CONVERGENCE OF A FULLY DISCRETE MONOTONE FINITE VOLUME SCHEME FOR LÉVY DRIVEN BALANCE LAWS

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Abstract. In this paper, we study fully discrete monotone finite volume scheme for multidimensional stochastic balance law driven by multiplicative Lévy noise. The convergence of approximations is proved towards the unique entropy solution for the underlying problem by using Young measure technique in stochastic setting.

Contents

1. Introduction

In recent years, there has been a growing interest in the studies of stochastic partial differential equations. Though a vast literature is available on the subject, interest on hyperbolic conservation laws with noise have witnessed a surge and there are many interesting questions, related to analytical and numerical aspects, waiting to be explored. A formal description of our problem as follows. Let $\big(\Omega, \mathbb{P}, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\ge 0}\big)$ be a filtered probability space satisfying the usual hypotheses *i.e.* $\{\mathcal{F}_t\}_{t\ge0}$ is a right-continuous filtration such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null subsets of $(\Omega, \mathcal{F})$. In addition, let $N(dz, dt)$[1] be a time homogeneous Poisson random measure on $(\mathbf{E}, \mathcal{E})$ with intensity measure $m(dz)$ with respect to the same stochastic basis, where $(\mathbf{E}, \mathcal{E}, m)$ is a σ-finite measure space. In this paper, we are interested in the study of numerical scheme and numerical approximation for the multi-dimensional nonlinear Cauchy problem of the following type

$$
\begin{aligned}
du(t,x) + \operatorname{div}_x\big(\vec{v}(t,x) f(u(t,x))\big)\, dt &= \int_{\mathbf{E}} \eta(u(t,x); z)\, \tilde{N}(dz, dt), \quad (t,x) \in \Pi_T, \qquad (1.1)\\
u(0,x) &= u_0(x), \quad x \in \mathbb{R}^d,
\end{aligned}
$$

where $\Pi_T = [0,T) \times \mathbb{R}^d$ with $T > 0$ fixed. Here, $f : \mathbb{R} \to \mathbb{R}$ is a given real valued flux function, $\vec{v}$ is a given vector valued function, $u_0(x)$ is a given initial function and $\tilde{N}(dz, dt) = N(dz, dt) - m(dz)\, dt$, the compensated Poisson random measure. Furthermore, $(u, z) \mapsto \eta(u, z)$ is a given real valued function signifying the multiplicative nature of the noise.

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[1] For the definition of a time homogeneous Poisson random measure, we refer to see [12, pp. 631].

Equation (1.1) arises in many different fields where non-Gaussianity plays an important role. As for example, it has been used in models of neuronal activity accounting for synaptic transmissions occurring randomly in time as well as at different locations on a spatially extended neuron, chemicals reaction-diffusion systems, market fluctuations both for risk management and option pricing purpose, stochastic turbulence, etc. The study of well-posedness theory for this kind of equation is of great importance in the light of current applications in continuum physics.

If $\eta = 0$, the equation (1.1) reduces to a standard conservation law in $\mathbb{R}^d$. It is well-documented that solutions of deterministic conservation laws develop discontinuities(shocks) even in finite time, and hence weak solutions must be sought. But there are infinitely many weak solutions and therefore an admissible criteria is required. There exists a satisfactory well-posedness theory based on Kruzkov's pioneering idea to picks up the physically relevant solution in an unique way, called *entropy solution*. We refer to [23, 27, 32, 36] and references therein for more on entropy solution theory for deterministic conservation laws.

The study of stochastic balance laws driven by noise is comparatively new area of pursuit. Only recently balance laws with stochastic forcing have attracted the attention of many authors [2, 3, 4, 8, 9, 10, 11, 14, 16, 17, 18, 22, 26, 28, 35] and resulted in significant momentum in the theoretical development of such problems.

Due to nonlinear nature of the underlying problem, explicit solution formula is hard to obtain and hence robust numerical schemes for approximating such equation are very important. In the last decade, there has been a growing interest in numerical approximation and numerical experiments for entropy solution to the related Cauchy problem driven by stochastic forcing. The first documented development in this direction is [24], where the authors proposed an operator-splitting method and constructed an approximations to prove the existence of path-wise weak solution (possibly non-unique) to the Cauchy problem driven by Brownian noise in one space dimension. In [5], the author revisited [24], and generalized the operator-splitting method for the same Cauchy problem but in a bounded domain of $\mathbb{R}^d$. Using Young measure theory, author established the convergence of approximate solutions to an entropy solution. We also refer to see [25], where the time splitting method was analyzed for more general noise coefficient in the spirit of Malliavin calculus and Young measure theory. By using stochastic compensated compactness method, in [30], Kröker and Rohde established the convergence of a semi-discrete finite volume scheme for strongly monotone numerical flux. In a recent papers [6, 7], Bauzet et. al. have studied fully discrete scheme via flux-splitting finite volume scheme and monotone finite volume scheme for stochastic conservation law driven by multiplicative Brownian noise and established its convergence by using Young measure technique. We also refer to the recent articles by Vovelle et. al.[19, 20], where a general framework for the analysis of approximations to stochastic scalar conservation laws driven by cylindrical Brownian noise were developed via kinetic approach along with martingale methods.

The study of numerical schemes for stochastic balance laws driven by Lévy noise is more sparse than the previous case. A semi-discrete finite difference scheme for a conservation laws driven by a homogeneous multiplicative Lévy noise has been studied by Koley et. al.[29]. Using BV estimates, the authors showed the convergence of approximate solutions, generated by the finite difference scheme, to the unique entropy solution as the spatial mesh size $\Delta x \to 0$ and established rate of convergence which is of order $\frac{1}{2}$. In [31], the author has studied fully discrete flux-splitting scheme and established convergence of approximate solutions under certain stability condition on mesh sizes in both space and time.

In this paper, we wish to extend the result [7] in the case of multiplicative Lévy driven noise, and address the convergence of a fully discrete monotone finite volume scheme for (1.1). First we establish few essential *a-priori* estimates for approximate solutions and then using these estimates, we deduce entropy inequality for approximate solutions. Using Young measure theory, we conclude the convergence of approximate solutions towards a generalized entropy solution of (1.1).

The rest of the paper is organized as follows. In Sections 2 and 3, we collect all the assumptions for the subsequent analysis, define proposed numerical scheme and finally state the main result of this article. Sections 4 and 5 deal with few *a-priori* estimates on the approximate solutions, and stochastic version of Young measure theory. Using these *a-priori* estimates, in Section 6, we establish discrete and continuous version of entropy inequalities on approximate solutions. The final Section 7 is devoted for the proof of the main theorem.

2. Preliminaries and technical framework

It is well-known that due to nonlinear flux term in (1.1), solutions to (1.1) are not necessarily smooth even if initial data is smooth, and hence must be interpreted via weak sense. Let $\mathcal{P}_T$ be the predictable σ-field on $[0,T]\times\Omega$ i.e., the σ-field generated by the sets of the form: $\{0\}\times A$ and $(s,t]\times B$ for any $A\in\mathcal{F}_0; B\in\mathcal{F}_s,\ 0<s,t\le T$. The notion of stochastic weak solution is defined as follows:

Definition 2.1 (Weak solution)**.** A square integrable $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t: t\ge 0\}$-predictable stochastic process $u(t)=u(t,x)$ is called a stochastic weak solution of (1.1) if $\forall\,\psi\in C_c^\infty([0,T)\times\mathbb{R}^d)$ there holds

$$\int_{\mathbb{R}^d}\psi(0,x)u_0(x)\,dx+\int_{\Pi_T}\Big\{\partial_t\psi(t,x)u(t,x)+\vec{v}(t,x)f(u(t,x))\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx$$
$$+\int_{\Pi_T}\int_{\mathbf{E}}\eta(u(t,x);z)\psi(t,x)\,\tilde{N}(dz,dt)\,dx=0\quad \mathbb{P}-\text{a.s.}$$

However, since there are infinitely many weak solutions, one needs to define an extra admissibility criteria to select physically relevant solution in an unique way, and one such condition is called entropy condition. Let us begin with the notion of entropy flux pair.

Definition 2.2 (Entropy flux pair)**.** (β,ϕ) is called an entropy flux pair if $\beta\in C^2(\mathbb{R})$ and $\phi:\mathbb{R}\mapsto\mathbb{R}$ such that

$$\phi'(r)=\beta'(r)f'(r).$$

An entropy flux pair (β,ϕ) is called convex if $\beta''(s)\ge 0$.

Let $\mathcal{A}=\big\{\beta\in C^2(\mathbb{R}),\text{convex such that support of }\beta''\text{ is compact}\big\}$. In the sequel, we will use specific entropy flux pairs. For any $a\in\mathbb{R}$ and $\beta\in\mathcal{A}$, define

$$\phi(a)=\int_0^a\beta'(s)f'(s)\,ds.$$

Note that, $\phi(\cdot)$ is a Lipschitz continuous function on $\mathbb{R}$ and (β,ϕ) is an entropy flux-pair. To this end, we define the notion of stochastic entropy solution of (1.1).

Definition 2.3 (Stochastic entropy solution)**.** An $L^2(\mathbb{R}^d)$-valued $\{\mathcal{F}_t: t\ge 0\}$-predictable stochastic process $u(t)=u(t,x)$ is called a stochastic entropy solution of (1.1) if the following hold:

i) For each $T>0$

$$\sup_{0\le t\le T}\mathbb{E}\Big[||u(t,\cdot)||_2^2\Big]<+\infty.$$

ii) For each $0\le\psi\in C_c^\infty([0,\infty)\times\mathbb{R}^d)$ and $\beta\in\mathcal{A}$, there holds

$$\int_{\mathbb{R}^d}\psi(x,0)\beta(u_0(x))\,dx+\int_{\Pi_T}\Big\{\partial_t\psi(t,x)\beta(u(t,x))+\phi(u(t,x))\vec{v}(t,x)\cdot\nabla\psi(t,x)\Big\}\,dx\,dt$$
$$+\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\eta(u(t,x);z)\beta'\big(u(t,x)+\lambda\eta(u(t,x);z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx$$
$$+\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u(t,x);z)\beta''\big(u(t,x)+\lambda\eta(u(t,x);z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx\ge 0,\quad\mathbb{P}-\text{a.s.}$$

Due to nonlocal nature of the Itô-Lévy formula and the missing noise-noise interaction, Definition 2.3 alone does not give the L^1-contraction principle in the sense of average when one tries to compare two entropy solutions directly, and hence fails to give uniqueness. For the details, we refer to see [14, 22]. However, in view of [2, 9], we can look for so called *generalized entropy solution* which are $L^2\big(\mathbb{R}^d\times(0,1)\big)$-valued $\{\mathcal{F}_t: t\ge 0\}$-predictable stochastic process.

Definition 2.4 (Generalized entropy solution)**.** An $L^2\big(\mathbb{R}^d\times(0,1)\big)$-valued $\{\mathcal{F}_t: t\ge 0\}$-predictable stochastic process $u(t)=u(t,x,\alpha)$ is called a generalized stochastic entropy solution of (1.1) provided
(1) For each $T>0$

$$\sup_{0\le t\le T}\mathbb{E}\Big[||u(t,\cdot,\cdot)||_2^2\Big]<+\infty.$$

(2) For all test functions $0 \le \psi \in C_c^{1,2}([0,\infty) \times \mathbb{R}^d)$, and any $\beta \in \mathcal{A}$, the following inequality holds

$$\begin{aligned}
&\int_{\mathbb{R}^d} \beta(u_0(x))\psi(x,0)\,dx + \int_{\Pi_T}\int_0^1 \Big\{\beta(u(t,x,\alpha))\partial_t\psi(t,x) + \phi(u(t,x,\alpha))\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx \\
&+ \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1 \eta(u(t,x,\alpha);z)\beta'\big(u(t,x,\alpha)+\lambda\eta(u(t,x,\alpha);z)\big)\psi(t,x)\,d\alpha\,d\lambda\,\tilde{N}(dz,dt)\,dx \\
&+ \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1 (1-\lambda)\eta^2(u(t,x,\alpha);z)\beta''\big(u(t,x,\alpha)+\lambda\eta(u(t,x,\alpha);z)\big) \\
&\qquad\qquad \times \psi(t,x)d\lambda\, m(dz)\,dt\,dx \ge 0, \quad \mathbb{P}-\text{a.s.}
\end{aligned} \tag{2.1}$$

The aim of this paper is to establish convergence of approximate solutions, constructed via monotone finite volume scheme (cf. Section 3), to the unique entropy solution of (1.1) and we will do so under the following assumptions:

- **A**.1 $f : \mathbb{R} \mapsto \mathbb{R}$ is C^2 and Lipschitz continuous with $f(0) = 0$.
- **A**.2 $\vec{v} : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is a C^1 function with $\mathrm{div}_x \vec{v}(t,x) = 0$ for all $(t,x) \in \Pi_T$. Furthermore, there exists $V < +\infty$ such that $|\vec{v}(t,x)| \le V$ for all $(t,x) \in \Pi_T$.
- **A**.3 There exist positive constant $0 < \lambda^* < 1$ and $h_1(z) \in L^2(\mathbf{E}, m)$ with $0 \le h_1(z) \le 1$ such that for all $u, v \in \mathbb{R};\ z \in \mathbf{E}$
$$|\eta(u;z) - \eta(v;z)| \le \lambda^* |u - v| h_1(z).$$
Moreover, $\eta(0;z) = 0$ for all $z \in \mathbf{E}$. Furthermore, there exists $C^* > 0$ such that $|\eta(u,z)| \le C^* h_1(z)$ for all $u \in \mathbb{R}$ and $z \in \mathbf{E}$.
- **A**.4 The initial function $u_0(x)$ is a $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_0$ measurable random variable satisfying
$$\mathbb{E}\Big[||u_0(\cdot)||_2^2\Big] < +\infty.$$

Remark 2.1. Note that we need the assumption **A**.1 to get entropy solution for the initial data in $L^2(\mathbb{R}^d)$ to control the multi-linear integrals terms. The assumption **A**.3 is needed to handle the nonlocal nature of the entropy inequalities. Boundedness of η is needed to validate Proposition 6.3.

In view of [9, 31], we have the following existence and uniqueness results.

Theorem 2.1. *Suppose the assumptions* **A**.1-**A**.4 *hold true. Then there exists a generalized entropy solution of* (1.1) *in the sense of Definition* 2.4.

Theorem 2.2. *Under the assumptions* **A**.1-**A**.4*, the generalized entropy solution of* (1.1) *is unique. Moreover, it is the unique stochastic entropy solution.*

3. Monotone finite volume schemes and statement of the main result

Our main point of interest is to propose numerical approximations for the problem (1.1) and analyze its convergence. To this end, we introduce the space discretization by finite volumes (control volumes). For that we need to recall the definition of so called admissible meshes for finite volume scheme (cf. [21]).

