CONVERGENT FINITE ELEMENT BASED DISCRETIZATION OF THE
STOCHASTIC LANDAU-LIFSHITZ-GILBERT EQUATION

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Abstract. We propose a convergent finite element based discretization of the stochastic
Landau-Lifshitz-Gilbert equation. A main difficulty in the convergence analysis for this
nonlinear SPDE is to properly address the pointwise sphere condition, and the Stratonovich
noise in the fully discrete scheme. Approximates of the scheme proposed here satisfy a
sphere constraint at nodal points of the spatial discretization, have finite energies, and
their increments may be controlled uniformly with respect to discretization parameters.
Sequences of corresponding continuified processes may then be generated which construct
weak martingale solutions of the limiting equations.

Stochastic Landau-Lifshitz-Gilbert equation, finite element method, tightness, Stratonovich
equation
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1. Introduction

The phenomenological Landau-Lifshitz-Gilbert equation (LLG) describes the magnetiza-
tion of a ferromagnetic material occupying a bounded region \( D \subset \mathbb{R}^n, n = 2 \) or 3; cf. [35].
Let \( \alpha \geq 0 \) denote the damping parameter, and \( \mathbf{H}_{\text{eff}} : D_T := (0, T) \times D \to \mathbb{R}^3 \) the effective
field; then the magnetization \( \mathbf{m} : (0, T) \times D \to S^2 := \{ \mathbf{x} \in \mathbb{R}^3 : |\mathbf{x}| = 1 \} \), satisfies

\[
\mathbf{m}_t = -\alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}) + \mathbf{m} \times \mathbf{H}_{\text{eff}} \quad \text{in } D_T := (0, T) \times D,
\]

together with the following initial and boundary conditions,

\[
\begin{align*}
\mathbf{m}(0, \cdot) &= \mathbf{m}_0 \quad \text{on } D, \\
\partial_n \mathbf{m} &= 0 \quad \text{on } \partial D_T,
\end{align*}
\]

where \( \partial D_T = \partial D \times (0, T) \), and \( \mathbf{m}_0 \in W^{1,2}(D; S^2). \) — An important problem in the theory of
ferromagnetism is to describe transitions between different local equilibrium states, which are
induced by thermal noise; those random fluctuations allow for a non-vanishing probability of
magnetization switching between equilibrium states e.g. [6, 23, 34]. The noise is incorporated
into the Landau-Lifshitz-Gilbert equation by a perturbation of the effective field \( \mathbf{H}_{\text{eff}} \) ([8]).
A standard assumption in the physical literature is that the noise is uncorrelated both in
space and time [8, 23, 36, 6, 22]. The general theory of parabolic SPDEs as developed in
the book [17] is not applicable here because of the irregularity of the noise and, indeed, even
local existence results seem to be out of reach of the current theory. Furthermore, there is
an example in the recent preprint [30] which shows that even the additive space-time white
noise can lead to non-existence of solutions for SPDEs of parabolic type in dimensions higher
than one. In this paper we consider a simplified version of the problem by using noise which
is uncorrelated in time and correlated in space, i.e., the field \( \mathbf{H}_{\text{eff}} \) is perturbed to \( \mathbf{H}_{\text{eff}} + \tilde{\mathbf{W}}, \)

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where $W$ is a $Q$-Wiener process which represents colored noise; see Section 2 for notation. The goal of this work is to construct martingale solutions by an implementable space-time discretization of this SPDE.

Thus, the above assumptions lead to the following stochastic version of the Landau-Lifshitz-Gilbert equation ($H_{\text{eff}} = \Delta m$),

$$\begin{align*}
\text{d}m &= -\alpha m \times (m \times [\Delta m \text{d}t + \text{d}W]) + m \times [\Delta m \text{d}t + \text{d}W].
\end{align*}$$

It is not obvious how to understand the stochastic term in this equation. In order to accommodate for the pathwise sphere constraint $|m| = 1$ the stochastic term should be understood in the Stratonovich sense ([6],[23],[8],[36]). Moreover, by following the arguments in [23], we assume that $\alpha > 0$ is small, in which case the noise in the leading term on the right-hand side of (1.4) can be neglected.

Then, the stochastic version of the LLG equation that we are going to numerically approximate in this paper takes the form

$$\begin{align*}
\text{d}M(t, x) &= -\alpha M(t, x) \times (M(t, x) \times \Delta M(t, x)) \text{d}t + M(t, x) \times \Delta M(t, x) \text{d}t \\
&\quad + M(t, x) \times \circ \text{d}W(t) \quad \forall (t, x) \in D_T, \\
\partial_n M(t, x) &= 0 \quad \forall (t, x) \in \partial D_T, \\
M(0, x) &= M_0(x) \quad \forall x \in D.
\end{align*}$$

We refer to [6, 8, 23, 22, 35] and references therein for further physical background for this model. In [12], the existence of a weak martingale solution to (1.5) by a general abstract Faedo-Galerkin method is shown where corresponding (approximate) solutions $\{M_n\}_n \subset L^2(\Omega; C([0, T]; L^2))$ satisfy $\mathbb{P}$-almost surely for every $t \geq 0$ and every $n \geq 1$ that $|M_n(t, \cdot)|_{L^2} = |M_0|_{L^2}$. By a compactness argument, corresponding limits are then shown to satisfy (1.5) in a proper sense; see Definition 2.1 below. The goal of this work is to provide an alternative proof, by constructing a weak martingale solution to equation (1.5) as proper limit of iterates which solve the implementable finite element based Algorithm 1.2. Corresponding iterates satisfy the sphere constraint at nodal points of the spatial discretization, a property which was not available for the general Faedo-Galerkin method in [12]; compare part (i) of Theorem 2.1 and (3.10) in Theorem 3.5 of [12].

