lambda=0}^1 \big|g(x)\beta'(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times(1+|v(r,x,\alpha)|)\rho_{\delta_0}(r-s)\psi(s,y)\big|^p\,\varrho_\delta(x-y)\min(1,|z|^2)\,d\lambda\,d\alpha\,dx\,m(dz)\,dr\Big]\,dy\,dv\\
&\le CE\Big[\int_{|y|<C_\psi}\int_{r=0}^T\int_{|z|>0}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{|v|\le C_{\beta'}+|v(r,x,\alpha)|+|\eta(x,v(r,x,\alpha);z)|}(1+|v(r,x,\alpha)|^p)||\beta'||^p_\infty\\
&\qquad\times g^p(x)\rho^p_{\delta_0}(r-s)||\psi||^p_\infty\varrho_\delta(x-y)\min(1,|z|^2)\,dv\,d\alpha\,dx\,m(dz)\,dr\,dy\Big]\\
&\le C(\beta,\psi)E\Big[\int_{r=0}^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1 g^p(x)(1+|v(r,x,\alpha)|^p)\\
&\qquad\times(C_{\beta'}+|g(x)|(1+|v(r,x,\alpha)|))\rho^p_{\delta_0}(r-s)\,d\alpha\,dx\,dr\Big]\\
&\le C(\beta,\psi)\int_{r=0}^T\Big(1+E\big[||v(r,\cdot,\cdot)||^{p+1}_{p+1}\big]\Big)\rho^p_{\delta_0}(r-s)\,dr\\
&\le C(\beta,\psi)\Big(1+\sup_{0\le r\le T}E\big[||v(r,\cdot,\cdot)||^{p+1}_{p+1}\big]\Big)\int_{r=0}^T\rho^p_{\delta_0}(r-s)\,dr\\
&\le C(\beta,\psi)\Big(1+\sup_{0\le r\le T}E\Big[||v(r,\cdot)||^{p+1}_{p+1}\Big]\Big)||\rho_{\delta_0}||^{p-1}_\infty\int_{r=0}^T\rho_{\delta_0}(r-s)\,dr\\
&\le \frac{C(\beta,\psi)\Big(1+\sup_{0\le r\le T}E\Big[||v(r,\cdot)||^{p+1}_{p+1}\Big]\Big)}{\delta_0^{p-1}}.
\end{aligned}\tag{5.9}$$

Similarly, we can derive the following bounds:

$$E\Big[||\partial_v J[\beta,\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^p_p\Big]\le\frac{C(\beta'',\psi)}{\delta_0^{p-1}},\tag{5.10}$$

$$E\Big[||\partial_{y_k} J[\beta,\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^p_p\Big]\le\frac{C(\beta',\partial_{y_k}\psi,\delta)}{\delta_0^{p-1}}.\tag{5.11}$$

Therefore, in view of (5.9), (5.10), and (5.11), we have arrived at

$$E\Big[||J[\beta,\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^p_{W^{1,p}(\mathbb{R}^d\times\mathbb{R})}\Big]\le\frac{C(\beta',\psi,\delta)}{\delta_0^{p-1}}.$$

Finally, we use the Sobolev embedding along with Cauchy-Schwartz's inequality to arrive at (5.8). ☐

Lemma 5.6. *It holds that* $J_3=0$ *and*

$$\begin{aligned}
\lim_{l\to0}\lim_{\delta_0\to0} I_3 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1 &\Big(\beta(v(r,x,\alpha)+\eta(x,v(r,x,\alpha);z)-u_\varepsilon(r,y)-\eta_\varepsilon(y,u_\varepsilon(r,y);z))\\
&-\beta(v(r,x,\alpha)-u_\varepsilon(r,y)-\eta_\varepsilon(y,u_\varepsilon(r,y);z))+\beta(v(r,x,\alpha)-u_\varepsilon(r,y))\\
&-\beta(v(r,x,\alpha)+\eta(x,v(r,x,\alpha);z)-u_\varepsilon(r,y))\Big)\\
&\times\psi(r,y)\,\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dy\,dr\Big].
\end{aligned}$$

Proof. Note that

$$\begin{aligned}
J_3=\int_{\Pi_T}\int_{\alpha=0}^1\int_{\mathbb{R}_v} &E\Big[\varsigma_l(v(t,x,\alpha)-v)\\
&\times\int_{s=t}^{t+\delta_0}\int_{|z|>0}\int_{\mathbb{R}^d_y}\Big(\beta(u_\varepsilon(s,y)+\eta_\varepsilon(y,u_\varepsilon(s,y);z)-v)-\beta(u_\varepsilon(s,y)-v)\Big)
\end{aligned}$$

$$\times \phi_{\delta,\delta_0}\, dy\, \tilde{N}(dz,ds)\Big]\, dv\, d\alpha\, dx\, dt$$

$$= 0, \quad \text{thanks to (5.5).}$$

For all $y \in \mathbb{R}^d$, $u_\varepsilon(\cdot, y)$ solves

$$du_\varepsilon(s,y) = -\mathrm{div} F_\varepsilon(u_\varepsilon(s,y))ds + \varepsilon \Delta u_\varepsilon(s,y)\, ds + \int_{|z|>0} \eta_\varepsilon(y, u_\varepsilon(s,y); z)\, \tilde{N}(dz,ds).$$

Now we apply the Itô-Lévy formula to $\varsigma_l(u_\varepsilon(s,y) - v)$:

$$\begin{aligned}
&\varsigma_l(u_\varepsilon(s,y) - v) - \varsigma_l(u_\varepsilon(s-\delta_0, y) - v) \\
&= \int_{s-\delta_0}^{s} \varsigma_l'(u_\varepsilon(\sigma,y) - v)\big(- \mathrm{div} F_\varepsilon(u_\varepsilon(\sigma,y)) + \varepsilon \Delta u_\varepsilon(\sigma,y)\big)\, d\sigma \\
&\quad + \int_{s-\delta_0}^{s} \int_{|z|>0} \Big(\varsigma_l(u_\varepsilon(\sigma,y) + \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z) - v) - \varsigma_l(u_\varepsilon(\sigma,y) - v)\Big)\, \tilde{N}(dz,d\sigma) \\
&\quad + \int_{s-\delta_0}^{s} \int_{|z|>0} \int_{\lambda=0}^{1} (1-\lambda)|\eta_\varepsilon(y, u_\varepsilon(\sigma,y); z)|^2 \varsigma_l''(u_\varepsilon(\sigma,y) - v + \lambda \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z))\, d\lambda\, m(dz)\, d\sigma \\
&= -\frac{\partial}{\partial v} \int_{s-\delta_0}^{s} \varsigma_l(u_\varepsilon(\sigma,y) - v)\big(- \mathrm{div} F_\varepsilon(u_\varepsilon(\sigma,y)) + \varepsilon \Delta u_\varepsilon(\sigma,y)\big)\, d\sigma \\
&\quad + \int_{s-\delta_0}^{s} \int_{|z|>0} \Big(\varsigma_l(u_\varepsilon(\sigma,y) + \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z) - v) - \varsigma_l(u_\varepsilon(\sigma,y) - v)\Big)\, \tilde{N}(dz,d\sigma) \\
&\quad + \int_{s-\delta_0}^{s} \int_{|z|>0} \int_{\lambda=0}^{1} (1-\lambda)|\eta_\varepsilon(y, u_\varepsilon(\sigma,y); z)|^2 \varsigma_l''(u_\varepsilon(\sigma,y) - v + \lambda \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z))\, d\lambda\, m(dz)\, d\sigma.
\end{aligned}$$

Therefore, from (5.7), we have

$$\begin{aligned}
&I_3 = \\
&E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta, \phi_{\delta,\delta_0}](s;y,v)\Big\{ -\frac{\partial}{\partial v} \int_{s-\delta_0}^{s} \varsigma_l(u_\varepsilon(\sigma,y) - v)\big(- \mathrm{div} F_\varepsilon(u_\varepsilon(\sigma,y)) + \varepsilon \Delta u_\varepsilon(\sigma,y)\big)\, d\sigma \\
&\quad + \int_{s-\delta_0}^{s} \int_{|z|>0} \Big(\varsigma_l(u_\varepsilon(\sigma,y) + \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z) - v) - \varsigma_l(u_\varepsilon(\sigma,y) - v)\Big)\, \tilde{N}(dz,d\sigma) \\
&\quad + \int_{s-\delta_0}^{s} \int_{|z|>0} \int_{\lambda=0}^{1} (1-\lambda)|\eta_\varepsilon(y, u_\varepsilon(\sigma,y); z)|^2 \varsigma_l''(u_\varepsilon(\sigma,y) - v + \lambda \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z)) \\
&\qquad \times\, d\lambda\, m(dz)\, d\sigma \Big\}\, dy\, ds\, dv\Big]
\end{aligned}$$