Definition 3.1 (Admissible mesh)**.** An admissible mesh $\mathcal{T}$ of $\mathbb{R}^d$ is a family of disjoint polygonal connected subset of $\mathbb{R}^d$ satisfying the following:

i) $\mathbb{R}^d$ is the union of the closure of the elements (called control volume) of $\mathcal{T}$.
ii) The common interface of any two elements of $\mathcal{T}$ is included in a hyperplane of $\mathbb{R}^d$.
iii) There exists nonnegative constant α such that
$$\begin{cases} \alpha h^d \le |K| \\ |\partial K| \le \dfrac{1}{\alpha} h^{d-1} \quad \forall K \in \mathcal{T}, \end{cases} \tag{3.1}$$
where $h = \sup\big\{\mathrm{diam}(K) : K \in \mathcal{T}\big\} < +\infty$, $|K|$ denotes the d-dimensional Lebesgue measure of K, and $|\partial K|$ represents the $(d-1)$-dimensional Lebesgue measure of ∂K.

In the sequel, we denote the followings:

- $\mathcal{E}_K$: the set of interfaces of the control volume K.

- $\mathcal{N}(K)$: the set of control volumes neighbors of the control volume K.
- $\sigma_{K,L}$: the common interface between K and L for any $L \in \mathcal{N}(K)$.
- $\mathcal{E}$: the set of all the interfaces of the mesh $\mathcal{T}$.
- $n_{K,\sigma}$: the unit normal vector to interface $\sigma_{K,L}$, oriented from K to L, for any $L \in \mathcal{N}_K$.
- $d\nu$: the $d-1$ dimensional Lebesgue measure.

In view of (3.1), we note that

$$\frac{|\partial K|}{|K|} \le \frac{1}{\alpha^2 h} \tag{3.2}$$

holds. We will use this inequality (3.2) several times in the later analysis.

As we emphasized, we shall study the fully discrete monotone finite volume scheme for (1.1). For that, one needs to discretize time variable also. To do this, we proceed as follows: let N be a positive integer and $T > 0$ be a fixed time. Set $k = \frac{T}{N}$ and $t_n = nk, \quad n = 0, 1, \cdots, N$. Then $\{t_n : n = 0, 1, \cdots, N\}$ splits the time interval $[0, T]$ into equal step length k. Before defining the monotone finite volume scheme, let us recall the definition of a monotone numerical flux.

Definition 3.2 (Monotone numerical flux)**.** Any function $F \in C(\mathbb{R}^2; \mathbb{R})$ is said to be monotone numerical flux associated to the flux function f, if the following conditions hold:

(a) The function F satisfies $F(v, v) = f(v)$, for all $v \in \mathbb{R}$.

(b) The function F is non-decreasing with respect to the first argument and non-increasing with respect to the second argument, i.e.,

$$\frac{\partial}{\partial u} F(u, v) \ge 0, \quad \frac{\partial}{\partial v} F(u, v) \le 0, \qquad \text{for all } u, v \in \mathbb{R}.$$

(c) There exist two constant $F_1, F_2 > 0$ such that for any $a, b, c \in \mathbb{R}$, it holds that

$$\begin{cases} \big|F(a, b) - F(c, b)\big| \le F_1 \big|a - c|, \\ \big|F(a, b) - F(a, c)\big| \le F_2 \big|b - c|. \end{cases}$$

Following [13], we propose the following monotone finite volume scheme to approximate the solution of (1.1): for any $K \in \mathcal{T}$, and $n \in \{0, 1, 2, \cdots, N-1\}$, define the discrete unknowns u_K^n as

$$\begin{aligned} &\frac{|K|}{k}\big(u_K^{n+1} - u_K^n\big) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \Big\} \\ &\qquad\qquad = \frac{|K|}{k} \int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta(u_K^n; z) \tilde{N}(dz, dt), \\ &u_K^0 = \frac{1}{|K|} \int_K u_0(x)\, dx, \end{aligned} \tag{3.3}$$

where, F is a monotone numerical flux and

$$\begin{cases} v_{K,L}^n = \dfrac{1}{k|\sigma_{K,L}|} \displaystyle\int_{t_n}^{t_{n+1}} \int_{\sigma_{K,L}} \big(\vec{v}(t, x) \cdot n_{K,L}\big)^+ \, d\nu(x)\, dt, \\ v_{L,K}^n = \dfrac{1}{k|\sigma_{K,L}|} \displaystyle\int_{t_n}^{t_{n+1}} \int_{\sigma_{K,L}} \big(\vec{v}(t, x) \cdot n_{L,K}\big)^+ \, d\nu(x)\, dt. \end{cases}$$

We define approximate finite volume solutions on Π_T as piecewise constant generated by the discrete solutions u_K^n:

$$u_{\mathcal{T},k}^h(t, x) = u_K^n \text{ for } x \in K \text{ and } t \in [t_n, t_{n+1}). \tag{3.4}$$

Remark 3.1. In view of the properties of stochastic integral with respect to compensated Poisson random measure, each u_K^n is $\mathcal{F}_{nk}$ - measurable for $K \in \mathcal{T}$ and $n \in \{0, 1, \cdots, N\}$. Thus, $u_{\mathcal{T},k}^h(t, \cdot)$ is an $L^2(\mathbb{R}^d)$-valued $\mathcal{F}_t$- predictable stochastic process as u_0 satisfies **A**.4.

Main Theorem. Let the assumptions **A**.1-**A**.4 be true and $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size h in the sense of Definition 3.1. Let k be the time step as discussed above and assume that

$$\frac{k}{h} \to 0 \text{ as } h \to 0. \tag{3.5}$$

Let $u^h_{\mathcal{T},k}(t,x)$ be the approximate solutions prescribed by (3.4). Then, there exists a $L^2(\mathbb{R}^d\times(0,1))$-valued $\{\mathcal{F}_t : t\ge 0\}$-predictable process $u=\mathbf{u}(t,x,\alpha)$ such that

i) $\mathbf{u}(t,x,\alpha)$ is a generalized entropy solution of (1.1) and $u^h_{\mathcal{T},k}(t,x)\mapsto \mathbf{u}(t,x,\alpha)$ in the sense of Young measure.

(ii) $u^h_{\mathcal{T},k}(t,x)\mapsto \bar{u}(t,x)$ in $L^p_{\text{loc}}(\mathbb{R}^d;L^p(\Omega\times(0,T)))$ for $1\le p<2$, where $\bar{u}(t,x)=\int_0^1 \mathbf{u}(t,x,\alpha)\,d\alpha$ is the unique stochastic entropy solution of (1.1).

4. A-priori estimates

This section is devoted to *a-priori* estimates for $u^h_{\mathcal{T},k}(t,x)$, which will be very useful to prove the convergence of the proposed scheme. We use the letter C to denote various generic constant which may change line to line. The Euclidean norm on $\mathbb{R}^d$ is denoted by $|\cdot|$. We denote $c_\eta=\int_{\mathbf{E}} h_1^2(z)\,m(dz)$, which is a finite constant, thanks to the assumption **A**.3. We start with the following lemma which is essentially a uniform moment estimate.

Lemma 4.1. *Let the assumptions* **A**.1-**A**.4 *hold and* $T>0$ *be fixed. Consider an admissible mesh* $\mathcal{T}$ *on* $\mathbb{R}^d$ *with size* h *in the sense of Definition* 3.1. *Let* $k=\frac{T}{N}$ *be the time step for some* $N\in\mathbb{N}^*$*, satisfying the Courant-Friedrichs-Levy (CFL) condition*

$$k\le \frac{\alpha^2 h}{(F_1+F_2)V}. \tag{4.1}$$

Then the approximate solution $u^h_{\mathcal{T},k}$ *satisfies the following bound:*

$$\|u^h_{\mathcal{T},k}\|^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}\le e^{c_\eta T}\mathbb{E}\big[||u_0||_2^2\big]. \tag{4.2}$$

Consequently, one has

$$\|u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)}\le T\,e^{c_\eta T}\mathbb{E}\big[||u_0||_2^2\big].$$

Proof. To prove (4.2), it is enough to prove: for each $n\in\{0,1,\cdots,N-1\}$, the following property holds

$$\sum_{K\in\mathcal{T}}|K|\mathbb{E}\big[(u_K^n)^2\big]\le \big(1+kc_\eta\big)^n\mathbb{E}\big[||u_0||_2^2\big]. \tag{4.3}$$

We will use mathematical induction to prove (4.3). Observe that

$$\sum_{K\in\mathcal{T}}|K|\mathbb{E}\big[(u_K^0)^2\big]=\sum_{K\in\mathcal{T}}|K|\mathbb{E}\Big[\Big(\frac{1}{|K|}\int_K u_0(x)\,dx\Big)^2\Big]\le \mathbb{E}\big[||u_0||_2^2\big]=\big(1+kc_\eta\big)^0\mathbb{E}\big[||u_0||_2^2\big].$$

Set $n\in\big\{0,1,cdots,N-1\big\}$ and suppose that (4.3) holds for n. We will show that (4.3) holds for $n+1$. To this end, thanks to the assumption **A**.2, we observe that

$$\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\big(v^n_{K,L}-v^n_{L,K}\big)=0, \tag{4.4}$$

$$\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\big(v^n_{K,L}+v^n_{L,K}\big)\le V\,|\partial K|. \tag{4.5}$$

Invoking (4.4), the scheme (3.3) reduces to

$$\frac{|K|}{k}\big(u_K^{n+1}-u_K^n\big)+\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v^n_{K,L}\big(F(u^n_K,u^n_L)-f(u^n_K)\big)-v^n_{L,K}\big(F(u^n_L,u^n_K)-f(u^n_K)\big)\Big\}$$
$$=\frac{|K|}{k}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u^n_K;z)\tilde{N}(dz,dt). \tag{4.6}$$

Multiplying (4.6) by u^n_K, we have, after taking expectation together with Itô isometry

$$\frac{|K|}{2}\mathbb{E}\Big[(u_K^{n+1})^2-(u_K^n)^2\Big]$$
$$=\frac{(k)^2}{2|K|}\mathbb{E}\Big[\Big(\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v^n_{K,L}\big(F(u^n_K,u^n_L)-f(u^n_K)\big)-v^n_{L,K}\big(F(u^n_L,u^n_K)-f(u^n_K)\big)\Big\}\Big)^2\Big]$$

$$
\begin{aligned}
&- k\mathbb{E}\Big[\sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n \big(F(u_K^n, u_L^n) - f(u_K^n)\big) - v_{L,K}^n \big(F(u_L^n, u_K^n) - f(u_K^n)\big) \Big\} u_K^n \Big] \\
&\qquad + \frac{|K|}{2}\mathbb{E}\Big[\int_{t_n}^{t_{n+1}} \int_{\mathbf{E}} \eta^2(u_K^n; z)\, m(dz)\, dt \Big] \\
&\equiv \mathcal{B}_1 - \mathcal{B}_2 + \mathcal{B}_3.
\end{aligned} \tag{4.7}
$$

We will estimate each of the above term separately. First we consider the term $\mathcal{B}_2$. To that context, we define

$$
\mathcal{T}_n := \big\{ (K,L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n \big\}
$$

and

$$
\begin{aligned}
\mathcal{B}_4 := k \sum_{(K,L)\in\mathcal{T}_n} |\sigma_{K,L}| \mathbb{E}\Big[& v_{K,L}^n \big\{ u_K^n \big(F(u_K^n, u_L^n) - f(u_K^n)\big) - u_L^n \big(F(u_K^n, u_L^n) - f(u_L^n)\big) \big\} \\
& - v_{L,K}^n \big\{ u_K^n \big(F(u_L^n, u_K^n) - f(u_K^n)\big) - u_L^n \big(F(u_L^n, u_K^n) - f(u_L^n)\big) \big\} \Big].
\end{aligned}
$$

Note that $\sum_{K\in\mathcal{T}} \mathcal{B}_2 = \mathcal{B}_4$. Proceeding similarly as in the proof of [7, Proposition 1], we arrive at

$$
\begin{aligned}
\mathcal{B}_4 \ge k \sum_{(K,L)\in\mathcal{T}_n} |\sigma_{K,L}| \mathbb{E}\Bigg[& \frac{v_{K,L}^n}{2(F_1+F_2)} \Big\{ \big(f(u_K^n) - F(u_K^n, u_L^n)\big)^2 + \big(f(u_L^n) - F(u_K^n, u_L^n)\big)^2 \Big\} \\
& + \frac{v_{L,K}^n}{2(F_1+F_2)} \Big\{ \big(f(u_K^n) - F(u_L^n, u_K^n)\big)^2 + \big(f(u_L^n) - F(u_L^n, u_K^n)\big)^2 \Big\} \Bigg].
\end{aligned} \tag{4.8}
$$

Next we move on to estimate $\mathcal{B}_1$. Using Cauchy-Schwarz inequality, (3.2) and the CFL condition (4.1) along with the convexity of the function $x \mapsto x^2$ (cf. [7, Proposition 1]), one has

$$
\begin{aligned}
\sum_{K\in\mathcal{T}} \mathcal{B}_1 \le \sum_{(K,L)\in\mathcal{T}_n} \frac{k|\sigma_{K,L}|}{2(F_1+F_2)} \mathbb{E}\Bigg[& v_{K,L}^n \Big\{ \big(f(u_K^n) - F(u_K^n, u_L^n)\big)^2 + \big(f(u_L^n) - F(u_K^n, u_L^n)\big)^2 \Big\} \\
& + v_{L,K}^n \Big\{ \big(f(u_K^n) - F(u_L^n, u_K^n)\big)^2 + \big(f(u_L^n) - F(u_L^n, u_K^n)\big)^2 \Big\} \Bigg].
\end{aligned} \tag{4.9}
$$

Again, thanks to the assumption **A**.3, we obtain

$$
\mathcal{B}_3 \le \frac{k|K|}{2} c_\eta \mathbb{E}\big[(u_K^n)^2\big]. \tag{4.10}
$$

We combine (4.9), (4.8) and (4.10) in (4.7) to conclude

$$
\sum_{K\in\mathcal{T}} \frac{|K|}{2} \mathbb{E}\big[(u_K^{n+1})^2\big] \le \sum_{K\in\mathcal{T}} \frac{|K|}{2} \big(1 + k\, c_\eta\big) \mathbb{E}\big[(u_K^n)^2\big].
$$

Thus, we have

$$
\begin{aligned}
\sum_{K\in\mathcal{T}} |K| \mathbb{E}\big[(u_K^{n+1})^2\big] &\le \big(1 + k\, c_\eta\big) \sum_{K\in\mathcal{T}} |K| \mathbb{E}\big[(u_K^n)^2\big] \\
&\le \big(1 + k\, c_\eta\big)^{n+1} \mathbb{E}\big[||u_0||_2^2\big] \quad \text{(by induction hypothesis).}
\end{aligned}
$$