In this work, we propose a fully discrete finite element based discretization of (1.5) whose solutions construct weak martingale solutions of (1.5) for vanishing discretization parameters. The scheme uses a lowest order conforming finite element space, such that $V_h \subset W^{1,2}(D, \mathbb{R}^2)$, on a quasi-uniform triangulation $T_h$ of $D$, and a partition $I_k = \{t_j\}_{j=0,\ldots,J}$ of equidistant time-step size $k > 0$ of the time interval $[0, T]$. Key properties of the scheme are

(i) the (discrete) sphere constraint for the corresponding iterates at nodal points $E_h := \{x_\ell \in \bar{D} : \ell \in L\}$ of the triangulation $T_h$, as well as

(ii) relevant bounds which hold uniformly with respect to $k, h > 0$.

In [1], a corresponding program has been realized for the deterministic LLG (1.1), where iterates satisfy the sphere constraint at nodal points of the mesh $T_h$, as well as the discrete energy law. We denote, for $j \geq 0$ and a sequence $\{\varphi^j\}_j$,

$$\begin{align*}
d_j \varphi^{j+1} &= k^{-1} (\varphi^{j+1} - \varphi^j), \\
\varphi^{j+1/2} &= \frac{1}{2} (\varphi^{j+1} + \varphi^j).
\end{align*}$$
The deterministic algorithm proposed and studied in [1] then reads as follows.

**Algorithm 1.1.** Let $M^0 \in V_h$ be such that $|M^0(x)| = 1$ for all $x \in \mathcal{E}_h$. For every $j \geq 0$, determine $M^{j+1} \in V_h$ such that

\[
\begin{aligned}
(d_t M^{j+1}, \Phi)_h + \alpha \left( M^{j+1/2} \times [M^{j+1/2} \times \tilde{\Delta}_h M^{j+1/2}], \Phi \right)_h \\
- (M^{j+1/2} \times \tilde{\Delta}_h M^{j+1/2}, \Phi)_h = 0 \quad \forall \Phi \in V_h.
\end{aligned}
\]

Here, $(\cdot, \cdot)_h$ denotes a discrete version (reduced integration) of the inner product in $L^2(D, \mathbb{R}^3)$, and $\tilde{\Delta}_h : W^{1,2}(D, \mathbb{R}^3) \to V_h$ is a discrete version of the Laplacian; we refer to Section 2 below for further details.

In order to construct a convergent discretization of (1.5), we have to account for stochastic effects as well. It will turn out from the analysis below, that keeping the averages $M^{j+1/2}$ of subsequent iterates in the leading position of the two nonlinear terms is essential to ensure the discrete sphere constraint, while changing $\tilde{\Delta}_h M^{j+1/2}$ to $\tilde{\Delta}_h M^{j+1}$ in those terms is needed to allow for relevant a priori bounds; see Lemma 4.1.

**Algorithm 1.2.** Let $M^0 \in V_h$ be such that $|M^0(x)| = 1$ for all $x \in \mathcal{E}_h$. For every $j \geq 0$, and $\Delta_j W := W(t_{j+1}) - W(t_j) \sim \mathcal{N}(0, kQ)$, determine $M^{j+1} \in V_h$ such that

\[
\begin{aligned}
(M^{j+1} - M^j, \Phi)_h + \alpha k \left( M^{j+1/2} \times [M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}], \Phi \right)_h \\
- k (M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}, \Phi)_h = (M^{j+1/2} \times \Delta_j W, \Phi)_h \quad \forall \Phi \in V_h.
\end{aligned}
\]

This algorithm is computationally studied in [3], where the (regularizing) role of noise in the context of possible finite time blow-up behavior is discussed. Long time behaviour of the solution of the algorithm has been studied in [41].

There is a vast literature on approximations of linear and nonlinear stochastic PDEs, including [10, 27, 26, 25, 28, 18, 38, 29] and references therein. However, our paper differs from all these and other related papers in the following aspects:

(i) Problem (1.5) is a nonlinear SPDE, with a (nonconvex) pointwise sphere-constraint to hold. It is the interplay of geometric aspects and (multiplicative) stochastic forcing which requires specific numerical discretisations to conclude convergence.

(ii) The construction of weak martingale solutions to problem (1.5) uses the implementable Algorithm 1.2, which is a space-time discretization based on the finite element method. Unconditional convergence of (subsequences of) iterates to solutions of (1.5) is shown, which is stated as Theorem 2.1. It is due to possible (pathwise) finite-time blow-up behaviour of solutions (see [3]) that no regularity properties superior to basic ones may be expected in practical studies for ferromagnetism.