(by the Itô-Lévy product rule and integration by parts)

$$\begin{aligned}
&= E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta', \phi_{\delta,\delta_0}](s;y,v)\Big(\int_{s-\delta_0}^{s} \varsigma_l(u_\varepsilon(\sigma,y) - v)\, \mathrm{div} F_\varepsilon(u_\varepsilon(\sigma,y))\, d\sigma\Big)\, ds\, dy\, dv\Big] \\
&\quad - E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta', \phi_{\delta,\delta_0}](s;y,v)\Big(\int_{s-\delta_0}^{s} \varsigma_l(u_\varepsilon(\sigma,y) - v)\, \varepsilon \Delta u_\varepsilon(\sigma,y))\, d\sigma\Big)\, ds\, dy\, dv\Big] \\
&\quad + E\Big[\int_{\Pi_T} \int_{\mathbb{R}_v} \int_{r=s-\delta_0}^{s} \int_{\mathbb{R}_x^d} \int_{|z|>0} \int_{\alpha=0}^{1} \Big(\beta(v(r,x,\alpha) + \eta(x, v(r,x,\alpha); z) - v) - \beta(v(r,x,\alpha) - v)\Big) \\
&\qquad \times \Big(\varsigma_l(u_\varepsilon(r,y) + \eta_\varepsilon(y, u_\varepsilon(r,y); z) - v) - \varsigma_l(u_\varepsilon(r,y) - v)\Big) \\
&\qquad \times \rho_{\delta_0}(r-s)\, \psi(s,y)\, \varrho_\delta(x-y)\, d\alpha\, m(dz)\, dx\, dr\, dv\, dy\, ds\Big] \\
&\quad + E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta, \phi_{\delta,\delta_0}](s;y,v)\Big\{ \int_{s-\delta_0}^{s} \int_{|z|>0} \int_{\lambda=0}^{1} (1-\lambda)|\eta_\varepsilon(y, u_\varepsilon(\sigma,y); z)|^2 \\
&\qquad \times \varsigma_l''(u_\varepsilon(\sigma,y) - v + \lambda \eta_\varepsilon(y, u_\varepsilon(\sigma,y); z))\, d\lambda\, m(dz)\, d\sigma \Big\}\, dy\, ds\, dv\Big] \\
&=: A_1^{l,\varepsilon}(\delta,\delta_0) + A_2^{l,\varepsilon}(\delta,\delta_0) + B^{\varepsilon,l} + A_3^{l,\varepsilon}(\delta,\delta_0).
\end{aligned}$$

Claim 1:

$$A_1^{l,\varepsilon}(\delta,\delta_0) \to 0 \quad \text{as } \delta_0 \to 0.$$

Justification: Let

$$G_\varepsilon(u,v) = \int_{r=0}^{v} \beta''(u-r) F'_{\varepsilon,k}(r)\,dr \quad \text{for } u,v \in \mathbb{R}.$$

It is easy to check that there is a positive integer p such that

$$\sup_{\varepsilon>0} |G_\varepsilon(u,v)| \le C_\beta(1+|u|^p) \quad \text{for all } u,v\in\mathbb{R}. \tag{5.12}$$

Furthermore, define

$$X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v) := \int_{\mathbb{R}^d_x}\int_{r=0}^{T}\int_{|z|>0}\int_{\lambda=0}^{1}\int_{\alpha=0}^{1} \eta(x,v(r,x,\alpha);z))G_\varepsilon(v(r,x,\alpha)+\lambda\eta(x,v(r,x,\alpha);z),v) \times \phi_{\delta,\delta_0}(r,x;s,y)\,d\alpha\,d\lambda\,\tilde{N}(dz,dr)\,dx.$$

Once again by differentiating under the integral sign,

$$\begin{aligned}
&\partial_v X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v)\\
&= \int_{\mathbb{R}^d_x}\int_{r=0}^{T}\int_{|z|>0}\int_{\lambda=0}^{1}\int_{\alpha=0}^{1} \eta(x,v(r,x,\alpha);z))\partial_v G_\varepsilon(v(r,x,\alpha)+\lambda\eta(x,v(r,x,\alpha);z),v)\\
&\qquad\qquad \times \phi_{\delta,\delta_0}(r,x;s,y)\,d\alpha\,d\lambda\,\tilde{N}(dz,dr)\,dx.\\
&\partial_{y_k} X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v) = X_\varepsilon[\partial_{y_k}\phi_{\delta,\delta_0}](s;y,v).
\end{aligned}$$

One can argue as in Lemma 5.5 (with the aid of (5.12) and moment estimates) to arrive at the conclusion that there exists a constant $C=C(\beta,\psi)$ and $p\in\mathbb{N}$ such that

$$\sup_{\varepsilon>0}\sup_{0\le s\le T}\Big(E\Big[||X_\varepsilon[\partial_{y_k}\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^2_{L^\infty(\mathbb{R}^d\times\mathbb{R})}\Big]\Big) \le \frac{C(\beta,\psi)}{\delta_0^{\frac{2(p-1)}{p}}}. \tag{5.13}$$

Now we repeatedly use integration by parts to obtain

$$\begin{aligned}
&\int_{\mathbb{R}_v}\int_{\Pi_T} J[\beta',\phi_{\delta,\delta_0}](s,y,v)\Big(\int_{s-\delta_0}^{s} \varsigma_l(u_\varepsilon(\sigma,y)-v)F'_{\varepsilon,k}(v)\partial_{y_k}u_\varepsilon(\sigma,y)\,d\sigma\Big)\,ds\,dy\,dv\\
&= \int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s}\int_{\mathbb{R}^d_x}\int_{r=0}^{T}\int_{|z|>0}\int_{\lambda=0}^{1}\int_{\alpha=0}^{1} \beta''(v(r,x,\alpha)+\lambda\eta(x,v(r,x,\alpha);z)-v)\\
&\qquad \times \eta(x,v(r,x,\alpha);z)F'_{\varepsilon,k}(v)\phi_{\delta,\delta_0}(r,x;s,y)\varsigma_l(u_\varepsilon(\sigma,y)-v)\\
&\qquad \times \partial_{y_k}u_\varepsilon(\sigma,y)\,d\alpha\,d\lambda\,\tilde{N}(dz,dr)\,dx\,d\sigma\,ds\,dy\,dv\\
&= \int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s} \partial_v X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v)\varsigma_l(u_\varepsilon(\sigma,y)-v)\partial_{y_k}u_\varepsilon(\sigma,y)\,d\sigma\,ds\,dy\,dv\\
&= \int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s} X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v)\varsigma_l'(u_\varepsilon(\sigma,y)-v)\partial_{y_k}u_\varepsilon(\sigma,y)\,d\sigma\,ds\,dy\,dv\\
&= \int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s} X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v)\partial_{y_k}\varsigma_l(u_\varepsilon(\sigma,y)-v)\,d\sigma\,ds\,dy\,dv\\
&= -\int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s} \partial_{y_k}X_\varepsilon[\phi_{\delta,\delta_0}](s;y,v)\varsigma_l(u_\varepsilon(\sigma,y)-v)\,d\sigma\,ds\,dy\,dv\\
&= -\int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s} X_\varepsilon[\partial_{y_k}\phi_{\delta,\delta_0}](s;y,v)\varsigma_l(u_\varepsilon(\sigma,y)-v)\,d\sigma\,ds\,dy\,dv.
\end{aligned} \tag{5.14}$$

Therefore, from (5.14) and (5.13), we have

$$\big|A_1^{l,\varepsilon}(\delta,\delta_0)\big| \le \sum_k \Big|E\Big[\int_{\mathbb{R}_v}\int_{\Pi_T}\int_{\sigma=s-\delta_0}^{s} X_\varepsilon[\partial_{y_k}\phi_{\delta,\delta_0}](s;y,v)\varsigma_l(u_\varepsilon(\sigma,y)-v)\,d\sigma\,ds\,dy\,dv\Big]\Big|$$

$$\le C\delta_0 \frac{C(\beta,\phi,\delta)}{\delta_0^{\frac{p-1}{p}}} = C_1(\beta,\phi,\delta)\delta_0^{\frac{1}{p}} \to 0 \quad \text{as } \delta_0 \to 0.$$

Claim 2:

$$A_2^{l,\varepsilon}(\delta,\delta_0) \to 0 \quad \text{as } \delta_0 \to 0.$$

Justification: Clearly,

$$\begin{aligned}
&|A_2^{l,\varepsilon}(\delta,\delta_0)| \\
&\le E\Big[\int_v \int_{s=0}^T \int_{|y|\le C_\phi} \int_{s-\delta_0}^s ||J[\beta',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||_{L^\infty(\mathbb{R}^d\times\mathbb{R})} \varsigma_l(v-u_\varepsilon(\sigma,y))\,\varepsilon|\Delta u_\varepsilon(\sigma,y)|\,d\sigma\,dy\,ds\,dv\Big] \\
&= E\Big[\int_{s=0}^T \int_{|y|\le C_\phi} \int_{s-\delta_0}^s ||J[\beta',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||_{L^\infty(\mathbb{R}^d\times\mathbb{R})}\,\varepsilon|\Delta u_\varepsilon(\sigma,y)|\,d\sigma\,dy\,ds\Big] \\
&\le \varepsilon C(\psi) \int_{s=0}^T \int_{s-\delta_0}^s \Big(E\Big[||J[\beta',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^2_{L^\infty(\mathbb{R}^d\times\mathbb{R})}\Big]\Big)^{\frac12} \Big(E\Big[\int_{|y|\le C_\phi} |\Delta u_\varepsilon(\sigma,y)|^2\Big]\Big)^{\frac12}\,d\sigma\,dy\,ds \\
&\le C(\beta,\psi,\delta)\Big(\sup_{0\le s\le T} E\Big[||J[\beta',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^2_{L^\infty(\mathbb{R}^d\times\mathbb{R})}\Big]\Big)^{\frac12} \\
&\qquad\qquad \times \varepsilon \int_{s=0}^T \int_{\sigma=s-\delta_0}^s \Big(E\Big[\int_{|y|\le C_\phi} |\Delta u_\varepsilon(\sigma,y)|^2\,dy\Big]\Big)^{\frac12}\,d\sigma\,ds \\
&\le \frac{C(\beta,\varepsilon,\psi,T)}{\delta_0^{\frac{p-1}{p}}}\,\delta_0 \sup_{0\le r\le T} E\Big[||\Delta u_\varepsilon(r)||_2\Big] \\
&\le C(\beta,\psi,\varepsilon,T)\delta_0^{\frac1p} \quad \Big(\text{as } \sup_{0\le t\le T} E\Big[||\Delta u_\varepsilon(t,\cdot)||_{L^2(\mathbb{R}^d)}\Big] \le C(\varepsilon,T) \text{ by Lemma 3.7}\Big) \\
&\to 0 \text{ as } \delta_0 \to 0.
\end{aligned}$$