In other words, (4.2) holds as well. As a consequence

$$
\|u_{\mathcal{T},k}^h\|_{L^2(\Omega\times\Pi_T)}^2 = \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}} k|K| \mathbb{E}\big[(u_K^n)^2\big] \le T\, e^{c_\eta T} \mathbb{E}\big[||u_0||_2^2\big].
$$

This completes the proof. □

Lemma 4.2 (Weak BV estimate)**.** *Let $T>0$ and $\mathcal{T}$ be an admissible mesh with size h in the sense of Definition* 3.1*. Let $k=\frac{T}{N}$ be the time step for some $N\in\mathbb{N}^*$ satisfying the CFL condition*

$$k \le \frac{(1-\xi)\alpha^2 h}{(F_1+F_2)V}, \text{ for some } \xi\in(0,1). \tag{4.11}$$

Let $u_K^n: K\in\mathcal{T},\, n\in\{0,1,\cdots,N-1\}$ be discrete unknowns as in (3.3)*. Then the following hold:*

a) *There exists $C_1\equiv C_1(T,u_0,\xi,F_1,F_2,c_\eta)\in\mathbb{R}_+^*$ such that*

$$\sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}|\mathbb{E}\Big[v_{K,L}^n\big(f(u_K^n)-F(u_K^n,u_L^n)\big)^2 + v_{L,K}^n\big(f(u_L^n)-F(u_L^n,u_K^n)\big)^2\Big] \le C_1.$$

b) *Let R be a positive constant such that $h<R$. Define*

$$\begin{cases} \mathcal{T}_R = \big\{K\in\mathcal{T}: K\subset B(0,R)\big\} \\ \mathcal{T}_n^R = \big\{(K,L)\in\mathcal{T}_R^2: L\in\mathcal{N}(K) \text{ and } u_K^n > u_L^n\big\}. \end{cases}$$

Then there exists $C_2\equiv C_2(R,d,T,u_0,\xi,F_1,F_2,c_\eta)\in\mathbb{R}_+^*$ such that

$$\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n^R} |\sigma_{K,L}|\mathbb{E}\bigg[v_{K,L}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(d)\big) + \max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(c)\big)\Big\}$$
$$+ v_{L,K}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(f(d)-F(c,d)\big) + \max_{u_L^n\le c\le d\le u_K^n}\big(f(c)-F(c,d)\big)\Big\}\bigg] \le C_2 h^{-\frac{1}{2}}.$$

Proof. Consider the numerical scheme given in (4.6)

$$\frac{|K|}{k}\big(u_K^{n+1}-u_K^n\big) + \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big) - v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}$$
$$= \frac{|K|}{k}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}} \eta(u_K^n;z)\tilde{N}(dz,dt). \tag{4.12}$$

Multiplying (4.12) by $k\,u_K^n$, taking expectation and summation over $K\in\mathcal{T}$ and $n\in\{0,1,\cdots,N-1\}$, we obtain

$$\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}}\sum_{L\in\mathcal{N}(K)} k\,|\sigma_{K,L}|\mathbb{E}\Big[\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big) - v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}u_K^n\Big]$$
$$+ \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}} |K|\mathbb{E}\Big[\big(u_K^{n+1}-u_K^n\big)u_K^n\Big] = 0$$

i.e., $\mathcal{A}+\mathcal{C}=0.$

We will estimate each of the terms $\mathcal{A}$ and $\mathcal{C}$ separately. Let us start with $\mathcal{C}$. Using the formula $ab=\frac{1}{2}\big[(a+b)^2-a^2-b^2\big]$ and (4.12), we get, thanks to Itô isometry

$$\mathcal{C} = \frac{1}{2}\Bigg\{\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|\mathbb{E}\Big[(u_K^{n+1})^2-(u_K^n)^2\Big] - \sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}|K|\mathbb{E}\Big[\Big(\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta(u_K^n;z)\tilde{N}(dz,dt)$$
$$-\frac{k}{|K|}\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big) - v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}\Big)^2\Big]\Bigg\}$$
$$= -\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Bigg\{\frac{k^2}{|K|}\mathbb{E}\Big[\Big(\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big) - v_{L,K}^n\big(F(u_L^n,u_K^n)$$
$$-f(u_K^n)\big)\Big\}\Big)^2\Big] + |K|\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,dt\Big]\Bigg\}$$

$$+\frac{1}{2}\sum_{K\in\mathcal{T}}|K|\mathbb{E}\Big[(u_K^N)^2-(u_K^0)^2\Big]$$

$$\equiv \mathcal{C}_1+\mathcal{C}_2,$$

where

$$\mathcal{C}_1=-\frac{1}{2}\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Bigg\{\frac{k^2}{|K|}\mathbb{E}\Big[\Big(\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big)-v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}\Big)^2\Big]+|K|\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,dt\Big]\Bigg\},$$

$$\mathcal{C}_2=\frac{1}{2}\sum_{K\in\mathcal{T}}|K|\mathbb{E}\Big[(u_K^N)^2-(u_K^0)^2\Big].$$

First we estimate $\mathcal{C}_1$. To this end, we rewrite $\mathcal{C}_1$ as

$$\mathcal{C}_1=-\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\Big\{\mathcal{B}_1+\frac{1}{2}|K|\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,dt\Big]\Big\},$$

where $\mathcal{B}_1$ is defined in the proof of Lemma 4.1. A similar argument (cf. estimation of (4.9) with the CFL condition (4.11)) reveals that

$$\sum_{K\in\mathcal{T}}\mathcal{B}_1\le\sum_{(K,L)\in\mathcal{T}_n}(1-\xi)\frac{k|\sigma_{K,L}|}{2(F_1+F_2)}\mathbb{E}\Bigg[v_{K,L}^n\Big\{\big(f(u_K^n)-F(u_K^n,u_L^n)\big)^2+\big(f(u_L^n)-F(u_K^n,u_L^n)\big)^2\Big\}+v_{L,K}^n\Big\{\big(f(u_K^n)-F(u_L^n,u_K^n)\big)^2+\big(f(u_L^n)-F(u_L^n,u_K^n)\big)^2\Big\}\Bigg]$$

$$\le\sum_{(K,L)\in\mathcal{T}_n}(1-\xi)\mathbb{E}\Bigg[v_{K,L}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(d)\big)^2+\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(c)\big)^2\Big\}+v_{L,K}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(f(d)-F(c,d)\big)^2+\max_{u_L^n\le c\le d\le u_K^n}\big(f(c)-F(c,d)\big)^2\Big\}\Bigg]\frac{k|\sigma_{K,L}|}{2(F_1+F_2)}.$$

Again, thanks to the assumption **A**.3, we see that

$$|K|\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,dt\Big]\le c_\eta|K|k\,\mathbb{E}\big[(u_K^n)^2\big],$$

and hence, one has the following lower bound of $\mathcal{C}_1$:

$$\mathcal{C}_1\ge-\sum_{n=0}^{N-1}k\sum_{(K,L)\in\mathcal{T}_n}\mathbb{E}\Bigg[v_{K,L}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(d)\big)^2+\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(c)\big)^2\Big\}+v_{L,K}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(f(d)-F(c,d)\big)^2+\max_{u_L^n\le c\le d\le u_K^n}\big(f(c)-F(c,d)\big)^2\Big\}\Bigg]\frac{|\sigma_{K,L}|}{(F_1+F_2)}\frac{(1-\xi)}{2}-\frac{1}{2}\sum_{n=0}^{N-1}k\sum_{K\in\mathcal{T}}|K|\mathbb{E}\big[(u_K^n)^2\big].$$

Invoking (4.3) and the lower bound of $\mathcal{C}_1$, we arrive at the following lower bound of $\mathcal{C}$:

$$\mathcal{C}\ge-\sum_{n=0}^{N-1}k\sum_{(K,L)\in\mathcal{T}_n}\mathbb{E}\Bigg[v_{K,L}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(d)\big)^2+\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(c)\big)^2\Big\}+v_{L,K}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(f(d)-F(c,d)\big)^2+\max_{u_L^n\le c\le d\le u_K^n}\big(f(c)-F(c,d)\big)^2\Big\}\Bigg]\frac{|\sigma_{K,L}|}{(F_1+F_2)}\frac{(1-\xi)}{2}-\frac{1}{2}T\,c_\eta e^{T\,c_\eta}\mathbb{E}\big[||u_0||_2^2\big]-\frac{1}{2}\mathbb{E}\big[||u_0||_2^2\big]. \tag{4.13}$$

Next we consider $\mathcal{A}$. Arguing similarly as in the proof of Lemma 4.1 (cf. treatment of the term $\sum_{K\in\mathcal{T}} \mathcal{B}_2 = \mathcal{B}_4$), one can rewrite $\mathcal{A}$ as

$$\mathcal{A} = \sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n} |\sigma_{K,L}| \mathbb{E}\Big[v_{K,L}^n \int_{u_K^n}^{u_L^n} \big(f(s) - F(u_K^n, u_L^n)\big) ds + v_{L,K}^n \int_{u_L^n}^{u_K^n} \big(f(s) - F(u_L^n, u_K^n)\big) ds \Big].$$

To find a lower bound of $\mathcal{A}$, we follow the same calculations as in [7, Proposition 2], and arrive at

$$\begin{aligned}\mathcal{A} \ge \sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n} \mathbb{E}\Big[v_{K,L}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(c)\big)^2 \Big\} \\ + v_{L,K}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(f(d) - F(c,d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(f(c) - F(c,d)\big)^2 \Big\} \Big] \frac{|\sigma_{K,L}|}{2(F_1 + F_2)}. \qquad (4.14)\end{aligned}$$

Since $\mathcal{A} + \mathcal{C} = 0$, we obtain, from (4.13) and (4.14)

$$\begin{aligned}\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n} |\sigma_{K,L} \mathbb{E}\Big[v_{K,L}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(c)\big)^2 \Big\} \\ + v_{L,K}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(f(d) - F(c,d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(f(c) - F(c,d)\big)^2 \Big\} \Big] \le \frac{(F_1 + F_2)}{\xi} \tilde{C}_1,\end{aligned}$$

where $\tilde{C}_1 = \Big(1 + T\, c_\eta e^{T\, c_\eta}\Big) \mathbb{E}\big[||u_0||_2^2\big]$. In other words, there exists $C_1 \equiv C_1(T, u_0, \xi, F_1, F_2, c_\eta) \in \mathbb{R}_+^*$ such that

$$\begin{aligned}\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n} |\sigma_{K,L} \mathbb{E}\Big[v_{K,L}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(c)\big)^2 \Big\} \\ + v_{L,K}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(f(d) - F(c,d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(f(c) - F(c,d)\big)^2 \Big\} \Big] \le C_1.\end{aligned}$$

Re-ordering the summation validates the assertion of the first part of the lemma

$$\begin{aligned}&\sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}} \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E}\Big[v_{K,L}^n \big(f(u_K^n) - F(u_K^n, u_L^n)\big)^2 + v_{L,K}^n \big(f(u_L^n) - F(u_L^n, u_K^n)\big)^2 \Big] \\ &\le \sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n} |\sigma_{K,L} \mathbb{E}\Big[v_{K,L}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(c)\big)^2 \Big\} \\ &\qquad + v_{L,K}^n \Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(f(d) - F(c,d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(f(c) - F(c,d)\big)^2 \Big\} \Big] \le C_1.\end{aligned}$$

To prove the second part of the lemma, we proceed as follows. Let

$$\begin{cases} \mathcal{T}_1 = \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(d)\big) + \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(c)\big), \\ \mathcal{T}_2 = \max_{u_L^n \le c \le d \le u_K^n} \big(f(d) - F(c,d)\big) + \max_{u_L^n \le c \le d \le u_K^n} \big(f(c) - F(c,d)\big). \end{cases}$$

Then, one has

$$\begin{cases} \mathcal{T}_1^2 \le 2\Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(F(d,c) - f(c)\big)^2 \Big\}, \\ \mathcal{T}_2^2 \le 2\Big\{ \max_{u_L^n \le c \le d \le u_K^n} \big(f(d) - F(c,d)\big)^2 + \max_{u_L^n \le c \le d \le u_K^n} \big(f(c) - F(c,d)\big)^2 \Big\}. \end{cases}$$

Following calculations as in Bauzet et. al.[7, Proposition 2], we obtain

$$\Big(\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n^R} |\sigma_{K,L}| \mathbb{E}\Big[v_{K,L}^n \mathcal{T}_1 + v_{L,K}^n \mathcal{T}_2 \Big] \Big)^2 \le 2\, T\, C_1\, V \frac{h^{d-1}}{\alpha} \frac{|B(0,R)|}{\alpha h^d} \le \frac{\tilde{C}_2}{h},$$

where $\tilde{C}_2 = 2\,T\,C_1\,\frac{V}{\alpha^2}|B(0,R)|$ and C_1 is a constant as in Lemma 4.2, a). Equivalently, we obtain

$$\sum_{n=0}^{N-1} k \sum_{(K,L)\in\mathcal{T}_n^R} |\sigma_{K,L}|\mathbb{E}\Bigg[v_{K,L}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(d)\big)+\max_{u_L^n\le c\le d\le u_K^n}\big(F(d,c)-f(c)\big)\Big\}$$
$$+\,v_{L,K}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big(f(d)-F(c,d)\big)+\max_{u_L^n\le c\le d\le u_K^n}\big(f(c)-F(c,d)\big)\Big\}\Bigg]\le C_2 h^{-\frac{1}{2}}$$

for some constant $C_2 \equiv (R,d,T,u_0,\xi,F_1,F_2,c_\eta)\in\mathbb{R}_+^*$. This completes the proof. □

5. Young measure and convergence of approximate solutions

Our main aim of this article is to establish the convergence of approximate solutions to the unique entropy solution of (1.1). Note that *a-priori* estimates on $u_{\mathcal{T},k}^h(t,x)$ given by Lemma 4.1 only guarantee weak compactness of the family $\{u_{\mathcal{T},k}^h\}_{h>0}$, which is inadequate in view of the nonlinearities in the equation. The concept of Young measure theory is appropriate in this case. We now recapitulate the results we shall use from Young measure theory due to [15, 33] for the deterministic setting, and [1] for the stochastic version of the theory.

Roughly speaking a Young measure is a parametrized family of probability measures where the parameters are drawn from a measure space. Let (Θ,Σ,μ) be a σ-finite measure space and $\mathcal{P}(\mathbb{R})$ be the space of probability measures on $\mathbb{R}$.