(iii) Our problem is intrinsically a Stratonovich equation, which makes the analysis more difficult. This is related to the well-known Wong-Zakai approximation; note that the Wong-Zakai approximation for SLLG equations remains an open problem. Our approach encounters a similar difficulty as the proof of the various versions of the Wong-Zakai Theorem and overcomes it by a splitting of the noise terms, see formula (4.19) and the following analysis. A similar difficulty is encountered in [18] by De Bouard and Debussche who however deal only with time discretization and for a different nonlinear stochastic PDE.
There is some similarity between our approach and the papers by Funaki [21] and Tessitore & Zabczyk [48] in that the approximation we use satisfies some a priori estimates which together with some compactness argument implies the existence of a solution. This approach was also used in the paper [12] where a general Galerkin approximation but no time discretization is used. Let us point out that the methods to establish the a priori bounds in [12] are different to ours, where a related analysis has to cope with the limited regularity properties of finite element functions, in particular. As a result, the present study provides an implementable numerical scheme, for which convergence is shown.

2. Preliminaries and Main Result

Standard references which are used in this section are [14, 7]. Throughout this paper we assume that $D \subset \mathbb{R}^n$, $n = 2, 3$ is a polygonal or polyhedral bounded Lipschitz domain and $\mathcal{T}_h$ is a quasiuniform triangulation of $D$ into triangles or tetrahedra $K$ for $n = 2$ or $n = 3$, where the maximum mesh-size of $\mathcal{T}_h$ is $h = \max \{\text{diam}(K) : K \in \mathcal{T}_h\} > 0$. The set of nodes of the triangulation $\mathcal{T}_h$ will be denoted by $\mathcal{E}_h := \{x_\ell : \ell \in L\}$. Let the cardinality of the set of neighboring nodes of each node $x_\ell$ be bounded independent of $h > 0$, i.e.,

$$\# \{m \in L_h : \exists K \in \mathcal{T}_h \text{ such that } x_m, x_\ell \in K\} \leq C.$$  

By $L^2(D; \mathbb{R}^3)$ we denote the standard Lebesgue space of (equivalence classes of) square integrable functions $f : D \to \mathbb{R}^3$. For given functions $f, g \in L^2(D; \mathbb{R}^3)$, the $L^2(D; \mathbb{R}^3)$-scalar product of them will be denoted by

$$\langle f, g \rangle = \int_D \langle f, g \rangle \, dx,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^3$. By $W^{1,2}(D; \mathbb{R}^3)$ we denote the Banach space of those $f \in L^2(D; \mathbb{R}^3)$ whose weak first order partial derivatives belong to $L^2(D; \mathbb{R}^3)$ as well.

For each element $K \in \mathcal{T}_h$, let $\mathcal{P}_1(K; \mathbb{R}^3)$ denote the set of all polynomials of degree less or equal to one. We define the lowest order finite element space $V_h \subset W^{1,2}(D; \mathbb{R}^3)$ by

$$V_h = \{ \phi_h \in C(\bar{D}; \mathbb{R}^3) : \phi_h|_K \in \mathcal{P}_1(K; \mathbb{R}^3) \quad \forall K \in \mathcal{T}_h \}.$$

The nodal interpolation operator

$$I_h : C(\bar{D}; \mathbb{R}^3) \to V_h$$

is uniquely defined by the following condition

$$I_h \phi(x_\ell) = \phi(x_\ell) \quad \text{for all } \ell \in L.$$ 

Note that in particular the restriction of $I_h$ to $V_h$ is an identity on $V_h$. Moreover, see for instance [7], the following inequality holds

$$|I_h \phi|_{L^\infty(D)} \leq C |\phi|_{L^\infty(D)} \quad \forall \phi \in C(\bar{D}; \mathbb{R}^3).$$

We define the bilinear form $(\cdot, \cdot)_h : C(\bar{D}; \mathbb{R}^3) \times C(\bar{D}; \mathbb{R}^3) \to \mathbb{R}$ by

$$(\phi, \psi)_h = \int_D I_h(\langle \phi, \psi \rangle)(x) \, dx = \sum_{\ell \in L} \zeta_\ell \langle \phi(x_\ell), \psi(x_\ell) \rangle \quad \forall \phi, \psi \in C(\bar{D}; \mathbb{R}^3),$$

$$|\phi|_h^2 = (\phi, \phi)_h \quad \forall \phi \in C(\bar{D}; \mathbb{R}^3).$$
for certain weights \( \zeta_\ell > 0, \ell \in L \). For each \( \ell \in L \) we denote by \( \varphi_\ell \in C(D) \) the nodal basis function which is \( T_\ell \)-elementwise affine and satisfies \( \varphi_\ell(x) = 1 \) and \( \varphi_\ell(x_m) = 0 \) for all \( m \in L \setminus \{ \ell \} \). Then, we may write \( \zeta_\ell = \int_D \varphi_\ell \, dx \). For the next two inequalities we refer to [14, 7, Sec. 28], see also [16, Lemma 2.1] for the case \( n = 2 \),

\[
|\phi_h|_{L^2} \leq |\phi_h|_h \leq (n + 2)^{1/2} |\phi_h|_{L^2} \quad \forall \phi_h \in V_h ,
\]