Claim 3:

$$A_3^{l,\varepsilon}(\delta,\delta_0) \to 0 \quad \text{as } \delta_0 \to 0.$$

Justification: First, we use integration by parts to conclude

$$\begin{aligned}
&A_3^{l,\varepsilon}(\delta,\delta_0) \\
&= E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} J[\beta,\phi_{\delta,\delta_0}](s;y,v)\Big(\int_{s-\delta_0}^s \int_{|z|>0} \int_{\lambda=0}^1 \varsigma_l''(u_\varepsilon(\sigma,y)-v+\lambda\eta_\varepsilon(y,u_\varepsilon(\sigma,y);z)) \\
&\qquad\qquad \times (1-\lambda)\eta_\varepsilon^2(y,u_\varepsilon(\sigma,y);z)\,d\lambda\,m(dz)\,d\sigma\Big)\,ds\,dy\,dv\Big] \\
&= E\Big[\int_{\mathbb{R}_v} \int_{\Pi_T} \int_{\sigma=s-\delta_0}^s \int_{|z|>0} \int_{\alpha=0}^1 J[\beta'',\phi_{\delta,\delta_0}](s;y,v)\varsigma_l(u_\varepsilon(\sigma,y)-v+\lambda\eta_\varepsilon(y,u_\varepsilon(\sigma,y);z)) \\
&\qquad\qquad \times (1-\lambda)\ \eta_\varepsilon^2(y,u_\varepsilon(\sigma,y);z)\,d\lambda\,m(dz)\,d\sigma\,ds\,dy\,dv\Big],
\end{aligned}$$

and therefore

$$\begin{aligned}
&\big|A_3^{l,\varepsilon}(\delta,\delta_0)\big| \\
&\le E\Big[\int_{s=0}^T \int_{|y|<C_\psi} \int_{\sigma=s-\delta_0}^s \int_{|z|>0} ||J[\beta'',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||_\infty \eta_\varepsilon^2(y,u_\varepsilon(\sigma,y);z)\,m(dz)\,d\sigma\,dy\,ds\Big] \\
&\le CE\Big[\int_{s=0}^T \int_{|y|<C_\psi} \int_{\sigma=s-\delta_0}^s ||J[\beta'',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||_\infty g^2(y)(1+|u_\varepsilon(\sigma,y)|^2)\,d\sigma\,dy\,ds\Big] \\
&\le C\int_{s=0}^T \int_{\sigma=s-\delta_0}^s \Big(E\Big[||J[\beta'',\phi_{\delta,\delta_0}](s;\cdot,\cdot)||^2_\infty\Big]\Big)^{\frac12} \Big(E\Big[\int_{|y|<C_\psi} g^4(y)(1+|u_\varepsilon(\sigma,y)|^4\,dy\Big]\Big)^{\frac12}\,d\sigma\,ds \\
&\le \frac{C(\beta,\psi,\delta)}{\delta_0^{\frac{p-1}{p}}} \int_{s=0}^T \int_{\sigma=s-\delta_0}^s \Big(1+E\Big[||u_\varepsilon(\sigma)||_4^4\Big]\Big)^{\frac12}\,d\sigma\,ds
\end{aligned}$$

$$\le C(\beta,\psi,\delta)\delta_0^{\frac{1}{p}} T\Big(1+\sup_{\varepsilon>0}\sup_{0\le t\le T} E\Big[||u_\varepsilon(t,\cdot)||_4^4\Big]\Big)^{\frac12} \to 0 \quad \text{as } \delta_0\to 0.$$

The next claim is about $B^{l,\varepsilon}(\delta,\delta_0)$.

Claim 4:

$$\begin{aligned}
&\lim_{l\to 0}\lim_{\delta_0\to 0} B^{l,\varepsilon}(\delta,\delta_0)\\
=&E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1 \Big\{\beta(v(r,x,\alpha)+\eta(x,v(r,x,\alpha);z)-u_\varepsilon(r,y)-\eta_\varepsilon(y,u_\varepsilon;z))\\
&\qquad -\beta(v(r,x,\alpha)-u_\varepsilon(r,y)-\eta_\varepsilon(y,u_\varepsilon(r,y);z))\\
&\qquad -\beta(v(r,x,\alpha)+\eta(x,v(r,x,\alpha);z)-u_\varepsilon(r,y))\\
&\qquad +\beta(v(r,x,\alpha)-u_\varepsilon(r,y))\Big\}\psi(r,y)\,\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dy\,dr\Big]
\end{aligned}$$

Justification: Note that, using integration by parts, $B^{l,\varepsilon}(\delta,\delta_0)$ can be written as

$$\begin{aligned}
&B^{l,\varepsilon}(\delta,\delta_0)\\
&=E\Big[\int_{\Pi_T}\int_{\mathbb{R}_v}\int_{r=s-\delta_0}^{s}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times \eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\times \rho_{\delta_0}(r-s)\,\psi(s,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\,dx\,dr\,dv\,dy\,ds\Big]\\
&=E\Big[\int_{s=\delta_0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_v}\int_{r=s-\delta_0}^{s}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times \eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\times \rho_{\delta_0}(r-s)\,\psi(s,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\,dx\,dr\,dv\,dy\,ds\Big]\\
&\quad+E\Big[\int_{s=0}^{\delta_0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_v}\int_{r=s-\delta_0}^{s}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times \eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\times \rho_{\delta_0}(r-s)\,\psi(s,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\,dx\,dr\,dv\,dy\,ds\Big]\\
&=E\Big[\int_{r=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_v}\int_{s=r}^{r+\delta_0}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times \eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\times \rho_{\delta_0}(r-s)\,\psi(s,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\,dx\,ds\,dv\,dy\,dr\Big]+o(\delta_0),
\end{aligned}$$

where we have used Fubini's theorem to infer the last line. Hence

$$\begin{aligned}
&\Big|B^{l,\varepsilon}(\delta,\delta_0)-E\Big[\int_{\Pi_T}\int_{\mathbb{R}_v}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times \eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\times\ \psi(r,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\ dx\,dv\,dy\,dr\Big]\Big|\\
&=\Big|B^{l,\varepsilon}(\delta,\delta_0)-E\Big[\int_{\Pi_T}\int_{\mathbb{R}_v}\int_{s=r}^{r+\delta_0}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\times \eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\times\ \rho_{\delta_0}(r-s)\,\psi(r,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\ dx\,ds\,dv\,dy\,dr\Big]\Big|\\
&\le E\Big[\int_{r=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_v}\int_{s=r}^{r+\delta_0}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad \times |\eta(x,v(r,x,\alpha);z)|\,|\eta_\varepsilon(y,u_\varepsilon(r,y);z)|\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\rho_{\delta_0}(r-s)\\
&\qquad\qquad\quad \times |\psi(s,y)-\psi(r,y)|\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\,dx\,ds\,dv\,dy\,dr\Big]+o(\delta_0)\\
&\le E\Big[\int_{r=0}^T\int_{\mathbb{R}^d_y}\int_{s=r}^{r+\delta_0}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1 |\eta(x,v(r,x,\alpha);z)|\,||\beta''||_\infty|\eta_\varepsilon(y,u_\varepsilon(r,y);z)|\\
&\qquad\qquad \times \rho_{\delta_0}(r-s)\,|r-s|\,||\partial_t\psi||_\infty\,\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,ds\,dv\,dy\,dr\Big]+o(\delta_0)\\
&\le \delta_0 C(\beta'',\psi)E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1 |\eta(x,v(r,x,\alpha);z)\eta_\varepsilon(y,u_\varepsilon(r,y);z)|\\
&\qquad\qquad\qquad\qquad \times \varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dy\,dr\Big]+o(\delta_0)\\
&\le \delta_0 C(\beta'',\psi)\Big(1+\sup_{0\le r\le T}E\Big[||v(r,\cdot,\cdot)||_2^2\Big]+\sup_{\varepsilon>0}\sup_{0\le r\le T}E\Big[||u_\varepsilon(r,\cdot)||_2^2\Big]\Big)+o(\delta_0).
\end{aligned}
$$

and so

$$
\begin{aligned}
&\lim_{\delta_0\to 0} B^{l,\varepsilon}(\delta,\delta_0)\\
&= E\Big[\int_{\Pi_T}\int_{\mathbb{R}_v}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\Big(\beta(v(r,x,\alpha)+\eta(x,v(r,x,\alpha);z)-v)-\beta(v(r,x,\alpha)-v)\Big)\\
&\qquad\qquad \times\Big(\varsigma_l(u_\varepsilon(r,y)+\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)-\varsigma_l(u_\varepsilon(r,y)-v)\Big)\\
&\qquad\qquad \times\ \psi(r,y)\,\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dv\,dy\,dr\Big]\\
&\equiv E\Big[\int_{\Pi_T}\int_{\mathbb{R}_v}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1\int_{\theta=0}^1 \eta(x,v(r,x,\alpha);z)\beta''(v(r,x,\alpha)-v+\lambda\eta(x,v(r,x,\alpha);z))\\
&\qquad\qquad \times \eta_\varepsilon(y,u_\varepsilon(r,y);z)\varsigma_l(u_\varepsilon(r,y)+\theta\eta_\varepsilon(y,u_\varepsilon(r,y);z)-v)\\
&\qquad\qquad \times \psi(r,y)\,\varrho_\delta(x-y)\,d\theta\,d\lambda\,d\alpha\,m(dz)\,dx\,dv\,dy\,dr\Big],
\end{aligned}
\tag{5.15}
$$

where we have first re-written the terms using the fundamental theorem of integral calculus and then applied integration by parts with respect to v. It is now routine to pass to the limit $l\to 0$ in (5.15), and hence the conclusion follows. ☐

Next, we consider the term I_5+J_5 and prove the following lemma.