Definition 5.1 (Young Measure)**.** A Young measure from Θ into $\mathbb{R}$ is a map $\tau\mapsto\mathcal{P}(\mathbb{R})$ such that for any $\phi\in C_b(\mathbb{R})$, $\theta\mapsto\langle\tau(\theta),\phi\rangle := \int_{\mathbb{R}}\phi(\xi)\tau(\theta)(d\xi)$ is measurable from Θ to $\mathbb{R}$. The set of all Young measures from Θ into $\mathbb{R}$ is denoted by $\mathcal{R}(\Theta,\Sigma,\mu)$.

In this context, we mention that with an appropriate choice of (Θ,Σ,μ), the family $\{u_{\mathcal{T},k}^h\}_{h>0}$ can be thought of as a family of Young measures. We are interested in finding a subsequences out of this family that "converges" to a Young measure in a suitable sense. To this end, we set

$$\Theta=\Omega\times(0,T)\times\mathbb{R}^d,\quad \Sigma=\mathcal{P}_T\times\mathcal{L}(\mathbb{R}^d)\quad\text{and}\quad \mu=\mathbb{P}\otimes\lambda_t\otimes\lambda_x,$$

where λ_t and λ_x are respectively the Lebesgue measures on $(0,T)$ and $\mathbb{R}^d$. Moreover, for $M\in\mathbb{N}$, set $\Theta_M=\Omega\times(0,T)\times B_M$, where B_M be the ball of radius M around zero in $\mathbb{R}^d$. With the above setting at hand, we sum up the necessary results in the following proposition to carry over the subsequent analysis. For its proof, we refer to see [9].

Proposition 5.1. *Let $\{u_{\mathcal{T},k}^h(t,x)\}_{h>0}$ be a sequence of $L^2(\mathbb{R}^d)$-valued predictable processes such that (4.2) holds. Then there exists a subsequence $\{h_n\}$ with $h_n\to 0$ and a Young measure $\tau\in\mathcal{R}(\Theta,\Sigma,\mu)$ such that the following hold:*

i) *If $g(\theta,\xi)$ is a Carathéodory function on $\Theta\times\mathbb{R}$ such that $supp(g)\subset\Theta_M\times\mathbb{R}$ for some $M\in\mathbb{N}$ and $\{g(\theta,u_{\mathcal{T},k}^{h_n}(\theta))\}_n$ (where $\theta\equiv(\omega;t,x)$) is uniformly integrable, then*
$$\lim_{h_n\to 0}\int_\Theta g(\theta,u_{\mathcal{T},k}^{h_n}(\theta))\,\mu(d\theta)=\int_\Theta\Big[\int_{\mathbb{R}} g(\theta,\xi)\tau(\theta)(d\xi)\Big]\,\mu(d\theta).$$

ii) *Denoting a triplet $(\omega;t,x)\in\Theta$ by θ, we define*
$$\mathbf{u}(\theta,\alpha)=\inf\Big\{c\in\mathbb{R}:\tau(\theta)\big((-\infty,c)\big)>\alpha\Big\}\quad\text{for}\quad\alpha\in(0,1)\text{ and }\theta\in\Theta.$$
Then, the function $\mathbf{u}(\theta,\alpha)$ is non-decreasing, right continuous on $(0,1)$ and $\mathcal{P}_T\times\mathcal{L}(\mathbb{R}^d\times(0,1))$-measurable. Moreover, if $g(\theta,\xi)$ is a nonnegative Carathéodory function on $\Theta\times\mathbb{R}$, then
$$\int_\Theta\Big[\int_{\mathbb{R}} g(\theta,\xi)\tau(\theta)(d\xi)\Big]\,\mu\,(d\theta)=\int_\Theta\int_0^1 g(\theta,\mathbf{u}(\theta,\alpha))\,d\alpha\,\mu(d\theta).$$

Thanks to Proposition 5.1 and the uniform moment estimate (4.2), one can use Fatou's lemma to obtain
$$\sup_{0\le t\le T}\mathbb{E}\Big[||\mathbf{u}(t,\cdot,\cdot)||_2^2\Big]<+\infty.$$
Here we remark that, $\mathbf{u}(\theta,\alpha)$ will be a convenient generalized entropy solution of (1.1).

6. On entropy inequalities for approximate solutions

In this section, we establish entropy inequality for fully discrete monotone finite volume approximate solution. Since we are in stochastic set up, one needs to encounter the Itô calculus, and therefore it is natural to consider a time-continuous approximate solution constructed from $u^h_{\mathcal{T},k}$.

6.1. **Time-continuous approximate solution.** Let $\mathcal{T}$ be an admissible mesh in $\mathbb{R}^d$. Define a time-continuous discrete approximations, denoted by $v^n_K(\omega,s)$ on $\Omega\times[t_n,t_{n+1}]$, $n\in\{0,1,\cdots,N-1\}$ and $K\in\mathcal{T}$ from the discrete unknowns u^n_K prescribed by (4.6) as

$$v^n_K(\omega,s)=u^n_K-\int_{t_n}^s\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v^n_{K,L}\big(F(u^n_K,u^n_L)-f(u^n_K)\big)-v^n_{L,K}\big(F(u^n_L,u^n_K)-f(u^n_K)\big)\Big\}$$
$$+\int_{t_n}^s\int_{\mathbf{E}}\eta(u^n_K;z)\tilde{N}(dz,dt). \tag{6.1}$$

Observe that

$$\begin{cases}v^n_K(\omega,t_n)=u^n_K\\ v^n_K(\omega,t_{n+1})=u^{n+1}_K.\end{cases}$$

We drop ω and write $v^n_K(\cdot)$ instead of $v^n_K(\omega,\cdot)$. Now we define a time-continuous approximate solutions $v^h_{\mathcal{T},k}(t,x)$ on Π_T by

$$v^h_{\mathcal{T},k}(t,x)=v^n_K(t),\ x\in K,\ t\in[0,T). \tag{6.2}$$

It is now natural to estimate the L^2-error between $u^h_{\mathcal{T},k}$ and $v^h_{\mathcal{T},k}$. To this end, we have the following proposition.

Proposition 6.1. *Let the assumptions of Lemma 4.2 hold, and $u^h_{\mathcal{T},k}$ be the finite volume approximate solution defined by (3.3) and (3.4), and $v^h_{\mathcal{T},k}$ be the corresponding time-continuous approximate solution prescribed by (6.1) and (6.2). Then there exits a constant $C_3\in\mathbb{R}^*_+$, independent of h and k such that*

$$\|v^h_{\mathcal{T},k}-u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)}\le C_3\big(h+k\big).$$

Proof. Let $u^h_{\mathcal{T},k}$ (respectively $v^h_{\mathcal{T},k}$) be the approximate solution (respectively time-continuous approximate solution) defined by (3.3) and (3.4)(respectively (6.1) and (6.2)). Now

$$\begin{aligned}&\|v^h_{\mathcal{T},k}-u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)}\\
&=\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_K\mathbb{E}\Bigg[\Bigg(\frac{s-t_n}{|K|}\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v^n_{K,L}\big(F(u^n_K,u^n_L)-f(u^n_K)\big)\\
&\qquad-v^n_{L,K}\big(F(u^n_L,u^n_K)-f(u^n_K)\big)\Big\}-\int_{t_n}^s\int_{\mathbf{E}}\eta(u^n_K;z)\tilde{N}(dz,dt)\Bigg)^2\Bigg]\,dx\,ds\\
&\text{(by Itô isometry)}\\
&=\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_K\mathbb{E}\Big[\int_{t_n}^s\int_{\mathbf{E}}\eta^2(u^n_K;z)m(dz)\,dt\Big]\,dx\,ds\\
&\quad+\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\int_K\mathbb{E}\Big[\Big(\frac{s-k}{|K|}\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v^n_{K,L}\big(F(u^n_K,u^n_L)-f(u^n_K)\big)\\
&\qquad-v^n_{L,K}\big(F(u^n_L,u^n_K)-f(u^n_K)\big)\Big\}\Big)^2\Big]\,dx\,ds\\
&\text{(by Cauchy-Schwartz inequality)}\\
&\le\sum_{K\in\mathcal{T}}\sum_{n=0}^{N-1}\frac{k^3}{|K|}\Bigg\{\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\mathbb{E}\Big[v^n_{K,L}\big(F(u^n_K,u^n_L)-f(u^n_K)\big)^2+v^n_{L,K}\big(F(u^n_L,u^n_K)-f(u^n_K)\big)^2\Big]\end{aligned}$$

$$\times \Big(\sum_{\sigma\in\mathcal{E}_K} |\sigma_{K,L}| \big(v^n_{K,L} + v^n_{L,K}\big)\Big)\Bigg\} + c_\eta k \sum_{K\in\mathcal{T}} \sum_{n=0}^{N-1} k|K| \mathbb{E}\big[(u^n_K)^2\big]$$

(by the inequalities (3.2) and (4.5))

$$\le \sum_{K\in\mathcal{T}} \sum_{n=0}^{N-1} \frac{V\,k^3}{\alpha^2 h} \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| E\Big[v^n_{K,L}\big(F(u^n_K,u^n_L) - f(u^n_K)\big)^2 + v^n_{L,K}\big(F(u^n_L,u^n_K) - f(u^n_K)\big)^2\Big]$$
$$+ c_\eta k \|u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)}$$
$$\le \frac{k^2 V}{\alpha^2 h} C_1 + c_\eta k \|u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)} \text{ (by Lemma 4.2, a))}$$
$$\le h\frac{(1-\xi)^2\alpha^2}{(F_1+F_2)^2 V} C_1 + c_\eta k \|u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)} \text{ (by (4.11))}$$
$$\le C_3\big(h+k\big),$$

where $C_3 = \max\Big\{\frac{(1-\xi)^2\alpha^2}{(F_1+F_2)^2 V} C_1,\ c_\eta\, \|u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)}\Big\} \in \mathbb{R}^*_+$, independent of h and k. This completes the proof. □

6.2. **Entropy inequalities for the approximate solutions.** In this subsection, we derive the entropy inequalities for $u^h_{\mathcal{T},k}$ which will be used to prove the convergence of the numerical scheme, and hence the existence of entropy solution of (1.1). To do so, we first need to derive the entropy inequalities for the discrete unknowns $u^n_K,\ K\in\mathcal{T},\ n\in\{0,1,2,\dots,N-1\}$.

It is well-known that any monotone numerical flux could be decompose into a sum of Godunov flux and a modified Lax-Friedrichs flux. More precisely, we have the following lemma, whose proof could be found in [7, 21].

Lemma 6.2. *Any monotone numerical flux F can be written as a convex combination of a Godunov flux and a modified Lax-Friedrichs flux as follows: for any $a,b\in\mathbb{R}$, there exists $\theta(a,b)\in[0,1]$ such that*

$$F(a,b) = \theta(a,b)F^G(a,b) + \big(1-\theta(a,b)\big)F^{LF}_D(a,b),$$

where F^G is a Godunov flux given by

$$F^G(a,b) = \begin{cases} \min_{s\in[a,b]} f(s) & \text{if } a\le b\\ \max_{s\in[a,b]} f(s) & \text{if } a\ge b\end{cases}$$

and F^{LF}_D is a modified Lax-Friedrichs flux with parameter $D=\max\{F_1,\,F_2\}$ satisfying

$$F^{LF}_D(a,b) = \frac{f(a)+f(b)}{2} - D(b-a).$$

Remark 6.1. The modified Lax-Friedrichs scheme corresponds to a decomposition of a flux function $f(x) = f_1(x) + f_2(x)$, where $f_1(x) = f(x)/2 + Dx$ and $f_2(x) = f(x)/2 - Dx$. In other words, it is an example of a flux-spitting scheme.

In view of Lemma 6.2, from now on, we assume that $F(\cdot,\cdot)$ is the Godunov flux, i.e.

$$F(a,b) = \begin{cases} \min_{s\in[a,b]} f(s) & \text{if } a\le b\\ \max_{s\in[a,b]} f(s) & \text{if } a\ge b.\end{cases} \tag{6.3}$$

Let $s(a,b)\in\big[\min(a,b),\,\max(a,b)\big]$ for any $a,b\in\mathbb{R}$. Since $F(\cdot,\cdot)$ is a Godunov flux, we see that for some $s(a,b)$,

$$\begin{cases} F(a,b) = f(s(a,b))\\ G(a,b) = \phi(s(a,b))\end{cases}$$

where G is the corresponding entropy numerical flux. Note also that $G(a,a)=\phi(a)\quad \forall a\in\mathbb{R}$.

Proposition 6.3 (Discrete entropy inequalities)**.** *Let the assumptions* **A**.1-**A**.4 *hold, and that* $F(\cdot,\cdot)$ *is the Godunov flux defined by* (6.3). *Consider an admissible mesh* $\mathcal{T}$ *on* $\mathbb{R}^d$ *with size* h*, and time step* $k=\frac{T}{N}$ *on* $[0,T]$ *such that* (3.5) *holds. Then* $\mathbb{P}$*-a.s., for any* $\beta\in\mathcal{A}$ *and* $0\le\psi\in C_c^\infty([0,T)\times\mathbb{R}^d)$*, there holds*

$$\begin{aligned}
&-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\big(\beta(u_K^{n+1})-\beta(u_K^n)\big)\psi(t_n,x)\,dx+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K\phi(u_K^n)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt\\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta(u_K^n;z)\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,\tilde{N}(dz,dt)\\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,m(dz)\,dt\\
&\quad\ge R^{h,k},
\end{aligned}\tag{6.4}$$

where $R^{h,k}$ *satisfies the following condition: for any* $\mathbb{P}$*-measurable set* B*,*

$$\mathbb{E}\Big[\mathbf{1}_B R^{h,k}\Big]\to 0\ \text{ as } h\to 0.$$

Proof. We prove this proposition into several steps.

Step 1: Let $T>0$ be fixed and $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size h. Let $k=\frac{T}{N}$ be the time step for some $N\in\mathbb{N}^*$ and $t_n=nk$, $n\in\{0,1,\cdots,N\}$. Applying Itô-Lévy formula to $\beta(v_K^n)$, where v_K^n is prescribed by (6.1), and $\beta\in\mathcal{A}$, we have $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\beta(v_K^n(t_{n+1}))-\beta(v_K^n(t_n))\\
&=\frac{1}{|K|}\int_{t_n}^{t_{n+1}}\beta'(v_K^n(t))\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big)-v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}\\
&\quad+\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1\eta(u_K^n;z)\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)d\lambda\,\tilde{N}(dz,dt)\\
&\quad+\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)d\lambda\,m(dz)\,dt.
\end{aligned}\tag{6.5}$$

Let $0\le\psi\in C_c^\infty([0,T)\times\mathbb{R}^d)$. Then there exists $R>h$ such that supp $\psi\subset B(0,R-h)\times[0,T)$. Define $\mathcal{T}_R=\big\{K\in\mathcal{T}: K\subset B(0,R)\big\}$ and $\mathcal{T}_n^R=\big\{(K,L)\in\mathcal{T}_R^2: L\in\mathcal{N}(K)\,\text{and}\,u_K^n>u_L^n\big\}$. We multiply (6.5) by $|K|\psi_K^n$ where $\psi_K^n=\frac{1}{|K|}\int_K\psi(t_n,x)\,dx$ and then sum over all $K\in\mathcal{T}_R$ and $n\in\{0,1,\cdots,N-1\}$. The resulting expression reads to

$$\begin{aligned}
&\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\big[\beta(u_K^{n+1})-\beta(u_K^n)\big]|K|\psi_K^n\\
&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\beta'(v_K^n(t))\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^nF(u_K^n,u_L^n)-v_{L,K}^nF(u_L^n,u_K^n)\Big\}\psi_K^n\,dt\\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1\eta(u_K^n;z)\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)|K|\psi_K^n\,d\lambda\,\tilde{N}(dz,dt)\\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)|K|\psi_K^n\,d\lambda\,m(dz)\,dt
\end{aligned}$$

i.e., $\mathbf{A}^{h,k}=\mathbf{B}^{h,k}+\mathbf{M}^{h,k}+\mathbf{D}^{h,k}$.