(2.7)

\[
|\langle \phi_h, \psi_h \rangle_h - \langle \phi_h, \psi_h \rangle| \leq Ch|\phi_h|_{L^2}|\psi_h|_{W^{1,2}} \quad \forall \phi_h, \psi_h \in V_h .
\]

(2.8)

We define the discrete Laplace operator \( \tilde{\Delta}_h : V_h \to V_h \) by the following variational identity

\[
-(\tilde{\Delta}_h \phi_h, x_h)_h = (\nabla \phi_h, \nabla x_h)_L^2 \quad \forall \phi_h, x_h \in V_h .
\]

(2.9)

It is well-known that there exists a constant \( C > 0 \) such that

\[
|\nabla \phi_h|_{L^2} \leq Ch^{-1}|\phi_h|_{L^2} \quad \forall \phi_h \in V_h .
\]

(2.10)

Choosing \( x_h = \tilde{\Delta}_h \phi_h \) in (2.9) and using (2.7), (2.10) we observe that for all \( \phi_h \in V_h \) holds

\[
|\tilde{\Delta}_h \phi_h|_h ^2 = -(\nabla \phi_h, \nabla \tilde{\Delta}_h \phi_h) \leq |\nabla \phi_h|_{L^2} |\nabla \tilde{\Delta}_h \phi_h|_{L^2} \leq C h^{-1} |\nabla \phi_h|_{L^2} |\tilde{\Delta}_h \phi_h|_h .
\]

(2.11)

Given \( \phi_h \in V_h \) and a node \( x_\ell \) for some \( \ell \in L \) we obtain from using \( x_h = \varphi_\ell \tilde{\Delta}_h \phi_h(x_\ell) \) in (2.9) that

\[
|\tilde{\Delta}_h \phi_h(x_\ell)|^2 = \zeta_\ell^{-1}(\tilde{\Delta}_h \phi_h, x_h)_h = \sum_{m \in L: \exists k \in T_h: x_m, x_\ell \in k} \langle \phi_h(x_m), \tilde{\Delta}_h \phi_h(x_\ell) \rangle (\nabla \varphi_\ell, \nabla \varphi_m)
\]

(2.12)

where we use equality (2.9).

Let \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \) be a complete probability space with a filtration \( \mathbb{F} = \{ \mathcal{F}_t; t \in [0, T] \} \).

Let \( \mathcal{K} \) be a Hilbert space. We assume that \( W = \{ W(t); t \in [0, T] \} \) is a \( \mathcal{K} \)-valued Wiener process. We denote by \( Q \) the covariance operator of \( W \). It is well known that \( Q \) is a symmetric and non-negative operator on \( \mathcal{K} \) and that \( Q \) belongs to \( S_1(\mathcal{K}) \), the space of trace class operators on \( \mathcal{K} \). Hence, there exists a sequence of i.i.d. \( \mathbb{R} \)-valued Brownian motions \( \{ \beta^j(t); t \in [0, T] \} \) on \( (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}) \), such that

\[
W(t) = \sum_{l=1}^\infty \sqrt{\alpha_l} \beta^l(t) e_l \quad \forall t \in [0, T] ,
\]

(2.13)

where \( \{ e_j \}_{j \geq 1} \) is an orthonormal basis of \( \mathcal{K} \) consisting of eigenfunctions of \( Q \), with eigenvalues \( \{ q_j \}_{j \geq 1} \subset \mathbb{R}^+ \). See e.g. [17, Chapter 4]. Sometimes, a Wiener process \( W \) with the covariance operator \( Q \) is called a \( Q \)-Wiener process.

For \( p \geq 1 \) and \( \mathcal{H} \) being a Hilbert space, consider the space \( M^p([0, T], \mathbb{F}; \mathcal{H}) := M^p(\Omega \times [0, T], \mathbb{F}; \mathcal{H}) \) of equivalence classes of \( \mathbb{F} \)-progressively measurable processes \( u : [0, T] \times \Omega \to \mathcal{H} \) such that

\[
\mathbb{E} \left[ \int_0^T \| u(t) \|_{\mathcal{H}}^p \, dt \right] < \infty .
\]

Let us denote by \( \mathcal{L}(\mathcal{K}, \mathcal{H}) \) the space of linear bounded operators from \( \mathcal{K} \) to \( \mathcal{H} \), and by \( S_2(\mathcal{K}, \mathcal{H}) \) the space of linear Hilbert-Schmidt operators from \( \mathcal{K} \) to \( \mathcal{H} \). We define the stochastic
integrate $\left\{ \int_0^t \varphi(s) \, dW(s) ; t \in [0,T] \right\}$ for any $\varphi \in M^2([0,T], F; S_2(\mathcal{K}, \mathcal{H}))$ as the continuous $\mathcal{H}$-valued $\mathbb{F}$-martingale, such that if $\varphi$ is a step process belonging to $M^2([0,T], F; \mathcal{L}(\mathcal{K}, \mathcal{H}))$, then

$$
\int_0^t \varphi(s) \, dW(s) = \sum_{m=1}^M \varphi(t_{m-1}) \left( W(t \wedge t_m) - W(t \wedge t_{m-1}) \right) \quad \forall t \in [0,T],
$$

see e.g. [17, Chapter 4]. The stochastic integral satisfies the Itô isometry (see [17, Proposition 4.5]), i.e., for each $\varphi \in M^2([0,T], F; S_2(\mathcal{K}, \mathcal{H}))$

$$(2.14) \quad \mathbb{E} \left[ \left\| \int_0^t \varphi(s) \, dW(s) \right\|_{\mathcal{H}}^2 \right] = \mathbb{E} \left[ \int_0^t \| \varphi(s) Q^{1/2} \|_{S_2(\mathcal{K}, \mathcal{H})}^2 \, ds \right] \quad \forall t \in [0,T].$$