Lemma 5.7. *Assume that $\vartheta\to 0,\delta\to 0$ and $\frac{\vartheta}{\delta}\to 0$. Then*

$$
\lim_{\frac{\vartheta}{\delta}\downarrow 0,\ \vartheta\downarrow 0,\ \delta\downarrow 0}\Big[\lim_{\varepsilon_n\downarrow 0}\lim_{l\downarrow 0}\lim_{\delta_0\downarrow 0}(I_5+J_5)\Big]=0.
$$

Proof. Note that

$$
\begin{aligned}
&\Big|I_5 - E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k} F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\Big|\\
&=\Big|E\Big[\int_{s=0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\Big(F^\beta(v(t,x,\alpha),k)-F^\beta(v(s,x,\alpha),k)\Big)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\rho_{\delta_0}(t-s)\\
&\qquad\qquad \times\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big]\\
&\quad + E\Big[\int_{s=0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k} F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\rho_{\delta_0}(t-s)\\
&\qquad\qquad \times\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big]\\
&\quad - E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k} F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\,\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\Big|
\end{aligned}
$$

$$\le E\Big[\int_{s=0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\big|F^\beta(v(t,x,\alpha),k)-F^\beta(v(s,x,\alpha),k)\big||\nabla_x\varrho_\delta(x-y)|\psi(s,y)\rho_{\delta_0}(t-s)$$
$$\times\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big]$$
$$+\Big|E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\Big(1-\int_{t=0}^T\rho_{\delta_0}(t-s)\,dt\Big)$$
$$\times\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]\Big|$$
$$\le E\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\big|F^\beta(v(t,x,\alpha),k)-F^\beta(v(s,x,\alpha),k)\big||\nabla_x\varrho_\delta(x-y)|\psi(s,y)\rho_{\delta_0}(t-s)$$
$$\times\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big]+o(\delta_0)$$
$$+E\Big[\int_{s=0}^{\delta_0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\big|F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\big|\psi(s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]$$

(we have used the fact that $\int_{t=0}^T\rho_{\delta_0}(t-s)\,dt\le 1$, equality holds if $s\ge\delta_0$)

$$\le CE\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\big|v(t,x,\alpha)-v(s,x,\alpha)\big|\big(1+|v(t,x,\alpha)|^p+|v(s,x,\alpha)|^p\big)|\nabla_x\varrho_\delta(x-y)|$$
$$\times\psi(s,y)\rho_{\delta_0}(t-s)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,dt\,ds\Big]+o(\delta_0)$$
$$+E\Big[\int_{s=0}^{\delta_0}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\big|F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\big|\psi(s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]$$

(we have used the Lipschitz continuity of $F^\beta(\cdot,k)$ in above)

$$\le C\Big(E\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1|v(t,x,\alpha)-v(s,x,\alpha)|^2\rho_{\delta_0}(t-s)\,d\alpha\,dx\,dt\,ds\Big]\Big)^{\frac12}+o(\delta_0)$$
$$\le C\Big(E\Big[\int_{r=0}^1\int_{\Pi_T}\int_{\alpha=0}^1|v(t+\delta_0\,r,x,\alpha)-v(t,x,\alpha)|^2\rho(-r)\,d\alpha\,dt\,dx\,dr\Big]\Big)^{\frac12}+o(\delta_0).$$

Note that $\lim_{\delta_0\downarrow 0}\int_{t=0}^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1|v(t+\delta_0 r,x,\alpha)-v(t,x,\alpha)|^2\,d\alpha\,dx\,dt\to 0$ almost surely for all $r\in[0,1]$. Therefore, by the bounded convergence theorem,

$$\lim_{\delta_0\downarrow 0}E\Big[\int_{t=0}^T\int_{r=0}^1\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1|v(t+\delta_0 r,x,\alpha)-v(t,x,\alpha)|^2\rho(-r)\,d\alpha\,dx\,dr\,dt\Big]=0.$$

This implies that

$$\lim_{\delta_0\to 0}I_5$$
$$=E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}F^\beta(v(s,x,\alpha),k)\cdot\nabla_x\varrho_\delta(x-y)\,\psi(s,y)\varsigma_l(u_\varepsilon(s,y)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]$$
$$=-E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}F^\beta(v(s,x,\alpha),u_\varepsilon(s,y)-k)\cdot\nabla_y\varrho_\delta(x-y)\,\psi(s,y)\varsigma_l(k)\,dk\,d\alpha\,dx\,dy\,ds\Big].$$

In a similar manner, we find

$$\lim_{\delta_0\to 0}J_5=E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_x}\int_{\mathbb{R}^d_y}\int_{\mathbb{R}_k}\int_{\alpha=0}^1F^\beta(u_\varepsilon(s,y),v(s,x,\alpha)-k)\cdot\nabla_y\varrho_\delta(x-y)\,\psi(s,y)\varsigma_l(k)\,d\alpha\,dk\,dx\,dy\,ds\Big].$$

Note that

$$I_5+J_5$$

$$=E\Big[\int_{\Pi_T}\int_{\Pi_T}\int_{\mathbb{R}_k}\int_{\alpha=0}^{1}\Big(-F^\beta(v(t,x,\alpha),u_\varepsilon(s,y)-k)+F^\beta(u_\varepsilon(s,y),v(t,x,\alpha)-k)\Big)\cdot\nabla_y\varrho_\delta(x-y)$$
$$\times\psi(s,y)\rho_{\delta_0}(t-s)\varsigma_l(k)\,d\alpha\,dk\,dx\,dt\,dy\,ds\Big].$$

Hence,

$$\lim_{\delta_0\to 0}(I_5+J_5)$$
$$=E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\mathbb{R}_k}\int_{\alpha=0}^{1}\Big(-F^\beta(v(t,x,\alpha),u_\varepsilon(t,y)-k)+F^\beta(u_\varepsilon(t,y),v(t,x,\alpha)-k)\Big)\cdot\nabla_y\varrho_\delta(x-y)$$
$$\times\psi(t,y)\varsigma_l(k)\,d\alpha\,dk\,dx\,dt\,dy\Big]. \tag{5.16}$$

There exists $p\in N$ such that for all $a,b,c\in\mathbb{R}$

$$|F^\beta(a,b)-F^\beta(a,c)|\le K|b-c|(1+|b|^p+|c|^p)\text{ and }|F^\beta(b,a)-F^\beta(c,a)|\le K|b-c|(1+|b|^p+|c|^p). \tag{5.17}$$

In view of (5.17), we can routinely pass to the limit $l\to 0$ in (5.16) and conclude

$$\lim_{l\to 0}\lim_{\delta_0\to 0}(I_5+J_5)$$
$$=E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^{1}\Big(F^\beta\big(u_\varepsilon(t,y),v(t,x,\alpha)\big)-F^\beta\big(v(t,x,\alpha),u_\varepsilon(t,y)\big)\Big)\cdot\nabla_y\varrho_\delta(x-y)$$
$$\times\psi(t,y)\,d\alpha\,dx\,dt\,dy\Big].$$

Note that

$$|F_k^{\beta_\vartheta}(a,b)-F_k^{\beta_\vartheta}(b,a)|\le|F_k^{\beta_\vartheta}(a,b)-\operatorname{sign}(a-b)(F_k(a)-F_k(b))|$$
$$+|F_k^{\beta_\vartheta}(b,a)-\operatorname{sign}(b-a)(F_k(b)-F_k(a))|\le C\vartheta(1+|a|^p+|b|^p), \tag{5.18}$$

and therefore

$$\Big|E\Big[\int_{\mathbb{R}^d_y}\int_{\Pi_T}\int_{\alpha=0}^{1}\Big\{F^\beta(u_\varepsilon(t,y),v(t,x,\alpha))-F^\beta(v(t,x,\alpha),u_\varepsilon(t,y))\Big\}\cdot\nabla_y\varrho_\delta(x-y)\psi(t,y)\,d\alpha\,dx\,dt\,dy\Big]\Big|$$
$$\le\vartheta\,CE\Big[\int_{\mathbb{R}^d_y}\int_{\Pi_T}\int_{\alpha=0}^{1}\big(1+|u_\varepsilon(t,y)|^p+|v(t,x,\alpha)|^p\big)|\nabla_y\varrho_\delta(x-y)|\psi(t,y)\,d\alpha\,dx\,dt\,dy\Big]$$
$$\le\frac{\vartheta}{\delta}C\to 0\quad\text{when }(\vartheta,\frac{\vartheta}{\delta},\delta)\to(0,0,0).$$

Hence the lemma follows. □

Lemma 5.8. *It holds that*

$$J_6\underset{\delta_0\to 0}{\to}E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^{1}\int_{\mathbb{R}_k}F^\beta(u_\varepsilon(s,y),k)\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]$$
$$\underset{l\to 0}{\to}E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^{1}F^\beta(u_\varepsilon(s,y),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx\,dy\,ds\Big]$$
$$\underset{\varepsilon\to 0}{\to}E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^{1}\int_{\gamma=0}^{1}F^\beta(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big],$$

and

$$\lim_{(\vartheta,\delta)\to(0,0)}E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^{1}\int_{\gamma=0}^{1}F^\beta(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]$$
$$=E\Big[\int_{\Pi_T}\int_{\alpha=0}^{1}\int_{\gamma=0}^{1}F(u(s,y,\gamma),v(s,y,\alpha))\cdot\nabla_y\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big].$$

Proof. The first part of the proof is divided into three steps.