We will study each of the above terms separately.

Step 2: In this step, we will study $\mathbf{B}^{h,k}$. We express $\mathbf{B}^{h,k}$ as

$$\mathbf{B}^{h,k}=\big(\mathbf{B}^{h,k}-\mathbf{B}_1^{h,k}\big)+\big(\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}\big)+\big(\mathbf{B}_2^{h,k}-\mathbf{B}_3^{h,k}\big)+\mathbf{B}_3^{h,k},$$

where

$$\begin{cases} \mathbf{B}_1^{h,k} = \displaystyle\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \beta'(u_K^n) \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \Big\} \psi_K^n \, dt, \\ \mathbf{B}_2^{h,k} = \displaystyle\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n G(u_K^n, u_L^n) - v_{L,K}^n G(u_L^n, u_K^n) \Big\} \psi_K^n \, dt, \\ \mathbf{B}_3^{h,k} = -\displaystyle\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \int_K \phi(u_K^n) v(x,t) \cdot \nabla_x \psi(x, t_n) \, dx \, dt. \end{cases}$$

Claim 1: For any $\mathbb{P}$-measurable set B

$$\mathbb{E}\big[\mathbf{1}_B(\mathbf{B}^{h,k} - \mathbf{B}_1^{h,k})\big] \to 0 \quad (h \to 0).$$

<u>Proof of the Claim 1</u>. Observe that for a.s. $\omega \in \Omega$, there exists $\tau_K^n(\omega) \in \mathbb{R}$ such that

$$\begin{aligned} &\mathbf{B}^{h,k} - \mathbf{B}_1^{h,k} \\ &= \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \beta''(\tau_K^n)\big(v_K^n(t) - u_K^n\big) \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \Big\} \psi_K^n \, dt \\ &= -\sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \beta''(\tau_K^n) \frac{t - t_n}{|K|} \Big(\sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \Big\} \Big)^2 \psi_K^n \, dt \\ &\quad + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \Bigg\{ \int_{t_n}^{t_{n+1}} \beta''(\tau_K^n) \Big(\int_{t_n}^{t} \eta(u_K^n; z) \tilde{N}(dz, ds) \Big) \\ &\qquad\qquad \times \Big(\sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n F(u_K^n, u_L^n) - v_{L,K}^n F(u_L^n, u_K^n) \Big\} \Big) \psi_K^n \, dt \Bigg\} \\ &\equiv \mathbf{T}_1^{h,k} + \mathbf{T}_2^{h,k}. \end{aligned}$$

Let us first study $\mathbb{E}\big[\mathbf{1}_B \mathbf{T}_1^{h,k}\big]$. Thanks to the Cauchy-Schwartz inequality, Lemma 4.2, (3.2), (4.5) and (4.4), we obtain

$$\begin{aligned} \Big|\mathbb{E}\big[\mathbf{1}_B \mathbf{T}_1^{h,k}\big]\Big| &= \Bigg| \mathbb{E}\Bigg[\mathbf{1}_B \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \frac{1}{|K|} \int_K \beta''(\tau_K^n) \frac{t - t_n}{|K|} \psi(t_n, x) \, dx \, dt \\ &\qquad \times \Big(\sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \Big\{ v_{K,L}^n \big(F(u_K^n, u_L^n) - f(u_K^n)\big) - v_{L,K}^n \big(F(u_L^n, u_K^n) - f(u_K^n)\big) \Big\} \Big)^2 \Bigg] \Bigg| \\ &\le ||\beta''||_\infty ||\psi||_\infty \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \Bigg\{ \frac{k^2}{|K|} \Big(\sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \big(v_{K,L}^n + v_{L,K}^n\big) \Big) \\ &\qquad \times \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E}\Big[v_{K,L}^n \big(F(u_K^n, u_L^n) - f(u_K^n)\big)^2 + v_{L,K}^n \big(F(u_L^n, u_K^n) - f(u_K^n)\big)^2 \Big] \Bigg\} \\ &\le ||\beta''||_\infty ||\psi||_\infty \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \Bigg\{ \frac{k^2}{|K|} V \, |\partial K| \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E}\Big[v_{K,L}^n \big(F(u_K^n, u_L^n) - f(u_K^n)\big)^2 \\ &\qquad\qquad + v_{L,K}^n \big(F(u_L^n, u_K^n) - f(u_K^n)\big)^2 \Big] \Bigg\} \\ &\le \sum_{n=0}^{N-1} k \sum_{K\in\mathcal{T}_R} \sum_{L\in\mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E}\Big[v_{K,L}^n \big(F(u_K^n, u_L^n) - f(u_K^n)\big)^2 + v_{L,K}^n \big(F(u_L^n, u_K^n) - f(u_K^n)\big)^2 \Big] \end{aligned}$$

$$\times ||\beta''||_\infty ||\psi||_\infty \frac{V}{\alpha^2}\frac{k}{h}$$
$$\le C_1 \frac{V}{\alpha^2}||\beta''||_\infty ||\psi||_\infty \frac{k}{h} \to 0 \text{ as } h \to 0.$$

Next we estimate $\mathbb{E}\big[\mathbf{1}_B \mathbf{T}_2^{h,k}\big]$. By using Cauchy-Schwarz inequality, Lemmas 4.1 and 4.2 along with (3.2), (4.5) and (4.4), we get

$$\begin{aligned}
&\Big(\mathbb{E}\big[\mathbf{1}_B \mathbf{T}_1^{h,k}\big]\Big)^2 \\
&= \Bigg(\mathbb{E}\Bigg[\mathbf{1}_B \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\Bigg\{\int_{t_n}^{t_{n+1}}\frac{1}{|K|}\int_K \beta''(\tau_K^n)\Big(\int_{t_n}^t \eta(u_K^n;z)\tilde{N}(dz,ds)\Big)\psi(t_n,x)\,dx\,dt \\
&\qquad\qquad \times \sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big)-v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}\Bigg\}\Bigg]\Bigg)^2 \\
&\le \mathbb{E}\Bigg[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\frac{1}{|K|}\Bigg\{\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Big\{v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big) \\
&\qquad\qquad\qquad\qquad - v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)\Big\}\Bigg\}^2\,dt\Bigg] \\
&\qquad \times \mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\Big(\int_{t_n}^t\int_{\mathbf{E}}\int_K \beta''(\tau_K^n)\psi(t_n,x)\eta(u_K^n;z)\tilde{N}(dz,ds)\,dx\Big)^2\,dt\Big] \\
&\le \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\frac{k}{|K|}\Bigg\{\Big(\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\big(v_{K,L}^n+v_{L,K}^n\big)\Big) \\
&\quad \times \sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\mathbb{E}\Big[v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big)^2+v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)^2\Big]\Bigg\} \\
&\qquad \times ||\beta''||_\infty^2||\psi||_\infty^2\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}|K|\int_{t_n}^{t_{n+1}}\int_{t_n}^t\int_{\mathbf{E}}\eta^2(u_K^n;z)\,m(dz)\,ds\,dt\Big] \\
&\le \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}k\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\mathbb{E}\Big[v_{K,L}^n\big(F(u_K^n,u_L^n)-f(u_K^n)\big)^2+v_{L,K}^n\big(F(u_L^n,u_K^n)-f(u_K^n)\big)^2\Big] \\
&\qquad\qquad \times ||\beta''||_\infty^2||\psi||_\infty^2\frac{V}{\alpha^2 h}c_\eta k\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}|K|k\mathbb{E}\big[(u_K^n)^2\big] \\
&\le C_1||\beta''||_\infty^2||\psi||_\infty^2\frac{V}{\alpha^2 h}c_\eta k\|u_{\mathcal{T},k}^h\|_{L^2(\Omega\times\Pi_T)}^2 = C\big(T,\beta,\psi,u_0,\xi,F_1,F_2,\alpha,c_\eta,V\big)\frac{k}{h}.
\end{aligned}$$

Hence, $\mathbb{E}\big[\mathbf{1}_B\mathbf{T}_2^{h,k}\big]\to 0$ as $\frac{k}{h}\to 0$. This proves the Claim 1 .

Next we move on to estimate $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}$. In view of (4.4), we note that

$$\sum_{L\in\mathcal{N}(K)}\big(v_{K,L}^n-v_{L,K}^n\big)f(u_K^n)=0=\sum_{L\in\mathcal{N}(K)}\big(v_{K,L}^n-v_{L,K}^n\big)\phi(u_K^n),$$

and hence

$$\begin{aligned}
&\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k} \\
&= \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\frac{k}{|K|}\sum_{L\in\mathcal{N}(K)}|\sigma_{K,L}|\Bigg\{v_{K,L}^n\Big[\beta'(u_K^n)\big(F(u_K^n,u_L^n)-f(u_K^n)\big)-\big(G(u_K^n,u_L^n)-\phi(u_K^n)\big)\Big]
\end{aligned}$$

$$- v_{K,L}^n\Big[\beta'(u_K^n)\big(F(u_L^n,u_K^n)-f(u_K^n)\big)-\big(G(u_L^n,u_K^n)-\phi(u_K^n)\big)\Big]\Bigg\}\int_K \psi(t_n,x)\,dx.$$

Claim 2: $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}\ge 0$ almost surely.

Proof of the Claim 2. In view of the above expression of $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}$, it suffices to show that

$$v_{K,L}^n\Big[\beta'(u_K^n)\big(F(u_K^n,u_L^n)-f(u_K^n)\big)-\big(G(u_K^n,u_L^n)-\phi(u_K^n)\big)\Big]$$
$$- v_{K,L}^n\Big[\beta'(u_K^n)\big(F(u_L^n,u_K^n)-f(u_K^n)\big)-\big(G(u_L^n,u_K^n)-\phi(u_K^n)\big)\Big]\ge 0. \tag{6.6}$$

Let us first consider the case $u_K^n<u_L^n$. The case $u_K^n>u_L^n$ will be similar. Since $F(\cdot,\cdot)$ is a Godunov flux, there exists $s(u_K^n,u_L^n)\in[u_K^n,u_L^n]$ such that

$$F(u_K^n,u_L^n)=f(s(u_K^n,u_L^n))=\min_{t\in[u_K^n,u_L^n]} f(t).$$

Thus, by using integration by parts formula, we have

$$\begin{aligned}
&\beta'(u_K^n)\big(F(u_K^n,u_L^n)-f(u_K^n)\big)-\big(G(u_K^n,u_L^n)-\phi(u_K^n)\big)\\
&=\beta'(u_K^n)\big(f(s(u_K^n,u_L^n))-f(u_K^n)\big)-\big(\phi(s(u_K^n,u_L^n))-\phi(u_K^n)\big)\\
&=\int_{u_K^n}^{s(u_K^n,u_L^n)} f'(t)\beta'(u_K^n)\,dt-\int_{u_K^n}^{s(u_K^n,u_L^n)} f'(t)\beta'(t)\,dt=\int_{u_K^n}^{s(u_K^n,u_L^n)} f'(t)\big(\beta'(u_K^n)-\beta'(t)\big)\,dt\\
&=f(s(u_K^n,u_L^n))\big(\beta'(u_K^n)-\beta'(s(u_K^n,u_L^n))\big)+\int_{u_K^n}^{s(u_K^n,u_L^n)} f(t)\beta''(t)\,dt\\
&\ge f(s(u_K^n,u_L^n))\big(\beta'(u_K^n)-\beta'(s(u_K^n,u_L^n))\big)+\int_{u_K^n}^{s(u_K^n,u_L^n)} f(s(u_K^n,u_L^n))\beta''(t)\,dt=0.
\end{aligned}$$

Again, there exists $s(u_L^n,u_K^n)\in[u_K^n,u_L^n]$ such that

$$F(u_L^n,u_K^n)=f(s(u_L^n,u_K^n))=\max_{t\in[u_K^n,u_L^n]} f(t).$$

Thanks to integration by parts formula, we obtain

$$\begin{aligned}
&\beta'(u_K^n)\big(F(u_L^n,u_K^n)-f(u_K^n)\big)-\big(G(u_L^n,u_K^n)-\phi(u_K^n)\big)\\
&=f(s(u_L^n,u_K^n))\big(\beta'(u_K^n)-\beta'(s(u_L^n,u_K^n))\big)+\int_{u_K^n}^{s(u_L^n,u_K^n)} f(t)\beta''(t)\,dt\\
&\le f(s(u_L^n,u_K^n))\big(\beta'(u_K^n)-\beta'(s(u_L^n,u_K^n))\big)+\int_{u_K^n}^{s(u_L^n,u_K^n)} f(s(u_L^n,u_K^n))\beta''(t)\,dt=0.
\end{aligned}$$

Putting things together yields (6.6). This completes the proof of the claim.

Next we move on to estimate $\mathbb{E}\Big[\mathbf{1}_B\big(\mathbf{B}_2^{h,k}-\mathbf{B}_3^{h,k}\big)\Big]$. Performing the calculations as in the proof of point 2.3 of [7, Proposition 4], we obtain, thanks to Lemma 4.2, b)

$$\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_B\big(\mathbf{B}_2^{h,k}-\mathbf{B}_3^{h,k}\big)\Big]\\
&\le C\,h\sum_{n=0}^{N-1}k\sum_{(K,L)\in\mathcal{T}_n^R}|\sigma_{K,L}|\mathbb{E}\Bigg[v_{K,L}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big|F(d,c)-f(d)\big|+\max_{u_L^n\le c\le d\le u_K^n}\big|F(d,c)-f(c)\big|\Big\}\\
&\qquad+v_{L,K}^n\Big\{\max_{u_L^n\le c\le d\le u_K^n}\big|f(d)-F(c,d)\big|+\max_{u_L^n\le c\le d\le u_K^n}\big|f(c)-F(c,d)\big|\Big\}\Bigg]+C\,h\\
&\le C\sqrt{h}\to 0 \text{ as } h\to 0,
\end{aligned}$$

where $C>0$ is a constant depending only on $f,\xi,\alpha,F_1,F_2,c(\vec{v}),\psi,c_\eta,\beta,T$.