We also recall that our Wiener process $W$ satisfies the following inequality

$$(2.15) \quad \mathbb{E} \left[ |W(t) - W(s)|_{\mathcal{K}}^{2n} \right] \leq C_n (t-s)^n (\text{Tr} Q)^n \quad \forall n \in \mathbb{N},$$

where for $n = 1$ we have equality and $C_n = 1$; see for instance [32, Corollary 1.1].

We summarize the assumptions needed below for $Q$ and $\mathcal{K}$, which will be fixed from now on.

1. $\mathcal{K} \subset W^{1,\infty}(D, \mathbb{R}^3) \cap W^{2,2}(D, \mathbb{R}^3)$ and the embedding is continuous,
2. $Q^{1/2} \in \mathcal{S}_1(\mathcal{K})$ is a symmetric, non-negative operator.

Notice that property (S1) implies

$$(2.16) \quad |f|_{W^{1,\infty}(D,\mathbb{R}^3) \cap W^{2,2}(D,\mathbb{R}^3)} \leq |f|_{\mathcal{K}}, \quad \forall f \in \mathcal{K}.$$

We recall the definition of a weak martingale solution to equation (1.5).

**Definition 2.1.** Let $T > 0$. A weak martingale solution $(\Omega, \mathcal{F}, \mathbb{P}, W, M)$ of problem (1.5) consists of

1. a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$, where $\mathcal{F}_t = \{ \mathcal{F}_t \}_{t \in [0,T]}$ is a filtration satisfying the usual conditions,
2. a $\mathcal{K}$-valued $\mathbb{Q}$-Wiener process $W = \{W(t)\}_{t \in [0,T]}$, and
3. a progressively measurable process $M : [0,T] \times \Omega \rightarrow \mathbb{L}^2(D)$ such that
   - (a) for $\mathbb{P}$-almost every $\omega \in \Omega$, $M(\cdot, \omega) \in C([0,T]; H^{-1,2})$.
   - (b) $\mathbb{E} \left[ \sup_{t \in [0,T]} |\nabla M(t, \cdot)|_{L^2}^2 \right] < \infty$, and the equality $|M(\cdot, \cdot)| = 1$ is satisfied Lebesgue almost everywhere.
   - (c) for every $\varphi \in C^\infty(D, \mathbb{R}^3)$ and every $t \geq 0$ the following equation is satisfied $\mathbb{P}$-almost surely,

$$
(M(t, \cdot), \varphi) - (M_0, \varphi) = \int_0^t \sum_{i=1}^n \int_D \left\langle \frac{\partial M}{\partial x_i}(s, x), \frac{\partial \varphi}{\partial x_i}(x) \times M(s, x) \right\rangle \, dx \, ds
$$

$$
- \alpha \int_0^t \sum_{i=1}^n \int_D \left\langle \frac{\partial M}{\partial x_i}(s, x), \frac{\partial (M \times \varphi)}{\partial x_i}(s, x) \times M(s, x) \right\rangle \, dx \, ds
$$

$$
+ \left\langle \left( \int_0^t \langle M(s) \times \circ dW(s), \varphi \rangle \right), W^{1,2} \times \mathcal{K}, \mathcal{L} \right\rangle \mathcal{L}^{1/2},
$$

$$(2.17)$$
where $\langle \cdot, \cdot \rangle_{(W^{1,2}), W^{1,2}}$ is a natural duality.

Here we understand the Stratonovich integral as a sum of the $(W^{1,2})^*$-valued Itô integral
\[ \int_0^t M(s) \times dW(s), \]
and the corresponding Itô correction term which in this case is equal to
\[ \frac{1}{2} \sum_{l=1}^{\infty} q_l \int_0^t \int_D \langle M(s, x), e_l \times (e_l \times \varphi(x)) \rangle \, dx \, ds, \]
where convergence of the series follows from assumptions $(S_1)$, $(S_2)$, and part (b) of the definition of the weak martingale solution. Indeed,
\[ \frac{1}{2} \sum_{l=1}^{\infty} q_l \int_0^t \int_D \langle M(s, x), e_l \times (e_l \times \varphi(x)) \rangle \, dx \, ds \leq \frac{t}{2} \sum_{l=1}^{\infty} q_l |e_l|^2 \|\varphi\|_{L^2} \leq C t \|\varphi\|_{L^2} \text{Tr} Q. \]

The main result of this paper is to show convergence of iterates of Algorithm 1.2 to weak martingale solution of problem 1.5, which is made precise in the following theorem.