Step 1: We will justify the $\delta_0 \to 0$ limit. Define

$$
\begin{aligned}
\mathcal{B}_1 := \Big| E\Big[\int_{\Pi_T} \int_{\Pi_T} \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & F^\beta(u_\varepsilon(s,y), v(t,x,\alpha) - k) \cdot \nabla_y \psi(s,y)\, \rho_{\delta_0}(t-s) \varrho_\delta(x-y) \\
& \times \varsigma_l(k)\, d\alpha\, dk\, dy\, ds\, dx\, dt \Big] \\
- E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_y} \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & F^\beta(u_\varepsilon(s,y), k) \cdot \nabla_y \psi(s,y)\, \varrho_\delta(x-y) \varsigma_l(v(s,x,\alpha)-k) d\alpha\, dk\, dy\, dx\, ds \Big] \Big| \\
= \Big| E\Big[\int_{\Pi_T} \int_{\Pi_T} \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & \Big(F^\beta(u_\varepsilon(s,y), v(t,x,\alpha)-k) - F^\beta(u_\varepsilon(s,y), v(s,x,\alpha)-k) \Big) \cdot \nabla_y \psi(s,y) \\
& \times \rho_{\delta_0}(t-s) \varrho_\delta(x-y) \varsigma_l(k)\, d\alpha\, dk\, dy\, ds\, dx\, dt \Big] \\
- E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_y} \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & F^\beta(u_\varepsilon(s,y), v(s,x,\alpha)-k) \cdot \nabla_y \psi(s,y)\, \varrho_\delta(x-y) \\
& \times \Big(1 - \int_{t=0}^T \rho_{\delta_0}(t-s)\, dt \Big) \varsigma_l(k) d\alpha\, dk\, dy\, dx\, ds \Big] \Big| \\
\le CE\Big[\int_{\Pi_T} \int_{\Pi_T} \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & \varsigma(k)\, |\nabla_y \psi(s,y)|\, \rho_{\delta_0}(t-s) \varrho_\delta(x-y) \\
& \times |v(s,x,\alpha) - v(t,x,\alpha)| \big(1 + |v(s,x,\alpha)|^p + |v(t,x,\alpha)|^p + |k|^p \big)\, d\alpha\, dk\, dy\, ds\, dx\, dt \Big] \\
+ E\Big[\int_{s=0}^{\delta_0} \int_{\mathbb{R}^d_x} \int_{\mathbb{R}^d_y} \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & \big| F^\beta(u_\varepsilon(s,y), k) \cdot \nabla_y \psi(s,y) \big|\, \varrho_\delta(x-y) \varsigma_l(v(s,x,\alpha)-k) d\alpha\, dk\, dy\, dx\, ds \Big]
\end{aligned}
$$

(we used the inequality(5.17))

$$
\begin{aligned}
\le CE\Big[\int_{s=\delta_0}^T \int_{\mathbb{R}^d_x} \int_{t=0}^T \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & |v(s,x,\alpha) - v(t,x,\alpha)| \big(1 + |v(s,x,\alpha)|^p + |v(t,x,\alpha)|^p + |k|^p \big) \\
& \times \varsigma_l(k) \rho_{\delta_0}(t-s)\, d\alpha\, dk\, dt\, dx\, ds \Big] \\
+ CE\Big[\int_{s=0}^{\delta_0} \int_{\mathbb{R}^d_x} \int_{t=0}^T \int_{\mathbb{R}_k} \int_{\alpha=0}^1 & |v(s,x,\alpha) - v(t,x,\alpha)| \big(1 + |v(s,x,\alpha)|^p + |v(t,x,\alpha)|^p + |k|^p \big) \\
& \times \varsigma_l(k) \rho_{\delta_0}(t-s)\, d\alpha\, dk\, dt\, dx\, ds \Big] + o(\delta_0) \\
\le C(\beta,\psi,l) \Big(E\Big[\int_{s=\delta_0}^T \int_{t=0}^T \int_{\mathbb{R}^d} \int_{\alpha=0}^1 & |v(s,x,\alpha) - v(t,x,\alpha)|^2 \rho_{\delta_0}(t-s) d\alpha\ dx\, dt\, ds \Big] \Big)^{\frac{1}{2}} + o(\delta_0) \\
= C(\beta,\psi,l) \Big(E\Big[\int_{r=0}^1 \int_{\mathbb{R}^d} \int_{t=0}^T \int_{\alpha=0}^1 & |v(t+\delta_0 r, x, \alpha) - v(t,x,\alpha)|^2 \rho(-r) d\alpha\, dt\, dx\, dr \Big] \Big)^{\frac{1}{2}} + o(\delta_0),
\end{aligned}
$$

where we have used the Schwartz's inequality with respect to the measure $\rho_{\delta_0}(t-s)\, d\alpha\, dx\, dt\, ds\, dP(\omega)$. We recall that $\lim_{\delta_0 \downarrow 0} \int_{t=0}^T \int_{\mathbb{R}^d_x} \int_{\alpha=0}^1 |v(t+\delta_0 r, x,\alpha) - v(t,x,\alpha)|^2\ d\alpha\, dx\, dt \to 0$ almost surely, for all $r \in [0,1]$. Therefore, by the bounded convergence theorem,

$$
\lim_{\delta_0 \downarrow 0} E\Big[\int_{r=0}^1 \int_{\mathbb{R}^d_x} \int_{t=0}^T \int_{\alpha=0}^1 |v(t+\delta_0 r, x, \alpha) - v(t,x,\alpha)|^2 \rho(-r) d\alpha\, dt\, dx\, dr \Big] = 0,
$$

and therefore the first step follows.

Step 2: We will justify the $l \to 0$ limit. Let

$$
\begin{aligned}
\mathcal{B}_2 := & E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_x} \int_{\alpha=0}^1 \int_{\mathbb{R}_k} F^\beta(u_\varepsilon(s,y), k) \cdot \nabla_y \psi(s,y)\, \varrho_\delta(x-y) \varsigma_l(v(s,x,\alpha)-k)\, dk\, d\alpha\, dx\, dy\, ds \Big] \\
& - E\Big[\int_{\Pi_T} \int_{\mathbb{R}^d_x} \int_{\alpha=0}^1 F^\beta(u_\varepsilon(s,y), v(s,x,\alpha)) \cdot \nabla_y \psi(s,y)\, \varrho_\delta(x-y)\, d\alpha\, dx\, dy\, ds \Big]
\end{aligned}
$$

$$= E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\Big(F^\beta(u_\varepsilon(s,y),k) - F^\beta(u_\varepsilon(s,y),v(s,x,\alpha))\Big)\cdot\nabla_y\psi(s,y)$$
$$\times\,\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big].$$

Therefore, by (5.17), there exists a natural number p such that

$$|\mathcal{B}_2| \le CE\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\mathbb{R}_k}|v(s,x,\alpha)-k|\big(1+|v(s,x,\alpha)|^p + |v(s,x,\alpha)-k|^p\big)|\nabla_y\psi(s,y)|$$
$$\times\,\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,dy\,ds\Big]$$
$$\le C||\nabla_y\psi(s,\cdot)||_\infty E\Big[\int_0^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}|v(s,x,\alpha)-k|(1+|v(s,x,\alpha)|^p)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,ds\Big]$$
$$+ C||\nabla_y\psi(s,\cdot)||_\infty E\Big[\int_0^T\int_{\mathbb{R}^d_x}\int_{|y|\le C_\psi}\int_{\alpha=0}^1\int_{\mathbb{R}_k}|v(s,x,\alpha)-k|^{p+1}\varsigma_l(v(s,x,\alpha)-k)$$
$$\times\,\varrho_\delta(x-y)\,dk\,d\alpha\,dy\,dx\,ds\Big]$$
$$\le C||\nabla_y\psi(s,\cdot)||_\infty E\Big[\int_0^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\mathbb{R}_k}|v(s,x,\alpha)-k|(1+|v(s,x,\alpha)|^p)\varsigma_l(v(s,x,\alpha)-k)\,dk\,d\alpha\,dx\,ds\Big]$$
$$+ C\,T\,||\nabla_y\psi(s,\cdot)||_\infty\, l^{p+1}\int_{|y|\le C_\psi}\int_{\mathbb{R}^d_x}\varrho_\delta(x-y)\,dx\,dy$$
$$\le C\,T\,||\nabla_y\psi(s,\cdot)||_\infty\, l\Big(l^p + \sup_{0\le s\le T}E\Big[||v(s,\cdot,\cdot)||_p^p\Big]\Big)\to 0 \quad \text{as } l\to 0.$$

Step 3: We now justify the passage to the limit $\varepsilon_n \to 0$. Let

$$G_x(s,y,\omega,\xi) = \int_{\mathbb{R}^d_x}\int_{\alpha=0}^1 F^\beta(\xi, v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx.$$

As in Lemma 5.2, $G_x(s,y,\omega,\xi)$ is a Caratheodory function for every $x\in\mathbb{R}^d$ and $\{G_x(s,y,\omega,u_{\varepsilon_n}(s,y))\}_n$ is bounded and uniformly integrable. This allows us to conclude that

$$\lim_{\varepsilon_n\to 0}E\Big[\int_{\mathbb{R}^d_y}\int_{\Pi_T}\int_{\alpha=0}^1 F^\beta(u_{\varepsilon_n}(s,y),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\alpha\,dx\,ds\,dy\Big]$$
$$= E\Big[\int_{\mathbb{R}^d_y}\int_{\Pi_T}\int_{\alpha=0}^1\int_{\gamma=0}^1 F^\beta(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,ds\,dy\Big].$$

This completes the proof of the first half of the lemma.