Step 3: This step is regarding the study of the stochastic term $\mathbf{M}^{h,k}$. We decompose $\mathbf{M}^{h,k}$ as follows:

$$\mathbf{M}^{h,k}=\mathbf{M}^{h,k}-\mathbf{M}_1^{h,k}+\mathbf{M}_1^{h,k},$$

where

$$\mathbf{M}_1^{h,k}=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta(u_K^n;z)\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,\tilde{N}(dz,dt).$$

Claim 3: For any $\mathbb{P}$-measurable set B

$$\mathbb{E}\Big[\mathbf{1}_B\big(\mathbf{M}_1^{h,k}-\mathbf{M}^{h,k}\big)\Big]\rightarrow 0\quad(h\rightarrow 0).$$

Proof of Claim 3. In view of triangle inequality, one has

$$\begin{aligned}
&\Big|\mathbb{E}\big[\mathbf{1}_B\big(\mathbf{M}_1^{h,k}-\mathbf{M}^{h,k}\big)\big]\Big|\\
&\le\Bigg|\mathbb{E}\Big[\mathbf{1}_B\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta(u_K^n;z)\Big\{\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)-\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)\Big\}\\
&\qquad\qquad\times\big(\psi(t_n,x)-\psi(t,x)\big)\,d\lambda\,dx\,\tilde{N}(dz,dt)\Big]\Bigg|\\
&+\Bigg|\mathbb{E}\Big[\mathbf{1}_B\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta(u_K^n;z)\Big\{\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)-\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)\Big\}\\
&\qquad\qquad\times\psi(t,x)\,d\lambda\,dx\,\tilde{N}(dz,dt)\Big]\Bigg|\\
&\equiv\mathcal{M}_1^{h,k}+\mathcal{M}_2^{h,k}.
\end{aligned}$$

First we consider $\mathcal{M}_1^{h,k}$. Note that $\operatorname{supp}\psi\subset B(0,R-h)\times[0,T)$ for some $R>h$. Thanks to Cauchy-Schwarz inequality, the assumptions **A**.1-**A**.4, and Itô-Lévy isometry, we get

$$\begin{aligned}
\big|\mathcal{M}_1^{h,k}\big|^2&\le\Bigg(\sum_{n=0}^{N-1}\Bigg\{\sum_{K\in\mathcal{T}_R}\int_K\mathbb{E}\Big[\Big(\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1\Big\{\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)-\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)\Big\}\\
&\qquad\times\eta(u_K^n;z)\big(\psi(t_n,x)-\psi(t,x)\big)\,d\lambda\tilde{N}(dz,dt)\Big)^2\Big]\,dx\Bigg\}^{\frac12}\Bigg)^2|B(0,R)|\\
&\le 2\|\beta'\|_{L^\infty}^2|B(0,R)|\Big(\sum_{n=0}^{N-1}\Big\{\sum_{K\in\mathcal{T}_R}\int_K\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\eta^2(u_K^n;z)\big(\psi(t_n,x)-\psi(t,x)\big)^2\,m(dz)\,dt\Big]\,dx\Big\}^{\frac12}\Big)^2\\
&\le 2k\|\beta'\|_{L^\infty}^2|B(0,R)|\,\|\partial_t\psi\|_{L^\infty}^2c_\eta\Big(\sum_{n=0}^{N-1}k\Big\{\sum_{K\in\mathcal{T}_R}|K|\mathbb{E}\big[(u_K^n)^2\big]\Big\}^{\frac12}\Big)^2\\
&\le kC\big(\beta',\partial_t\psi,c_\eta,|B(0,R)|\big)\|u_{\mathcal{T},k}^h\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}^2\\
&\le hC\big(\xi,\alpha,c_f,V,\beta',\partial_t\psi,c_\eta,|B(0,R)|\big)\|u_{\mathcal{T},k}^h\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d))}^2\ (\text{by }(4.11)).
\end{aligned}$$

Thus, we conclude that

$$\mathcal{M}_1^{h,k}\longrightarrow 0\ \text{ as } h\rightarrow 0.$$

Let us turn our focus on the term $\mathcal{M}_2^{h,k}$. Here we note that the boundedness of η i.e., $|\eta(u,z)|\le Ch_1(z)$ for any $u\in\mathbb{R}$ and $z\in\mathbf{E}$ is crucial. In view of the Cauchy-Schwarz inequality and Itô-Lévy isometry, we obtain

$$\begin{aligned}
\Big|\mathcal{M}_2^{h,k}\Big|^2&\le|B(0,R)|\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_0^1\Big\{\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)-\beta'\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)\Big\}^2\\
&\qquad\times\eta^2(u_K^n;z)\psi^2(t,x)\,d\lambda\,m(dz)\,dt\Big]\,dx
\end{aligned}$$

$$\le C(R)||\beta''||_\infty^2||\psi||_\infty^2 \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \mathbb{E}\Big[\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}} |u_K^n - v_K^n(t)|^2\eta^2(u_K^n;z)\,m(dz)dt\Big]\,dx$$

(by the boundedness of η)

$$\le C\big(R,\beta'',c_\eta,\psi,\big)\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\int_{t_n}^{t_{n+1}}\mathbb{E}\big[(u_K^n - v_K^n(t))^2\big]\,dt\,dx$$

$$= C\big(R,\beta'',\psi,c_\eta\big)\|u_{\mathcal{T},k}^h - v_{\mathcal{T},k}^h\|^2_{L^2(\Omega\times\Pi_T)} \longrightarrow 0 \,\text{as}\; h\to 0,$$

where in the last line, we invoke (4.11) and Proposition 6.1.

Step 4: In this step, we consider the additional term $\mathbf{D}^{h,k}$. We rewrite $\mathbf{D}^{h,k}$ as

$$\mathbf{D}^{h,k} = \mathbf{D}^{h,k} - \mathbf{D}_1^{h,k} + \mathbf{D}_1^{h,k},$$

where

$$\mathbf{D}_1^{h,k} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 (1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,m(dz)\,dt.$$

Claim 4: For any $\mathbb{P}$-measurable set B,

$$\mathbb{E}\Big[\mathbf{1}_B\big(\mathbf{D}_1^{h,k} - \mathbf{D}^{h,k}\big)\Big] \to 0 \quad (h\to 0).$$

<u>Proof of Claim 4</u>. In view of the assumptions **A**.1-**A**.4, Proposition 6.1 and the CFL condition (4.11), we have, for some constant $C \equiv C(\xi,\alpha,F_1,F_2,\psi,c_\eta,\beta,T)\in\mathbb{R}_+^*$

$$\begin{aligned}
&\Big|\mathbb{E}\big[\mathbf{1}_B\big(D_1^{h,k} - D^{h,k}\big)\big]\Big|\\
&= \Bigg|\mathbb{E}\Big[\mathbf{1}_B\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \big\{\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big) - \beta''\big(v_K^n(t)+\lambda\eta(u_K^n;z)\big)\big\}\\
&\qquad\qquad \times(1-\lambda)\eta^2(u_K^n;z)\psi(t_n,x)\,d\lambda\,dx\,m(dz)\,dt\Big]\Bigg|\\
&\le ||\beta'''||_\infty\,||\psi||_\infty\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_{B(0,R)}\eta^2(u_K^n;z)\big|u_K^n - v_K^n(t)\big|dx\,m(dz)\,dt\Big]
\end{aligned}$$

(by the boundedness of η)

$$\begin{aligned}
&\le C\,\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{B(0,R)}\big|u_K^n - v_K^n(t)\big|dx\,dt\Big]\\
&= C\,\|u_{\mathcal{T},k}^h - v_{\mathcal{T},k}^h\|_{L^1\big(\Omega\times B(0,R)\times[0,T)\big)} \longrightarrow 0 \text{ as } h\to 0.
\end{aligned}$$

Step 5: In this final step, we wrap up all the analysis done in **Step** 1- **Step** 4. Define

$$R^{h,k} = (\mathbf{B}^{h,k} - \mathbf{B}_1^{h,k}) + \big(\mathbf{B}_2^{h,k} - \mathbf{B}_3^{h,k}\big) + \big(\mathbf{M}_1^{h,k} - \mathbf{M}^{h,k}\big) + \big(\mathbf{D}_1^{h,k} - \mathbf{D}^{h,k}\big).$$

Since $\mathbb{P}$-a.s., $\mathbf{A}^{h,k} = \mathbf{B}^{h,k} + \mathbf{M}^{h,k} + \mathbf{D}^{h,k}$, summarizing all we infer that

$$\begin{aligned}
&-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\big(\beta(u_K^{n+1}) - \beta(u_K^n)\big)\psi(x,t_n)\,dx\\
&+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K \phi(u_K^n)v(x,t)\cdot\nabla_x\psi(x,t_n)\,dx\,dt\\
&+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \eta(u_K^n;z)\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(x,t_n)d\lambda\,dx\,\tilde{N}(dz,dt)
\end{aligned}$$

$$+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(x,t_n)d\lambda\,dx\,m(dz)\,dt$$
$$\ge R^{h,k}.$$

Again, thanks to Claim 1, Claim 3 and Claim 4, we see that for any $\mathbb{P}$-measurable set B,

$$\mathbb{E}\big[\mathbf{1}_B R^{h,k}\big]\to 0 \text{ as } h\to 0 \text{ (under (3.5))}.$$

This completes the proof. □

Note that, in the proof of Proposition 6.3, we only use the fact that $F(\cdot,\cdot)$ is Godunov flux to study the terms $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}$ and $\mathbf{B}_2^{h,k}-\mathbf{B}_3^{h,k}$. In view of Lemma 6.2, we may expect the same result (cf. Proposition 6.3) for a general monotone flux. Regarding this, we have the following proposition.

Proposition 6.4. *Let the assumptions* $\mathbf{A}.1$-$\mathbf{A}.4$ *hold and* $\mathcal{T}$ *be an admissible mesh on* $\mathbb{R}^d$ *with size* h. *Let* $T>0$ *be fixed and* $N\in\mathbb{N}^*$ *be such that the time step* $k=\frac{T}{N}$ *satisfies* (3.5). *Then* (6.4) *holds and for any* $\mathbb{P}$*-measurable set* B, $\mathbb{E}\Big[\mathbf{1}_B R^{h,k}\Big]\to 0$ *as* $h\to 0$, *where* $R^{h,k}$ *is described by Proposition* 6.3.

Proof. In view of the above discussion, it suffices to show: for any general monotone flux $F(\cdot,\cdot)$

i) $\mathbb{P}$-a.s., $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}\ge 0$.
ii) for any $\mathbb{P}$-measurable set B, $\mathbb{E}\Big[\mathbf{1}_B\big(\mathbf{B}_2^{h,k}-\mathbf{B}_3^{h,k}\big)\Big]\to 0$ as $h\to 0$.

Let F be a monotone flux. Then thanks to Lemma 6.2, there exists $\theta(a,b)\in(0,1)$ such that

$$F(a,b)=\theta(a,b)F^G(a,b)+\big(1-\theta(a,b)\big)F_D^{LF}(a,b),$$

where F^G is a Godunov flux and F_D^{LF} is a modified Lax-Friedrichs flux. Then, the numerical entropy flux G takes of the form

$$G(a,b)=\theta(a,b)G^G(a,b)+\big(1-\theta(a,b)\big)G_D^{LF}(a,b),$$

where $G^G(a,b)=\phi(s(a,b))$ and $G_D^{LF}(a,b)=\dfrac{\phi(a)+\phi(b)}{2}-D(\beta(b)-\beta(a))$. Under this decompositions, the expression $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}$ reads as

$$\begin{aligned}
&\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}\\
&=\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\frac{k}{|K|}\sum_{L\in\mathcal{N}(K)}\theta(u_K^n,u_L^n)|\sigma_{K,L}|\Bigg\{v_{K,L}^n\Big[\beta'(u_K^n)\big(F^G(u_K^n,u_L^n)-f(u_K^n)\big)-\big(G^G(u_K^n,u_L^n)\\
&\quad-\phi(u_K^n)\big)\Big]-v_{K,L}^n\Big[\beta'(u_K^n)\big(F^G(u_L^n,u_K^n)-f(u_K^n)\big)-\big(G^G(u_L^n,u_K^n)-\phi(u_K^n)\big)\Big]\Bigg\}\int_K\psi(t_n,x)\,dx\\
&\quad+\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\frac{k}{|K|}\sum_{L\in\mathcal{N}(K)}\big(1-\theta(u_K^n,u_L^n)\big)|\sigma_{K,L}|\Bigg\{v_{K,L}^n\Big[\beta'(u_K^n)\big(F_D^{LF}(u_K^n,u_L^n)-f(u_K^n)\big)\\
&\qquad-\big(G_D^{LF}(u_K^n,u_L^n)-\phi(u_K^n)\big)\Big]-v_{K,L}^n\Big[\beta'(u_K^n)\big(F_D^{LF}(u_L^n,u_K^n)-f(u_K^n)\big)\\
&\qquad-\big(G_D^{LF}(u_L^n,u_K^n)-\phi(u_K^n)\big)\Big]\Bigg\}\int_K\psi(t_n,x)\,dx\\
&\equiv\mathbf{B}_{1,G}^{h,k}-\mathbf{B}_{2,G}^{h,k}+\big(\mathbf{B}_{1,LF}^{h,k}-\mathbf{B}_{2,LF}^{h,k}\big).
\end{aligned}$$

Since F^G is a Godunov numerical flux, the same argument (cf. proof of $\mathbf{B}_1^{h,k}-\mathbf{B}_2^{h,k}\ge 0$) yields that $\mathbf{B}_{1,G}^{h,k}-\mathbf{B}_{2,G}^{h,k}\ge 0$ $\mathbb{P}$-a.s. To deal with second term, we use the fact that any modified Lax-Friedrichs scheme is basically a flux-splitting scheme, see Remark 6.1. Thus, by using a similar argument as in the proof of Discrete entropy inequality of [31] we conclude that

$$\mathbb{P}\text{-a.s.},\quad \mathbf{B}_{1,LF}^{h,k}-\mathbf{B}_{2,LF}^{h,k}\ge 0.$$