**Theorem 2.1.** Let $D \subset \mathbb{R}^n$, $n = 2, 3$ be a polyhedral bounded domain, $T > 0$, $M_0 \in W^{1,2}$ with $|M_0| = 1$ Lebesgue almost everywhere. Choose a Hilbert space $\mathcal{K}$ and an operator $Q : \mathcal{K} \to \mathcal{K}$ that satisfies assumptions $(S_1)$ and $(S_2)$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space, and $W$ be a $\mathcal{K}$-valued $Q$-Wiener process. For every finite $(k, h) > 0$, let $T_h$ be a quasuniform triangulation of $D$, and $I_k$ be an equidistant mesh covering $[0, T]$. There exists a solution $\{M^j\}_{j=0}^T \subset L^2(\Omega; L^\infty(0, T; W^{1,2}(D, \mathbb{R}^3)))$ of Algorithm 1.2 that satisfies

(i) $|M^j(x_l)| = 1$ for all $l \in L$, and all $1 \leq j \leq J$,

(ii) $\mathbb{E} \left[ \sup_{1 \leq j \leq J} |\nabla M^j|^2_{L^2} + k \sum_{j=1}^J |M^{j+1/2} \times \Delta_h M^{j+1/2}|^2_h \right] \leq C_T$,

(iii) $\mathbb{E} \left[ \sum_{j=0}^{J-1} \left( |M^{j+1} - M^j|^2_h + \frac{1}{k} |M^{j+1} - M^j|^4_h + \|\nabla[M^{j+1} - M^j]\|^2_{L^2} \right) \right] \leq C_T$,

(iv) For each $j \in \{0, \ldots, J\}$, the map $M^j : \Omega \to W^{1,2}(D, \mathbb{R}^3)$ is $F_{t_j}$-measurable.

Let $\tilde{M}_{k,h} : D_T \to \mathbb{R}^3$ be the continuous process obtained from the iterates $\{M^j\}_{j=0}^T$ in (6.10) for $(k, h) > 0$. Then, there exist a filtered probability space $(\Omega', \mathcal{F}', \mathbb{P}')$, an $\mathbb{F}'$-adapted $\mathcal{K}$-valued $Q$-Wiener process $W'$, progressively measurable process $m : [0, T] \times \Omega' \to L^2(D)$ such that $(\Omega', \mathcal{F}', \mathbb{P}', W', m)$ is a weak martingale solution of problem (1.5) and a subsequence $\{\tilde{M}_{k,h}\}_{k,h}$ such that for any $\kappa \in [1, \infty)$, and all $\alpha \in (0, \frac{1}{2})$, $r \in (1, 4)$ such that $\alpha > \frac{1}{r}$, $\mathbb{P}'$-almost surely
\[ \tilde{M}_{k,h} \to m \quad \text{in} \quad L^2(0, T; L^\kappa) \cap W^{\alpha,r}(0, T; (W^{1,2})^*) \quad (k, h \to 0). \]

The remainder of this work is organised as follows. In Section 3, we gather definitions and auxiliary results of the theory of Sobolev-Slobodetski spaces. In Section 4, we derive stability properties for iterates from the practicable Algorithm 1.2, see Lemma 4.1 and Theorem 4.1. Those results provide the basis for the proof of our main result. Sections 5 and 6 are
concerned with the proof of convergence of iterates from Algorithm 1.2, and identification of the limit as weak martingale solution. In particular, existence and regularity properties of the process \( \mathbf{m} \) are addressed in Proposition 5.1, Corollary 5.1 (part (a) of the definition of weak martingale solution) and identities (5.5)-(5.8) (part (b) of the definition of weak martingale solution). Furthermore, convergence and identification of deterministic integrals (drift term) for \( k, h \to 0 \) are provided in identities (5.9), (5.10), (5.11), and Lemma 5.2. Section 6 identifies a filtered probability space, a Wiener process, and the limit of the stochastic (Stratonovich) integral for vanishing discretization parameters. Hence we are able to show part (c) of the definition of weak martingale solution and thus conclude the proof of Theorem 2.1.

3. Some auxiliary results on Sobolev-Slobodetski spaces

**Definition 3.1.** Let \( E \) be a Banach space, and \( T > 0 \).

i) Fractional Sobolev spaces are defined for \( 0 < s < 1, \ 1 \leq p < \infty \) by

\[
W^{s,p}(0,T;E) = \{ f \in L^p(0,T;E) : |f|_{W^{s,p}} < \infty \},
\]

where

\[
|f|_{W^{s,p}} = \left( \int_0^T \int_0^T \frac{|f(r) - f(t)|_E^p \ dr \ dt}{|r-t|^s} \right)^{\frac{1}{p}}.
\]

ii) Lipschitz spaces are defined for \( 0 < s < 1 \) by

\[
\text{Lip}^s([0,T];E) = \{ f \in L^\infty(0,T;E) : |f|_{\text{Lip}^s} < \infty \},
\]

where

\[
|f|_{\text{Lip}^s} = \text{esssup}_{r,t \in [0,T]} \frac{|f(r) - f(t)|_E}{|r-t|^s}.
\]

iii) Nikolskii spaces are defined for \( 0 < s < 1, \ 1 \leq p < \infty \) by

\[
N^{s,p}(0,T;E) = \{ f \in L^p(0,T;E) : |f|_{N^{s,p}} < \infty \},
\]

where

\[
|f|_{N^{s,p}} = \sup_{k>0} k^{-s}|f(\cdot + k) - f(\cdot)|_{L^p(-k,T-k;E)}.
\]

We use here a standard convention that for \( f \in L^p(0,T;E) \), we put \( f = 0 \) outside of the interval \((0,T)\).