To prove the second half of the lemma, let us denote

$$\mathcal{B}(\vartheta,\delta) := \Big|E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\gamma=0}^1 F^{\beta_\vartheta}(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dy\,dx\,ds\Big]$$
$$- E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\gamma=0}^1 F(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big]\Big|$$
$$\le E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_y}\int_{\alpha=0}^1\int_{\gamma=0}^1\sum_{k=1}^d\Big|F_k^{\beta_\vartheta}(u(s,y,\gamma),v(s,x,\alpha)) - F_k(u(s,y,\gamma),v(s,x,\alpha))\Big|$$
$$\times\,|\partial_{y_k}\psi(s,y)|\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dy\,dx\,ds\Big].$$

By (5.18), we conclude

$$\mathcal{B}(\vartheta,\delta) \le C\vartheta\Big(1+\sup_{0\le s\le T}E\Big[||v(s,\cdot,\cdot)||_p^p\Big] + \sup_{\varepsilon>0}\sup_{0\le s\le T}E\Big[||u_\varepsilon(s)||_p^p\Big]\Big).$$

Now we estimate as follows:

$$
\begin{aligned}
&\Big| E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\gamma=0}^1 F(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big] \\
&\quad - E\Big[\int_{\Pi_T}\int_{\alpha=0}^1\int_{\gamma=0}^1 F(u(s,y,\gamma),v(s,y,\alpha))\cdot\nabla_y\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big]\Big| \\
&\le E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\gamma=0}^1 \big|F(u(s,y,\gamma),v(s,x,\alpha))-F(u(s,y,\gamma),v(s,y,\alpha))\big| \\
&\qquad\qquad \times|\nabla_y\psi(s,y)|\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big] \\
&\qquad \big(\text{by the inequality (5.17)}\big) \\
&\le CE\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\gamma=0}^1 |v(s,y,\gamma)-v(s,x,\gamma)|\big(1+|v(s,y,\gamma)|^p+|u(s,y,\alpha)|^p\big) \\
&\qquad\qquad \times|\nabla_y\psi(s,y)|\,\varrho_\delta(x-y)\,d\alpha\,d\gamma\,dx\,dy\,ds\Big] \\
&\qquad (\text{by Cauchy-Schwartz's inequality w.r.t. the measure } \varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\,dP(\omega)) \\
&\le C\Big(E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y}\int_{\mathbb{R}^d_x}\int_{\gamma=0}^1 |v(s,y,\gamma)-v(s,x,\gamma)|^2\varrho_\delta(x-y)\,d\gamma\,dx\,dy\,ds\Big]\Big)^{\frac12} \\
&= C\Big(E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_z}\int_{\mathbb{R}^d_y}\int_{\gamma=0}^1 |v(s,y,\gamma)-v(s,y+\delta z,\gamma)|^2\varrho(z)\,d\gamma\,dy\,dz\,ds\Big]\Big)^{\frac12} = o(\delta).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\Big| E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1\int_{\gamma=0}^1 F^\beta(u(s,y,\gamma),v(s,x,\alpha))\cdot\nabla_y\psi(s,y)\,\varrho_\delta(x-y)\,d\gamma\,d\alpha\,dx\,dy\,ds\Big] \\
&\quad - E\Big[\int_{\Pi_T}\int_{\alpha=0}^1\int_{\gamma=0}^1 F(u(s,y,\gamma),v(s,y,\alpha))\cdot\nabla_y\psi(s,y)\,d\gamma\,d\alpha\,dy\,ds\Big]\Big| \\
&\le \mathrm{Const}(\psi)\,\vartheta + o(\delta)\to 0 \quad \text{as } (\vartheta,\delta)\to(0,0).
\end{aligned}
$$

□

Lemma 5.9. *It holds that*

$$
\begin{aligned}
\lim_{l\to0}\lim_{\delta_0\to0} J_4 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\lambda=0}^1\int_{\alpha=0}^1 &(1-\lambda)\beta''\big(u_\varepsilon(s,y)-v(s,x,\alpha)+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big) \\
&\times|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\varrho_\delta(x-y)\,d\alpha\,d\lambda\,m(dz)\,dx\,dy\,ds\Big], \qquad (5.19)
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{l\to0}\lim_{\delta_0\to0} I_4 = E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\lambda=0}^1\int_{\alpha=0}^1 &(1-\lambda)\beta''\big(v(s,x,\alpha)-u_\varepsilon(s,y)+\lambda\eta(x,v(s,x,\alpha);z)\big) \\
&\times|\eta(x,v(s,x,\alpha);z)|^2\psi(s,x)\varrho_\delta(x-y)\,d\alpha\,d\lambda\,m(dz)\,dx\,dy\,ds\Big]. \qquad (5.20)
\end{aligned}
$$

Proof. We will establish (5.19) in detail. The proof of (5.20) is very similar, and thus left to the reader. Note that J_4 can be rewritten as

$$
\begin{aligned}
J_4 = E\Big[\int_{\Pi_T\times\Pi_T}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1\int_{\lambda=0}^1 &(1-\lambda)\beta''\big(u_\varepsilon(s,y)-k+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big) \\
&\times|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\varsigma_l(v(t,x,\alpha)-k)\,d\lambda\,d\alpha\,dk\,m(dz)\,dx\,dt\,dy\,ds\Big] \\
= E\Big[\int_{\Pi_T\times\Pi_T}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1\int_{\lambda=0}^1 &(1-\lambda)\beta''\big(u_\varepsilon(s,y)-v(t,x,\alpha)+k+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big) \\
&\times|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\varsigma_l(k)\,d\lambda\,d\alpha\,dk\,m(dz)\,dx\,dt\,dy\,ds\Big].
\end{aligned}
$$

Furthermore,

$$\Big|J_4 - E\int_{\Pi_T}\int_{\mathbb{R}^d_x}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1\int_{\lambda=0}^1 (1-\lambda)\beta''(u_\varepsilon(s,y) - v(s,x,\alpha) + k + \lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z))$$

$$\times |\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\varrho_\delta(x-y)\varsigma_l(k)\,d\lambda\,d\alpha\,dk\,m(dz)\,dx\,dy\,ds\Big|$$

$$=\Big|E\Big[\int_{\Pi_T\times\Pi_T}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1\int_{\lambda=0}^1(1-\lambda)\Big(\beta''\big(u_\varepsilon(s,y) - v(t,x,\alpha) + k + \lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)$$

$$-\beta''\big(u_\varepsilon(s,y) - v(s,x,\alpha) + k + \lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)\Big)$$

$$\times|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\varsigma_l(k)\,d\lambda\,d\alpha\,dk\,m(dz)\,dx\,dt\,dy\,ds\Big]$$

$$-E\Big[\int_{s=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\mathbb{R}_k}\int_{\lambda=0}^1(1-\lambda)|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\beta''\big(u_\varepsilon(s,y) - k + \lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)$$

$$\times\psi(s,y)\Big(1-\int_{t=0}^T\rho_{\delta_0}(t-s)\,dt\Big)\,\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)d\lambda\,dk\,d\alpha\,m(dz)\,dx\,dy\,ds\Big]\Big|$$

$$\le ||\beta'''||_\infty\,E\Big[\int_{s=0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2|v(t,x,\alpha)-v(s,x,\alpha)|$$

$$\times\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\,d\alpha\,m(dz)\,dx\,dy\,dt\,ds\Big] + o(\delta_0)$$

$$\le \mathrm{Const}(\beta,\eta)\,E\Big[\int_{s=0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x}\int_{\alpha=0}^1 g^2(y)(1+|u_\varepsilon(s,y)|^2)|v(t,x,\alpha)-v(s,x,\alpha)|$$

$$\times\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)d\alpha\,dx\,dy\,dt\,ds\Big] + o(\delta_0)$$

$$\le \mathrm{Const}(\beta,\eta)\,E\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x}\int_{\alpha=0}^1 g^2(y)(1+|u_\varepsilon(s,y)|^2)|v(t,x,\alpha)-v(s,x,\alpha)|$$

$$\times\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)d\alpha\,dx\,dy\,dt\,ds\Big]$$

$$+\mathrm{Const}(\beta,\eta)\,E\Big[\int_{s=0}^{\delta_0}\int_{t=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x}\int_{\alpha=0}^1 g^2(y)(1+|u_\varepsilon(s,y)|^2)|v(t,x,\alpha)-v(s,x,\alpha)|$$

$$\times\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)d\alpha\,dx\,dy\,dt\,ds\Big] + o(\delta_0)$$

(by the Cauchy-Schwartz's inequality)

$$\le \mathrm{Const}(\beta,\eta)\sqrt{E\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x} g^4(y)(1+|u_\varepsilon(s,y)|^4)\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\,dx\,dy\,dt\,ds\Big]}$$

$$\times\sqrt{E\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_y\times\mathbb{R}^d_x}\int_{\alpha=0}^1|v(t,x,\alpha)-v(s,x,\alpha)|^2\psi(s,y)\rho_{\delta_0}(t-s)\varrho_\delta(x-y)\,d\alpha dx dy\,dt\,ds\Big]} + o(\delta_0)$$

$$\le \mathrm{Const}(\beta,\eta,\psi)\sqrt{E\Big[\int_{s=\delta_0}^T\int_{t=0}^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1|v(t,x,\alpha)-v(s,x,\alpha)|^2\rho_{\delta_0}(t-s)\,d\alpha\,dx\,dt\,ds\Big]} + o(\delta_0)$$

$$\le \mathrm{Const}(\beta,\eta,\psi)\sqrt{E\Big[\int_{r=0}^1\int_{t=0}^T\int_{\mathbb{R}^d_x}\int_{\alpha=0}^1|v(t,x,\alpha)-v(t+r\delta_0,x,\alpha)|^2\rho(-r)\,d\alpha\,dx\,dt\,dr\Big]} + o(\delta_0).$$

As before $\lim_{\delta_0\downarrow 0}\int_{r=0}^T\int_{\mathbb{R}^d}\int_{\alpha=0}^1|v(t+\delta_0 r,x,\alpha)-v(t,x,\alpha)|^2\,d\alpha\,dx\,dt = 0$ for all r. Therefore, by the dominated convergence theorem, $E\Big[\int_{r=0}^1\int_{t=0}^T\int_{\mathbb{R}^d}\int_{\alpha=0}^1|v(t,x,\alpha)-v(t+r\delta_0,x,\alpha)|^2\rho(-r)\,d\alpha\,dx\,dt\,dr\Big]\to 0$ as $\delta_0\to 0$.