Once again, using the above form of the numerical flux and its corresponding entropy flux, we obtain

$$
\begin{aligned}
\mathbf{B}_2^{h,k} - \mathbf{B}_3^{h,k} = & \Bigg\{ \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \frac{k}{|K|} \sum_{L\in\mathcal{N}(K)} \theta(u_K^n, u_L^n)|\sigma_{K,L}| \Big\{ v_{K,L}^n \big(G^G(u_K^n, u_L^n) - \phi(u_K^n) \big) \\
& \qquad - v_{L,K}^n \big(G^G(u_L^n, u_K^n) - \phi(u_K^n) \big) \Big\} \int_K \psi(t_n, x)\, dx \\
& + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \int_K \theta(u_K^n, u_L^n) \phi(u_K^n) \vec{v}(t,x) \cdot \nabla_x \psi(t_n, x)\, dx\, dt \Bigg\} \\
+ & \Bigg\{ \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \frac{k}{|K|} \sum_{L\in\mathcal{N}(K)} \big(1 - \theta(u_K^n, u_L^n)\big)|\sigma_{K,L}| \Big\{ v_{K,L}^n \big(G_D^{LF}(u_K^n, u_L^n) - \phi(u_K^n) \big) \\
& \qquad - v_{L,K}^n \big(G_D^{LF}(u_L^n, u_K^n) - \phi(u_K^n) \big) \Big\} \int_K \psi(t_n, x)\, dx \\
& + \sum_{n=0}^{N-1} \sum_{K\in\mathcal{T}_R} \int_{t_n}^{t_{n+1}} \int_K \big(1 - \theta(u_K^n, u_L^n)\big) \phi(u_K^n) \vec{v}(t,x) \cdot \nabla_x \psi(t_n, x)\, dx\, dt \Bigg\} \\
\equiv & \big(\mathbf{B}_{2,G}^{h,k} - \mathbf{B}_{3,G}^{h,k} \big) + \big(\mathbf{B}_{2,LF}^{h,k} - \mathbf{B}_{3,LF}^{h,k} \big).
\end{aligned}
$$

As G^G is the numerical flux corresponding to a Godunov flux, one has (cf. Step 2 of Proposition 6.3)

$$
\mathbb{E}\Big[\mathbf{1}_B \big(\mathbf{B}_{2,G}^{h,k} - \mathbf{B}_{3,G}^{h,k} \big) \Big] \to 0 \quad (h \to 0) \quad \text{for any } \mathbb{P}\text{-measurable set } B.
$$

Once again, we use similar technique as in [31] for a flux-splitting scheme and conclude the convergence of $\mathbb{E}\Big[\mathbf{1}_B \big(\mathbf{B}_{2,LF}^{h,k} - \mathbf{B}_{3,LF}^{h,k} \big) \Big]$. This finishes the proof. □

To prove convergence of the monotone finite volume scheme, it is required to have a continuous entropy inequality on the discrete solutions. We have the following proposition which essentially gives the entropy inequality for $u_{\mathcal{T},k}^h$.

Proposition 6.5 (Entropy Inequality for Approximate Solution)**.** *Let the assumptions* **A**.1-**A**.4 *hold, and $T > 0$ be fixed. Let $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size h in the sense of Definition* 3.1*. Let $k = \frac{T}{N}$ be the time step for some $N \in \mathbb{N}^*$ satisfying* (3.5)*. Then the following hold:*

a) *For any $\beta \in \mathcal{A}$ and $0 \le \psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$, there holds*

$$
\begin{aligned}
& \int_{\mathbb{R}^d} \beta(u_0(x))\psi(0,x)\, dx + \int_{\Pi_T} \Big\{ \beta(u_{\mathcal{T},k}^h)\partial_t \psi(t,x) + \phi(u_{\mathcal{T},k}^h)\vec{v}(t,x)\cdot\nabla_x\psi(t,x) \Big\}\, dt\, dx \\
& + \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1 \eta(u_{\mathcal{T},k}^h; z)\beta'\big(u_{\mathcal{T},k}^h + \lambda\eta(u_{\mathcal{T},k}^h; z)\big)\psi(t,x)\, d\lambda\, \tilde{N}(dz,dt)\, dx \\
& + \int_{\Pi_T}\int_{\mathbf{E}}\int_0^1 (1-\lambda)\eta^2(u_{\mathcal{T},k}^h; z)\beta''\big(u_{\mathcal{T},k}^h + \lambda\eta(u_{\mathcal{T},k}^h; z)\big)\psi(t,x)\, d\lambda\, m(dz)\, dt\, dx \ge \mathcal{R}^{h,k} \quad \mathbb{P}\text{-}a.s. \qquad (6.7)
\end{aligned}
$$

b) *For any $\mathbb{P}$-measurable set B,*

$$
\mathbb{E}\big[\mathbf{1}_B \mathcal{R}^{h,k} \big] \to 0 \quad (h \to 0).
$$

Proof. We will prove this proposition into two steps. In the first step, we will show the inequality (6.7) for a convenient $\mathcal{R}^{h,k}$, and establish the required convergence for $\mathcal{R}^{h,k}$ in the final step.

Step 1: Let the assumptions of the proposition hold true. Since (3.5) holds, we may assume that the CFL condition (4.11) holds as well. Therefore the estimates given in Lemmas 4.1 and 4.2 and Proposition 6.3 hold as well. Let $\psi \in C_c^\infty([0,T) \times \mathbb{R}^d)$ be a nonnegative test function. Then there exists $R > h$ such that supp $\psi \subset B(0, R-h) \times [0,T)$. Also define $\mathcal{T}_R = \{K \in \mathcal{T} : K \subset B(0,R)\}$.

Note that $\psi(t_N, x) = 0$ for any $x \in \mathbb{R}^d$. Using the summation by parts formula,

$$
\sum_{n=1}^{N} a_n\big(b_n - b_{n-1}\big) = a_N b_N - a_0 b_0 - \sum_{n=0}^{N-1} b_n\big(a_{n+1} - a_n\big)
$$

we obtain

$$
\begin{aligned}
&-\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K \big(\beta(u_K^{n+1})-\beta(u_K^n)\big)\psi(t_n,x)\,dx \\
&= \sum_{K\in\mathcal{T}_R}\int_K \beta(u_K^0)\psi(0,x)\,dx + \int_k^T\int_{\mathbb{R}^d}\beta(u_{\mathcal{T},k}^h)\partial_t\psi(t-k,x)\,dx\,dt.
\end{aligned}
\tag{6.8}
$$

Invoking (6.8) in (6.4), we obtain (6.7) for a convenient $\mathcal{R}^{h,k}$ given by

$$
\begin{aligned}
\mathcal{R}^{h,k} =& R^{h,k} + \Big\{\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx - \sum_{K\in\mathcal{T}_R}\int_K\beta(u_K^0)\psi(0,x)\,dx\Big\} \\
&+\Big\{\int_{\Pi_T}\beta(u_{\mathcal{T},k})\partial_t\psi(t,x)\,dt\,dx - \int_k^T\int_{\mathbb{R}^d}\beta(u_{\mathcal{T},k}^h)\partial_t\psi(t-k,x)\,dx\,dt\Big\} \\
&+\Big\{\int_{\Pi_T}\phi(u_{\mathcal{T},k}^h)\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\,dt\,dx - \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K\phi(u_K^n)\vec{v}(t,x)\cdot\nabla_x\psi(t_n,x)\,dx\,dt\Big\} \\
&+\Bigg\{\int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1\eta(u_{\mathcal{T},k}^h;z)\beta'\big(u_{\mathcal{T},k}^h+\lambda\eta(u_{\mathcal{T},k}^h;z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx \\
&\quad -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1\eta(u_K^n;z)\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,\tilde{N}(dz,dt)\Bigg\} \\
&+\Bigg\{\int_{\mathbb{R}^d}\int_0^T\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u_{\mathcal{T},k}^h;z)\beta''\big(u_{\mathcal{T},k}^h+\lambda\eta(u_{\mathcal{T},k}^h;z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx \\
&\quad -\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big)\psi(t_n,x)\,d\lambda\,dx\,m(dz)\,dt\Bigg\} \\
&\equiv R^{h,k}+\mathcal{I}^{h,k}+\mathcal{T}^{h,k}+\mathcal{D}^{h,k}+\mathcal{M}^{h,k}+\mathcal{A}^{h,k}.
\end{aligned}
$$

Step 2: In this step, we will show the convergence of the following quantities:

$$
\mathbb{E}\big[\mathbf{1}_B R^{h,k}\big],\ \mathbb{E}\big[\mathbf{1}_B\mathcal{I}^{h,k}\big],\ \mathbb{E}\big[\mathbf{1}_B\mathcal{T}^{h,k}\big],\ \mathbb{E}\big[\mathbf{1}_B\mathcal{D}^{h,k}\big],\ \mathbb{E}\big[\mathbf{1}_B\mathcal{M}^{h,k}\big],\ \text{and}\ \mathbb{E}\big[\mathbf{1}_B\mathcal{A}^{h,k}\big]
$$

where B being a $\mathbb{P}$- measurable subset of Ω.

a). **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{I}^{h,k}\big]$:** Thanks to Lebesgue differentiation theorem, for almost all $x\in K$, $\big|u_0(x)-u_K^0\big|\to 0$ as diameter of K tends to zero (i.e., $h\to 0$). Now

$$
\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{I}^{h,k}\big]\Big| &= \Big|\mathbb{E}\Big[\mathbf{1}_B\sum_{K\in\mathcal{T}_R}\int_K\big(\beta(u_0(x))-\beta(u_K^0)\big)\psi(x,0)\,dx\Big]\Big| \\
&\le ||\beta'||_\infty\mathbb{E}\Big[\sum_{K\in\mathcal{T}_R}\int_K\big|u_0(x)-u_K^0\big|\psi(x,0)\,dx\Big],
\end{aligned}
$$

and hence $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{I}^{h,k}\big]\to 0$ as $h\to 0$.

b). **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{T}^{h,k}\big]$**: We use Lemma 4.1, the CFL condition (4.11) along with triangle inequality to have

$$
\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{T}^{h,k}\big]\Big| &= \Big|\mathbb{E}\Big[\mathbf{1}_B\int_k^T\int_{\mathbb{R}^d}\beta(u_{\mathcal{T},k}^h)\big(\partial_t\psi(t,x)-\partial_t\psi(t-k,x)\big)\,dx\,dt\Big] \\
&\qquad + \mathbb{E}\Big[\mathbf{1}_B\int_0^k\int_{\mathbb{R}^d}\beta(u_{\mathcal{T},k}^h)\partial_t\psi(t,x)\,dx\,dt\Big]\Big| \\
&\le ||\beta'||_\infty\,k\Big(||\partial_t\psi||_\infty\|u_{\mathcal{T},k}^h\|_{L^\infty\left(0,T;L^1(\Omega\times B(0,R))\right)} + ||\partial_{tt}\psi||_\infty\,\|u_{\mathcal{T},k}^h\|_{L^1\left(\Omega\times B(0,R)\times[0,T)\right)}\Big),
\end{aligned}
$$

and hence $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{T}^{h,k}\big]\to 0$ as $h\to 0$.

c). **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{D}^{h,k}\big]$:** In view of (3.4), we see that

$$
\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{D}^{h,k}\big]\Big| &= \Big|\mathbb{E}\Big[\mathbf{1}_B \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\int_{t_n}^{t_{n+1}} \phi(u_K^n)\vec{v}(t,x)\cdot\big(\nabla_x\psi(t,x)-\nabla_x\psi(t_n,x)\big)\,dt\,dx\Big]\Big| \\
&\le V||\nabla_x\partial_t\psi||_\infty k\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_K\int_{t_n}^{t_{n+1}}\mathbb{E}\Big[\big|\phi(u_K^n)\big|\Big]\,dx\,dt \\
&\le V||\nabla_x\partial_t\psi||_\infty k||\beta'||_\infty||f'||_\infty\sum_{n=0}^{N-1}\sum_{K\in\mathbb{R}_R}\int_K\int_{t_n}^{t_{n+1}}\mathbb{E}\big[|u_K^n|\big]\,dx\,dt \\
&\le Ck\|u^h_{\mathcal{T},k}\|_{L^1\left(\Omega\times B(0,R)\times[0,T)\right)} \to 0 \text{ as } h\to 0,
\end{aligned}
$$

where $C\equiv C(\beta,f,V,\psi)\in\mathbb{R}_+^*$ is a constant. In the above, we have used the boundedness condition of $\vec{v}(t,x)$, Lemma 4.1 and (4.11).

d). **Convergence of $E\big[\mathbf{1}_B\,\mathcal{M}^{h,k}\big]$:** In view of the definition of $u_{\mathcal{T},k}$, Cauchy-Schwarz inequality, Itô-Lévy isometry, the CFL condition (4.11) and Lemma 4.1, we see that

$$
\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{M}^{h,k}\big]\Big|^2 &= \Big|E\Big[\mathbf{1}_B\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \eta(u_K^n;z)\beta'\big(u_K^n+\lambda\eta(u_K^n;z)\big) \\
&\qquad\qquad \times\big(\psi(t,x)-\psi(t_n,x)\big)\,d\lambda\,dx\,\tilde{N}(dz,dt)\Big]\Big|^2 \\
&\le |B(0,R)|\Bigg(\sum_{n=0}^{N-1}\Bigg\{\mathbb{E}\Big[\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 \eta^2(u_K^n;z)\beta'^2\big(u_K^n+\lambda\eta(u_K^n;z)\big) \\
&\qquad\qquad \times\big(\psi(t,x)-\psi(t_n,x)\big)^2\,d\lambda\,dx\,m(dz)\,dt\Big]\Bigg\}^{\frac{1}{2}}\Bigg)^2 \\
&\le C(R,\psi,c_\eta,\beta)k\Big(\sum_{n=0}^{N-1}k\Big(\sum_{K\in\mathcal{T}_R}|K|\mathbb{E}\Big[(u_K^n)^2\Big]\Big)^{\frac{1}{2}}\Big)^2 \\
&\le C(R,c_\eta,\psi,\beta,T)\,k\|u^h_{\mathcal{T},k}\|^2_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d)))} \to 0 \text{ as } h\to 0.
\end{aligned}
$$

e). **Convergence of $\mathbb{E}\big[\mathbf{1}_B\,\mathcal{A}^{h,k}\big]$ and $\mathbb{E}\big[\mathbf{1}_B R^{h,k}\big]$:** To show the convergence, we proceed as follows. We rewrite $\mathcal{A}^{h,k}$ as

$$
\begin{aligned}
\mathcal{A}^{h,k} = \sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K\int_0^1 &(1-\lambda)\eta^2(u_K^n;z)\beta''\big(u_K^n+\lambda\eta(u_K^n;z)\big) \\
&\times\Big\{\psi(t,x)-\psi(t_n,x)\Big\}\,d\lambda\,dx\,m(dz)\,dt.
\end{aligned}
$$

Therefore, by using Lemma 4.1 and the CFL condition (4.11), we obtain

$$
\begin{aligned}
\Big|\mathbb{E}\big[\mathbf{1}_B\,\mathcal{A}^{h,k}\big]\Big| &\le ||\beta''||_\infty k||\partial_t\psi||_\infty\mathbb{E}\Big[\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_{\mathbf{E}}\int_K \eta^2(u_K^n;z)\,dx\,m(dz)\,dt\Big] \\
&\quad\text{(by the assumption } \mathbf{A}.3 \text{ on } \eta) \\
&\le ||\beta''||_\infty k||\partial_t\psi||_\infty c_\eta\sum_{n=0}^{N-1}\sum_{K\in\mathcal{T}_R}\int_{t_n}^{t_{n+1}}\int_K\mathbb{E}\big[(u_K^n)^2\big]\,dx\,dt \\
&\le ||\beta''||_\infty k||\partial_t\psi||_\infty c_\eta\|u^h_{\mathcal{T},k}\|^2_{L^2(\Omega\times\Pi_T)} \to 0 \text{ as } h\to 0.
\end{aligned}
$$

We have seen already that $\mathbb{E}\big[\mathbf{1}_B R^{h,k}\big]\to 0$ as $h\to 0$ (cf. Proposition 6.3). This completes the proof. $\square$

7. Proof of Main Theorem

Having all the necessary *a-priori* bounds and entropy inequalities on the approximate solution $u^h_{\mathcal{T},k}$ at hand, we are now in a position to prove Main theorem. As we mentioned earlier that $\mathbf{u}(\theta,\alpha)$ given by Proposition 5.1 will serve as a possible generalized entropy solution to (1.1). Since $\mathbf{u}(t,x,\alpha) \in L^\infty(0,T;L^2(\Omega\times\mathbb{R}^d\times(0,T)))$, it remains only to show that $\mathbf{u}(t,x,\alpha)$ satisfies the entropy inequalities (2.1).