The following properties are shown in [46].

(i) \( W^{s,p} \subseteq N^{s,p} \),

(ii) \( W^{s,p} \subseteq W^{r,p} \), as well as \( N^{s,p} \subseteq N^{r,p} \), for \( s \geq r \),

(iii) \( W^{s,p} \) and \( N^{s,p} \) are both included in \( W^{r,p} \) and \( N^{r,p} \), provided \( s > r \),

(iv) if \( s > \frac{1}{p} \), then both, \( W^{s,p} \) and \( N^{s,p} \) are included in \( \text{Lip}^{s-\frac{1}{p}} \), and are included in the set of continuous functions,

(v) if \( s - \frac{1}{p} \geq r - \frac{1}{q} \), then \( W^{s,p} \subseteq W^{r,q} \) and \( N^{s,p} \subseteq N^{r,q} \), provided \( 0 < r \leq s < 1 \), and \( 1 \leq p \leq q < \infty \).

Let \( E \) be a Banach space, and \( \mathcal{G} \in C([0,T];E) \) be piecewise affine on subintervals \([t_j, t_{j+1})\) of constant length \( k > 0 \) which cover \([0,T)\). The following criterion will be useful in the sequel, which may be considered as a generalization of [47, Lemma III.5.6].
Lemma 3.1. Assume that $k > 0$ and that $I_k = \{t_j\}_{j=0}^J$ is an equidistant mesh of size $k > 0$ covering $[0, T]$. Assume that $G \in C([0, T]; E)$ is such that for every $j \in \{0, \ldots, J-1\}$ the function

$$[t_j, t_{j+1}] \ni t \mapsto G(t) \in E$$

is affine. Assume that for some $p \geq 1$, $\alpha \in (0,1)$ and $C > 0$, and every $\ell \geq 1$,

$$k \sum_{j=0}^{J-\ell} |G(t_{j+\ell}) - G(t_j)|^p_E \leq C^p k^{\alpha \ell}.$$  

Then $G \in N^{\alpha,p}(0, T; E)$ and there exists a constant $C = C(T) > 0$ which does not depend on $k > 0$, such that

$$|G|_{N^{\alpha,p}(0, T; E)} \leq C.$$  

Proof. Note that $T = t_j = kJ$. We have to show that for some constant $C > 0$, independent of $k$,

$$\mathcal{X}_p(\delta) := \int_0^{T-\delta} |G(t + \delta) - G(t)|^p_E \ dt \leq C \delta^{\alpha p} \quad \forall \delta \in (0, T).$$  

For this purpose, we distinguish three cases.

i) $\delta \in (0, k)$. Then

$$\mathcal{X}_p(\delta) = \left( \sum_{j=0}^{J-2} \left[ \int_{t_j}^{t_{j+1} - \delta} + \int_{t_{j+1} - \delta}^{t_{j+1}} + \int_{t_{j-1}}^{t_{j+1} - \delta} \right] |G(t + \delta) - G(t)|^p_E \ dt \right).$$  

Since for each $j \in \{0, \ldots, J-1\}$, the function $G_{[t_j, t_{j+1}]}$ is affine, we infer that

$$|G(t) - G(s)|_E \leq \frac{|t-s|}{k} |G(t_{j+1}) - G(t_j)|_E, \quad \forall t, s \in [t_j, t_{j+1}].$$  

Let us fix $j \in \{0, \ldots, J-1\}$ and take $t \in [t_{j+1} - \delta, t_{j+1}] \subset [t_j, t_{j+1}]$. Then $|t_{j+1} - t| \leq \delta$. Moreover, $t + \delta \in [t_{j+1}, t_{j+1} + \delta] \subset [t_{j+1}, t_{j+2}]$ and $|(t + \delta) - t_{j+1}| \leq \delta$. Hence,

$$|G(t + \delta) - G(t)|_E \leq |G(t + \delta) - G(t_{j+1})|_E + |G(t_{j+1}) - G(t)|_E \leq \frac{\delta}{k} \left( |G(t_{j+2}) - G(t_{j+1})|_E + |G(t_{j+1}) - G(t_j)|_E \right).$$  

Accordingly, let now $t \in [t_{j+1} - \delta, t_{j+1}]$. Then also $t + \delta \in [t_j, t_{j+1}]$ and hence

$$|G(t + \delta) - G(t)|_E \leq \frac{\delta}{k} |G(t_{j+1}) - G(t_j)|_E.$$  

Consequently, by using equality (3.3), assumption (3.1) with $\ell = 1$ (so that $t_\ell = t_1 = k$) leads to

$$\mathcal{X}_p(\delta) \leq 2^p \left( \frac{\delta}{k} \right)^p k \sum_{j=0}^{J-1} |G(t_{j+1}) - G(t_j)|^p_E \leq 2^p C^p \left( \frac{\delta}{k} \right)^{\alpha p} \frac{\delta^{\alpha p}}{k^{\alpha p}},$$

which concludes the proof of (3.2).
Let $t \in [t_j, t_{j+1}] \cap [0, T - \delta]$. Then by the triangle inequality we have
\[
|G(t + \delta) - G(t)|_E 
\leq |G(t + \delta) - G(t + t_\ell)|_E + |G(t + t_\ell) - G(t_{j+\ell})|_E 
+ |G(t_{j+\ell}) - G(t_j)|_E + |G(t_j) - G(t)|_E
\] 
\[= I + II + III + IV.\]