Let

$$\mathcal{N} := E\Big[\int_{\Pi_T\times\mathbb{R}^d_x}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1\int_{\lambda=0}^1(1-\lambda)|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\beta''\big(u_\varepsilon(s,y)-k+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)$$

$$\times\, \psi(s,y)\varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,d\lambda\,d\alpha\,dk\,m(dz)\,dx\,dy\,ds\Big]$$
$$- E\Big[\int_{\Pi_T\times\mathbb{R}^d_x}\int_{|z|>0}\int_{\alpha=0}^1\int_{\lambda=0}^1 (1-\lambda)\beta''\big(u_\varepsilon(s,y)-v(s,x,\alpha)+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)$$
$$\times\, |\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\varrho_\delta(x-y)\,d\lambda\,d\alpha\,m(dz)\,dx\,dy\,ds\Big]$$
$$= E\Big[\int_{\Pi_T\times\mathbb{R}^d_x}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1\int_{\lambda=0}^1 (1-\lambda)|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\Big(\beta''\big(u_\varepsilon(s,y)-k+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)$$
$$-\beta''\big(u_\varepsilon(s,y)-v(s,x,\alpha)+\lambda\eta_\varepsilon(y,u_\varepsilon(s,y);z)\big)\Big)\psi(s,y)$$
$$\times\, \varrho_\delta(x-y)\varsigma_l(v(s,x,\alpha)-k)\,d\lambda\,d\alpha\,dk\,m(dz)\,dx\,dy\,ds\Big].$$

We estimate $\mathcal{N}$ as follows:

$$|\mathcal{N}| \le CE\Big[\int_{\Pi_T\times\mathbb{R}^d_x}\int_{|z|>0}\int_{\mathbb{R}_k}\int_{\alpha=0}^1 |\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2|v(s,x,\alpha)-k|\psi(s,y)\varrho_\delta(x-y)$$
$$\times\, \varsigma_l(v(s,x,\alpha)-k)\,d\alpha\,dk\,m(dz)\,dx\,dy\,ds\Big]$$
$$\le C\,l\,E\Big[\int_{\Pi_T\times\mathbb{R}^d_x}\int_{|z|>0}|\eta_\varepsilon(y,u_\varepsilon(s,y);z)|^2\psi(s,y)\varrho_\delta(x-y)\,m(dz)\,dx\,dy\,ds\Big]$$
$$\le C(\psi,\eta)\,l\to 0 \quad \text{as } l\to 0.$$

Hence, (5.19) is established. □

Lemma 5.10. *For fixed $\delta>0$ and β, it holds that*

$$\limsup_{(\varepsilon,\delta_0,l)\to 0}|J_7| = 0.$$

Proof. Note that

$$|J_7| \le \varepsilon||\beta'||_\infty\Big|E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}|\nabla_y u_\varepsilon(s,y)||\nabla_y[\psi(s,y)\varrho_\delta(x-y)|\,dx\,dy\,ds\Big]\Big|$$
$$\le \varepsilon\,||\beta'||_\infty E\Big[\int_{|y|\le K}\int_{t=0}^T\int_{\mathbb{R}^d_x}|\nabla_y u_\varepsilon(t,y)|\,|\nabla_y[\psi(t,y)\varrho_\delta(x-y)]|\,dx\,dt\,dy\Big]$$

(by the Cauchy-Schwartz's inequality)

$$\le C(\beta,\psi)\varepsilon^{\frac12}\Big(E\Big[\int_{\Pi_T}\varepsilon|\nabla_y u_\varepsilon(t,y)|^2\,dy\,dt\Big]\Big)^{\frac12}\Big(E\Big[\int_{|y|\le K}\int_0^T\big|\int_{\mathbb{R}^d_x}\nabla_y[\psi(t,y)\varrho_\delta(x-y)]\,dx\big|^2\,dt\,dy\Big]\Big)^{\frac12}$$
$$\le C(\beta,\psi,\delta)\,\varepsilon^{\frac12}\Big(\sup_{\varepsilon>0}E\Big[|\varepsilon\int_{t=0}^T\int_{\mathbb{R}^d_y}|\nabla_y u_\varepsilon(t,y)|^2\,dy\,dt|\Big]\Big)^{\frac12}$$
$$\le C(\beta,\psi,\delta)\,\varepsilon^{\frac12} \quad \text{(by (3.21))}$$
$$\to 0 \quad \text{as } \varepsilon\to 0.$$

Thus $\limsup_{(\varepsilon,\delta_0,l)\to 0}|J_7|=0$, which completes the proof. □

Lemma 5.11. *Assume that $\vartheta\to 0$, $\delta\to 0$, and $\vartheta^{-1}\delta^2\to 0$. Then*

$$\limsup_{\vartheta\to 0,\delta\to 0,\vartheta^{-1}\delta^2\to 0^+}\ \limsup_{\varepsilon\to 0}\Big[\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3)+(I_4+J_4)\Big)\Big]=0.$$

Proof. We combine Lemmas 5.6 and 5.9 to conclude that

$$\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3)+(I_4+J_4)\Big)$$
$$= E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\Big(\int_{|z|>0}\int_{\alpha=0}^1\Big\{\beta\big(v(t,x,\alpha)-u_\varepsilon(t,y)+\eta(x,v(t,x,\alpha);z)-\eta_\varepsilon(y,u_\varepsilon(t,y);z)\big)$$

$$
\begin{aligned}
&\qquad - \beta\big(v(t,x,\alpha) - u_\varepsilon(t,y)\big) - \big(\eta(x, v(t,x,\alpha); z) - \eta_\varepsilon(y, u_\varepsilon(t,y); z)\big) \\
&\qquad \times \beta'\big(v(t,x,\alpha) - u_\varepsilon(t,y)\big)\Big\}\, d\alpha\, m(dz)\Big)\psi(t,y)\varrho_\delta(x-y)\, dx\, dy\, dt\Big] \\
&\le E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\Big(\int_{|z|>0}\int_{\alpha=0}^1 \Big\{\beta\big(v(t,x,\alpha) - u_\varepsilon(t,y) + \eta(x, v(t,x,\alpha); z) - \eta(y, u_\varepsilon(t,y); z)\big) \\
&\qquad - \beta\big(v(t,x,\alpha) - u_\varepsilon(t,y)\big) - \big(\eta(x, v(t,x,\alpha); z) - \eta(y, u_\varepsilon(t,y); z)\big) \\
&\qquad \times \beta'\big(v(t,x,\alpha) - u_\varepsilon(t,y)\big)\Big\}\, d\alpha\, m(dz)\Big)\psi(t,y)\varrho_\delta(x-y)\, dx\, dy\, dt\Big] \\
&\quad + C(\beta,\psi)(\varepsilon + \varepsilon^2) \\
&= E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\Big(\int_{|z|>0}\int_{\alpha=0}^1 \big(\beta(a+b) - \beta(a) - b\beta'(a)\big)\, d\alpha\, m(dz)\Big)\,\psi(t,y)\varrho_\delta(x-y)\, dx\, dy\, dt\Big] \\
&\qquad + C(\beta,\psi)o(\varepsilon) \\
&= E\Big[\int_{\Pi_T}\int_{\mathbb{R}^d_x}\Big(\int_{|z|>0}\int_{\alpha=0}^1\int_{\theta=0}^1 b^2(1-\theta)\beta''(a+\theta\, b)\, d\theta\, d\alpha\, m(dz)\Big)\,\psi(t,y)\varrho_\delta(x-y)\, dx\, dy\, dt\Big] \\
&\qquad + C(\beta,\psi)o(\varepsilon) \qquad (5.21)
\end{aligned}
$$

where $a = v(t,x,\alpha) - u_\varepsilon(t,y)$ and $b = \eta(x, v(t,x,\alpha); z) - \eta(y, u_\varepsilon(t,y); z)$. Note that

$$
\begin{aligned}
b^2\beta''(a+\theta\, b) &= \big(\eta(x, v(t,x,\alpha); z) - \eta(y, u_\varepsilon(t,y); z)\big)^2\, \beta''\Big(a + \theta\,\big(\eta(x, v(t,x,\alpha); z) - \eta(y, u_\varepsilon(t,y); z)\big)\Big) \\
&\le \Big(|v(t,x,\alpha) - u_\varepsilon(t,y)|^2 + K|x-y|^2\Big)(1\wedge|z|^2)\,\beta''(a+\theta\, b) \\
&= \Big(a^2 + K^2|x-y|^2\Big)\,\beta''(a+\theta\, b)\,(1\wedge|z|^2). \qquad (5.22)
\end{aligned}
$$

We need to find a suitable upper bound on $a^2\,\beta''(a+\theta\, b)$. Note that β'' is nonnegative and symmetric around zero. Thus, we can assume without loss of generality that $a \ge 0$. Then, by assumption (**A**.2),

$$
v(t,x,\alpha) - u_\varepsilon(t,y) + \theta\, b \ge -K|x-y| + (1-\lambda^*)(v(t,x,\alpha) - u_\varepsilon(t,y))
$$

for $\theta \in [0,1]$. In other words

$$
0 \le a \le (1-\lambda^*)^{-1}(a + \theta b + K|x-y|). \qquad (5.23)
$$

We substitute $\beta = \beta_\vartheta$ in (5.22), and use (5.23) to obtain

$$
\begin{aligned}
b^2\beta_\vartheta''(a+\theta\, b) &\le (1-\lambda^*)^{-2}(a+\theta\, b + K|x-y|)^2\,\beta_\vartheta''(a+\theta b)\,(|z|^2\wedge 1) + \frac{K|x-y|^2}{\vartheta}\,(|z|^2\wedge 1) \\
&\le 2(1-\lambda^*)^{-2}(a+\theta b)^2\beta_\vartheta''(a+\theta b)(|z|^2\wedge 1) + C(K,\lambda^*)\frac{|x-y|^2}{\vartheta}(|z|^2\wedge 1) \\
&\le \Big[2(1-\lambda^*)^{-2}C\vartheta + C(K,\lambda^*)\frac{|x-y|^2}{\vartheta}\Big](|z|^2\wedge 1), \qquad (5.24)
\end{aligned}
$$

as $\sup_{r\in\mathbb{R}} r^2\beta_\vartheta''(r) \le \vartheta$ by (2.6).