7.1. **Proof of Main Theorem.** Let the assumptions **A**.1-**A**.4 be true and $\mathcal{T}$ be an admissible mesh on $\mathbb{R}^d$ with size h in the sense of Definition 3.1. Again, let $k=\frac{T}{N}$ be the time step, for some $N\in\mathbb{N}^*$ with fixed $T>0$ such that (3.5) holds. In this way, we assume that the CFL condition (4.11) holds and hence the results of Lemmas 4.1, 4.2, Propositions 6.1, 6.3 and 6.5 hold. Let $\psi\in C_c^\infty([0,T)\times\mathbb{R}^d)$ be a nonnegative test function. Then there exists $R>h$ such that supp $\psi\subset B(0,R-h)\times[0,T)$. Define $\mathcal{T}_R=\{K\in\mathcal{T}: K\subset B(0,R)\}$. Let $B\in\mathcal{F}_T$. In view of (6.7), we have

$$
\begin{aligned}
&\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\Big\{\beta(u^h_{\mathcal{T},k})\partial_t\psi(t,x)+\phi(u^h_{\mathcal{T},k})\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx\Big]\\
&\quad+\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\eta(u^h_{\mathcal{T},k};z)\beta'\big(u^h_{\mathcal{T},k}+\lambda\eta(u^h_{\mathcal{T},k};z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx\Big]\\
&\qquad+\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u^h_{\mathcal{T},k};z)\beta''\big(u^h_{\mathcal{T},k}+\lambda\eta(u^h_{\mathcal{T},k};z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx\Big]\\
&\qquad\quad+\mathbb{E}\Big[\mathbf{1}_B\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx\Big]\ge\mathbb{E}\Big[\mathbf{1}_B\mathcal{R}^{h,k}\Big]
\end{aligned}
$$

$$
\text{i.e.,}\qquad \mathcal{H}_{1,h}+\mathcal{H}_{2,h}+\mathcal{H}_{3,h}+\mathbb{E}\Big[\mathbf{1}_B\int_{\mathbb{R}^d}\beta(u_0(x))\psi(0,x)\,dx\Big]\ge\mathbb{E}\Big[\mathbf{1}_B\mathcal{R}^{h,k}\Big]. \tag{7.1}
$$

We would like to pass to the limit in (7.1) as $h\to0$. For this, we use the technique of Young measure theory in stochastic setting. Let (Θ,Σ,μ) be a σ- finite measure space prescribed in Section 5. Note that $L^2(\Theta,\Sigma,\mu)$ is a closed subspace of the larger space $L^2\big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\big)$ and hence the weak convergence in $L^2\big(\Theta,\Sigma,\mu\big)$ would imply weak convergence in $L^2\big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\big)$. Now, for any $B\in\mathcal{F}_T$, the functions $\mathbf{1}_B\partial_t\psi(t,x)$, $\mathbf{1}_B\partial_{x_i}\psi(t,x)$, $\mathbf{1}_B\psi(t,x)$ are all members of $L^2\big(0,T;L^2((\Omega,\mathcal{F}_T),L^2(\mathbb{R}^d))\big)$. Therefore, in view of Proposition 5.1 and the above discussion, one has

$$
\begin{aligned}
\lim_{h\to0}\mathcal{H}_{1,h}&=\lim_{h\to0}\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\Big\{\beta(u^h_{\mathcal{T},k})\partial_t\psi(t,x)+\phi(u^h_{\mathcal{T},k})\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,dt\,dx\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_0^1\Big\{\beta(\mathbf{u}(t,x,\alpha))\partial_t\psi(t,x)+\phi(\mathbf{u}(t,x,\alpha))\vec{v}(t,x)\cdot\nabla_x\psi(t,x)\Big\}\,d\alpha\,dt\,dx\Big].
\end{aligned}\tag{7.2}
$$

Next we want to pass to the limit in $\mathcal{H}_{3,h}$. For this, we fix (λ,z), and define a Carathéodory function

$$
G_{\lambda,z}(r,x,\omega,\xi)=\mathbf{1}_B(\omega)(1-\lambda)\eta^2(\xi,z)\beta''\big(\xi+\lambda\eta(\xi,z)\big)\psi(r,x).
$$

Note that $\{G_{\lambda,z}(r,x,\omega,u^{h_n}_{\mathcal{T},k}(r,x,\omega))\}_n$ is uniformly integrable in $L^1((\Theta,\Sigma,\mu);\mathbb{R})$. Thus, in view of Proposition 5.1 we have, for fixed $(\lambda,z)\in(0,1)\times\mathbf{E}$

$$
\begin{aligned}
&\lim_{h\to0}\mathbb{E}\Big[\int_{\Pi_T}\mathbf{1}_B(1-\lambda)\eta^2(u^h_{\mathcal{T},k};z)\beta''\big(u^h_{\mathcal{T},k}+\lambda\eta(u^h_{\mathcal{T},k};z)\big)\psi(t,x)\,dt\,dx\Big]\\
&=\mathbb{E}\Big[\int_{\Pi_T}\int_0^1\mathbf{1}_B(1-\lambda)\eta^2(\mathbf{u}(t,x,\alpha);z)\beta''\big(\mathbf{u}(t,x,\alpha)+\lambda\eta(\mathbf{u}(t,x,\alpha);z)\big)\psi(t,x)\,d\alpha\,dt\,dx\Big].
\end{aligned}
$$

Thanks to the assumption **A**.3, and Lemma 4.1, we invoke dominated convergence theorem and have

$$
\begin{aligned}
\lim_{h\to0}\mathcal{H}_{3,h}&=\lim_{h\to0}\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1(1-\lambda)\eta^2(u^h_{\mathcal{T},k};z)\beta''\big(u^h_{\mathcal{T},k}+\lambda\eta(u^h_{\mathcal{T},k};z)\big)\psi(t,x)\,d\lambda\,m(dz)\,dt\,dx\Big]\\
&=\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1(1-\lambda)\eta^2(\mathbf{u}(t,x,\alpha);z)\beta''\big(\mathbf{u}(t,x,\alpha)+\lambda\eta(\mathbf{u}(t,x,\alpha);z)\big)\\
&\qquad\qquad\times\psi(t,x)\,d\alpha\,d\lambda\,m(dz)\,dt\,dx\Big].
\end{aligned}\tag{7.3}
$$

Now passage to the limit in the martingale term requires some additional reasoning. Let $\Gamma = \Omega\times[0,T]\times\mathbf{E}$, $\mathcal{G} = \mathcal{P}_T \times \mathcal{L}(\mathbf{E})$ and $\varsigma = \mathbb{P}\otimes\lambda_t\otimes m(dz)$, where $\mathcal{L}(\mathbf{E})$ represents a Lebesgue σ- algebra on $\mathbf{E}$. The space $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ represents the space of square integrable predictable integrands for Itô-Lévy integrals with respect to the compensated Poisson random measure $\tilde{N}(dz,dt)$. Moreover, by Itô-Lévy isometry and martingale representation theorem, it follows that Itô-Lévy integral defines isometry between two Hilbert spaces $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ and $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. In other words, if $\mathcal{I}$ denotes the Itô-Lévy integral operator, i.e., the application

$$\begin{aligned}\mathcal{I} : L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big) &\to L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)\\ v &\mapsto \int_0^T\int_{\mathbf{E}} v(\omega,z,r)\tilde{N}(dz,dr)\end{aligned}$$

and $\{X_n\}_n$ be a sequence in $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ converges weakly to X; then $\mathcal{I}(X_n)$ will converge weakly to $\mathcal{I}(X)$ in $L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big)$. Note that, for fixed $z\in\mathbf{E}$, $G(t,x,\omega,\xi) = \Big(\beta\big(\xi+\eta(\xi;z)\big)-\beta(\xi)\Big)\psi(t,x)$ is a Carathéodory function and $\{G(t,x,\omega,u_{\mathcal{T},k}^{h_n}(t,x,\omega))\}_n$ is uniformly integrable in $L^1((\Theta,\Sigma,\mu);\mathbb{R})$. Therefore, one can apply Proposition 5.1 and conclude that for $m(dz)$-almost every $z\in\mathbf{E}$ and $g(t,z)\in L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$,

$$\begin{aligned}&\lim_{h\to 0}\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}\Big(\beta\big(u_{\mathcal{T},k}^h+\eta(u_{\mathcal{T},k}^h;z)\big)-\beta\big(u_{\mathcal{T},k}^h\big)\Big)\psi(r,x)g(r,z)\,dx\,dr\Big]\\ &\quad=\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}^d}\int_0^1\Big(\beta\big(\mathbf{u}(r,x,\alpha)+\eta(\mathbf{u}(r,x,\alpha);z)\big)-\beta\big(\mathbf{u}(r,x,\alpha)\big)\Big)\psi(r,x)g(r,z)\,d\alpha\,dx\,dr\Big].\end{aligned}$$

We apply dominated convergence theorem along with Lemma 4.1 and the assumption **A**.3 to have

$$\begin{aligned}&\lim_{h\to 0}\mathbb{E}\Big[\int_0^T\int_{\mathbf{E}}\int_{\mathbb{R}^d}\Big(\beta\big(u_{\mathcal{T},k}^h+\eta(u_{\mathcal{T},k}^h;z)\big)-\beta\big(u_{\mathcal{T},k}^h\big)\Big)\psi(r,x)h(r,z)\,dx\,m(dz)\,dr\Big]\\ &\quad=\mathbb{E}\Big[\int_0^T\int_{\mathbf{E}}\Big\{\int_{\mathbb{R}^d}\int_0^1\Big(\beta\big(\mathbf{u}(r,x,\alpha)+\eta(\mathbf{u}(r,x,\alpha);z)\big)-\beta\big(\mathbf{u}(r,x,\alpha)\big)\Big)\\ &\qquad\qquad\times\psi(r,x)h(r,z)\,d\alpha\,dx\Big\}\,m(dz)\,dr\Big].\end{aligned}$$

Hence, if we denote

$$X_n(t,z) = \int_{\mathbb{R}^d}\Big(\beta\big(u_{\mathcal{T},k}^h+\eta(u_{\mathcal{T},k}^h;z)\big)-\beta\big(u_{\mathcal{T},k}^h\big)\Big)\psi(t,x)\,dx$$

and

$$X(t,z) = \int_{\mathbb{R}^d}\int_0^1\Big(\beta\big(\mathbf{u}(t,x,\alpha)+\eta(\mathbf{u}(t,x,\alpha);z)\big)-\beta\big(\mathbf{u}(t,x,\alpha)\big)\Big)\psi(t,x)\,d\alpha\,dx$$

then, X_n converges to X in $L^2\big((\Gamma,\mathcal{G},\varsigma);\mathbb{R}\big)$ which implies, in view of the above discussion

$$\int_0^T\int_{\mathbf{E}}X_n(t,z)\tilde{N}(dz,dt)\rightharpoonup\int_0^T\int_{\mathbf{E}}X(t,z)\tilde{N}(dz,dt)\quad\text{in}\quad L^2\big((\Omega,\mathcal{F}_T);\mathbb{R}\big).$$

In other words, since $B\in\mathcal{F}_T$, we obtain

$$\begin{aligned}\lim_{h\to 0}\mathcal{H}_{2,h} &= \lim_{h\to 0}\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\eta(u_{\mathcal{T},k}^h;z)\beta'\big(u_{\mathcal{T},k}^h+\lambda\eta(u_{\mathcal{T},k}^h;z)\big)\psi(t,x)\,d\lambda\,\tilde{N}(dz,dt)\,dx\Big]\\ &=\mathbb{E}\Big[\mathbf{1}_B\int_{\Pi_T}\int_{\mathbf{E}}\int_0^1\int_0^1\eta(\mathbf{u}(t,x,\alpha);z)\beta'\big(\mathbf{u}(t,x,\alpha)+\lambda\eta(\mathbf{u}(t,x,\alpha);z)\big)\\ &\qquad\qquad\times\psi(t,x)\,d\alpha\,d\lambda\,\tilde{N}(dz,dt)\,dx\Big].\end{aligned}\tag{7.4}$$

By (7.2),(7.3) and (7.4), and the fact that $\mathbb{E}\big[\mathbf{1}_B\mathcal{R}^{h,k}\big]\to 0$ as $h\to 0$ for any $B\in\mathcal{F}_T$, thanks to Proposition 6.5, one can pass to the limit as $h\to 0$ in (7.1) yielding (2.1). This shows that $\mathbf{u}(x,t,\alpha)$ is a generalized entropy solution of (1.1). Again thanks to the uniqueness of generalized entropy solutions (cf. Theorem 2.2), $\mathbf{u}(t,x,\alpha)$ is independent of α and hence $\bar{u}(t,x)=\int_0^1\mathbf{u}(t,x,\alpha)\,d\alpha$ is the unique entropy solution of

(1.1). Moreover, the whole sequence of $u^h_{\mathcal{T},k}(t,x)$ converges to $\bar{u}(t,x)$ in $L^p_{\text{loc}}(\mathbb{R}^d; L^p(\Omega \times (0,T)))$ for any $1 \le p < 2$. This completes the proof.

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