Now combine (5.21)-(5.24) to write the following inequality:

$$
\begin{aligned}
&E\Big[\int_{(0,T]}\int_{x,y}\int_{|z|>0}\int_{\alpha=0}^1 \Big(\beta_\vartheta(a+b) - \beta_\vartheta(a) - b\beta_\vartheta'(a)\Big)\psi(t,y)\,\varrho_\delta(x-y)\, d\alpha\, m(dz)\, dx\, dy\, dt\Big] \\
&\quad \le C_1\Big(\vartheta + \vartheta^{-1}\delta^2\Big)T,
\end{aligned}
$$

where the constant C_1 depends only on ψ and is in particular independent of ε. We now let $\vartheta \to 0$, $\delta \to 0$ and $\vartheta^{-1}\delta^2 \to 0$, yielding

$$
\limsup_{\vartheta\to 0,\delta\to 0,\vartheta^{-1}\delta^2\to 0}\ \limsup_{\varepsilon\to 0}\Big[\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3) + (I_4+J_4)\Big)\Big] \le 0.
$$

This wraps up the proof, once we observe in (5.21) that $\lim_{l\to 0}\lim_{\delta_0\to 0}\Big((I_3+J_3)+(I_4+J_4)\Big)$ is nonnegative. □

All of the above results can be combined into the following proposition.

Proposition 5.12. *Let $v(t,x,\alpha)$ be a given generalized entropy solution of (1.2) with initial data $v(0,x)$ and $u(t,x,\gamma)$ be the generalized entropy solution with initial data $u(0,x)$, which has been extracted out of a Young measure valued subsequential limit of the sequence $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$ of viscous approximations. Then, for any nonnegative $H^1([0,\infty)\times\mathbb{R}^d)$ function $\psi(t,x)$ with compact support, it holds that*

$$\begin{aligned}0 \le & E\Big[\int_{\mathbb{R}^d_x} |v(0,x)-u(0,x)|\psi(0,x)\,dx\Big] + E\Big[\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1 |v(t,x,\alpha)-u(t,x,\gamma)|\partial_t\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big] \\ & + E\Big[\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1 F\Big(v(t,x,\alpha),u(t,x,\gamma)\Big)\cdot\nabla_x\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big]. \qquad (5.25)\end{aligned}$$

Proof. We add (5.1) and (5.3) and then take the limits

$$\lim_{\varepsilon_n\downarrow 0}\lim_{l\to 0}\lim_{\delta_0\downarrow 0},$$

invoking Lemmas 5.1, 5.2 , 5.6, 5.7, 5.8, 5.9, 5.10, and 5.11. In the resulting expression, we take $\delta=\vartheta^{\frac{2}{3}}$ and then send $\vartheta\to 0+$ with the second parts of Lemmas 5.1, 5.2, 5.7, 5.8, and 5.11 in mind, thereby arriving at

$$\begin{aligned}0 \le & E\Big[\int_{\mathbb{R}^d_x} |v(0,x)-u(0,x)|\psi(0,x)\,dx\Big] + E\Big[\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1 |v(t,x,\alpha)-u(t,x,\gamma)|\partial_t\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big] \\ & + E\Big[\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1 F\big(v(t,x,\alpha),u(t,x,\gamma)\big)\cdot\nabla_x\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big],\end{aligned}$$

which holds for any nonnegative $\psi\in C_c^2\big([0,\infty)\times\mathbb{R}^d\big)$. It now follows by a routine approximation argument that (5.25) holds for any ψ with compact support such that $\psi\in H^1([0,\infty)\times\mathbb{R}^d)$. □

Proof of Theorem 2.2. Let $v(t,x,\alpha)$ be a generalized entropy solution of (1.2) with initial data $v(0,x)$ and $u(t,x,\gamma)$ be the solution that has been obtained as the Young measure valued limit of the sequence $\{u_\varepsilon(t,x)\}_{\varepsilon>0}$, where u_ε solves (3.15) with initial data $u_0^\varepsilon(\cdot)$. Now from Proposition 5.12, for any nonnegative $\psi(t,x)\in H^1([0,\infty)\times\mathbb{R}^d)$ with compact support, it holds that

$$\begin{aligned}0 \le & E\Big[\int_{\mathbb{R}^d_x} |v(0,x)-u(0,x)|\psi(0,x)\,dx\Big] + E\Big[\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1 |v(t,x,\alpha)-u(t,x,\gamma)|\partial_t\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big] \\ & + E\Big[\int_{\Pi_T}\int_{\gamma=0}^1\int_{\alpha=0}^1 F\big(v(t,x,\alpha),u(t,x,\gamma)\big)\cdot\nabla_x\psi(t,x)\,d\alpha\,d\gamma\,dx\,dt\Big]. \qquad (5.26)\end{aligned}$$

For each $n\in\mathbb{N}$, define

$$\phi_n(x)=\begin{cases}1, & \text{if } |x|\le n\\ 2(1-\frac{|x|}{2n}), & \text{if } n<|x|\le 2n\\ 0, & \text{if } |x|>2n.\end{cases}$$

For each $h>0$ and fixed $t\ge 0$, define

$$\psi_h^t(s)=\begin{cases}1, & \text{if } s\le t\\ 1-\frac{s-t}{h}, & \text{if } t\le s\le t+h\\ 0, & \text{if } s>t+h.\end{cases}$$

Clearly (5.26) holds with $\psi(s,x)=\phi_n(x)\psi_h^t(s)$.

Let $\mathbb{T}$ be the set all points t in $[0,\infty)$ such that t is a right Lebesgue point of

$$A_n(s)=E\Big[\int_{\mathbb{R}^d_x}\int_{\gamma=0}^1\int_{\alpha=0}^1 \phi_n(x)|v(s,x,\alpha)-u(s,x,\gamma)|\,d\alpha\,d\gamma\,dx\Big],$$

for all n. Clearly, $\mathbb{T}^C$ has zero Lebesgue measure. Fix $t\in\mathbb{T}$. Thus, from (5.26) we have

$$\frac{1}{h}\int_t^{t+h} E\Big[\int_{\mathbb{R}^d_x}\int_{\gamma=0}^1\int_{\alpha=0}^1 |v(s,x,\alpha)-u(s,x,\gamma)|\phi_n(x)\,d\alpha\,d\gamma\,dx\Big]\,ds$$

$$\leq E\Big[\int_{\Pi_T}\int_{\gamma=0}^{1}\int_{\alpha=0}^{1} F\big(v(s,x,\alpha),u(s,x,\gamma)\big)\cdot\nabla_x\phi_n(x)\,\psi_h^t(s)\,d\alpha\,d\gamma\,dx\,ds\Big] + E\Big[\int_{\mathbb{R}_x^d}|v(0,x)-u(0,x)|\phi_n(x)\,dx\Big].$$

Taking limit as $h\to 0$, we obtain

$$\begin{aligned}
&E\Big[\int_{\mathbb{R}_x^d}\int_{\gamma=0}^{1}\int_{\alpha=0}^{1}|v(t,x,\alpha)-u(t,x,\gamma)|\phi_n(x)\,d\alpha\,d\gamma\,dx\Big]\\
&\quad\leq E\Big[\int_{\mathbb{R}^d}\int_{s=0}^{t}\int_{\gamma=0}^{1}\int_{\alpha=0}^{1} F\big(v(s,x,\alpha),u(s,x,\gamma)\big)\cdot\nabla_x\phi_n(x)\,d\alpha\,d\gamma\,ds\,dx\Big]\\
&\qquad + E\Big[\int_{\mathbb{R}_x^d}|v(0,x)-u(0,x)|\phi_n(x)\,dx\Big]\\
&\quad\leq C(T)\frac{1}{n}\Big(1+\sup_{0\leq s\leq T}E\Big[||v(s,\cdot,\cdot)||_p^p\Big]+\sup_{0\leq s\leq T}E\Big[||u(s,\cdot,\cdot)||_p^p\Big]\Big)\\
&\qquad + E\Big[\int_{\mathbb{R}_x^d}|v(0,x)-u(0,x)|\phi_n(x)\,dx\Big]. \qquad (5.27)
\end{aligned}$$

Letting $n\to\infty$, we obtain from (5.27) with $v(0,x)=u(0,x)$,

$$E\Big[\int_{\mathbb{R}^d}\int_{\gamma=0}^{1}\int_{\alpha=0}^{1}|v(t,x,\alpha)-u(t,x,\gamma)|\,d\alpha\,d\gamma\,dx\Big]=0.$$

From this the claims of the theorem follow in a standard way (see [10, 20]). □

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