We may proceed as in the first step to control the terms $I$, $II$, and $IV$; on using (3.1), we arrive at
\[
X_p(\delta) \leq 3 \times 4^{p-1} k \sum_{j=0}^{J-1} |G(t_{j+1}) - G(t_j)|_E^p + 4^{p-1} k \sum_{j=0}^{J-1} |G(t_{j+1}) - G(t_j)|_E^p 
\leq C^p 4^p (k^{op} + t_\ell^{op}) = (4C)^p k^{op} (1 + \ell^{op}) 
\leq (4C)^p k^{op} (1 + \ell)^{op} \leq (4C)^p k^{op} 2^{op} \ell^{op} \leq (2^{2+o} C)^p \delta^{op}.
\]

This concludes the proof. \qed

The following compactness results will be needed below; see e.g. [20] for related proofs.

**Lemma 3.2.** Assume that $B_0 \subset B \subset B_1$ are Banach spaces, $B_0$ and $B_1$ being reflexive. Assume that the embedding $B_0 \subset B$ is compact, $q \in (1, \infty)$, and $\alpha \in (0,1)$. Then the embedding
\[
L^q(0, T; B_0) \cap \mathcal{W}^{\alpha,q}(0, T; B_1) \hookrightarrow L^q(0, T; B)
\]
is compact.

**Lemma 3.3.** Assume that $X_0$, $X$ are Banach spaces such that the embedding $X_0 \subset X$ is compact. Assume that $q \in (1, \infty)$ and $0 < \alpha < \beta < 1$. Then the embedding
\[
\mathcal{W}^{\beta,q}(0, T; X_0) \subset \mathcal{W}^{\alpha,q}(0, T; X)
\]
is compact.

4. **Unconditional Stability of Algorithm 1.2**

Fix $h > 0$ and $k = \frac{T}{J}$, where $J \in \mathbb{N}^*$. The following lemma validates a discrete sphere constraint for $V_h$-valued iterates $\{\mathcal{M}^j\}_{j=1}^J$ of Algorithm 1.2.

**Lemma 4.1.** Suppose that the initial data $\mathcal{M}^0$ is such that $|\mathcal{M}^0(x_\ell)| = 1$ for all $\ell \in L$, and $|\nabla \mathcal{M}^0|_{L^2} \leq C$. Then the sequence $\{\mathcal{M}^j\}_{j=0}^J$ generated by Algorithm 1.2 satisfies properties (i)-(iv) formulated in Theorem 2.1.

**Proof of Lemma 4.1.** The proof will be divided into four steps.

**Step 1:** Solvability and proof of property (iv).

Existence of sequences $\{\mathcal{M}^j(\omega)\}_{j=1}^J \subset V_h$ that solve Algorithm 1.2 follows by induction from the Brouwer fixed point theorem. To justify this claim let us suppose that a sequence
By using the classical formula
\( (\Phi - M^j(\omega)) = \sum_{i=1}^{\infty} \sqrt{q_i} \Delta_j^i \mathcal{I} \left[ (\Phi \times e_i) \right] \)
(4.1)

By using the classical formula
\[ \langle a \times b, a \rangle = 0 \quad \forall a, b \in \mathbb{R}^3, \]
we infer that for every \( \Phi \in V_h \) such that \( |\Phi|_h \geq |M^j(\omega)|_h \)
(4.2)

Hence the Brouwer Theorem, see for instance [24, Corollary VI.1.1], implies the existence of \( \Phi^*(\omega) \in V_h \) such that \( F_j^* (\Phi^*) = 0 \) and \( |\Phi^*(\omega)|_h \leq |M^j(\omega)|_h \). Then \( M^{j+1}(\omega) := 2\Phi^*(\omega) - M^j(\omega) \) solves Algorithm 1.2. In fact, we have shown that for each \( M^j(\omega) \in V_h \) there exists \( M^{j+1}(\omega) \in V_h \) such that
\[ G_j (M^{j+1}, M^j, z) = 0, \]
where \( G_j : V_h \times V_h \times \mathcal{K} \to V_h \) is the function defined by the right hand side of (4.1), i.e.
\begin{align*}
(4.3) \quad G_j (\Phi, M^j, z) & = 2(\Phi - M^j) - \sum_{i=1}^{\infty} z^i \mathcal{I} \left[ (\Phi \times e_i) \right]. \\
& + k \mathcal{I} \left[ a \Phi \times \tilde{\Delta}_h (\Phi - M^j) \right] + \Phi \times \tilde{\Delta}_h (\Phi - M^j) \quad \forall \Phi \in V_h.
\end{align*}

However, if \( G_j (\Phi, M^j, z) = 0 \) for some \( (\Phi, M^j, z) \in V_h \times V_h \times \mathcal{K} \), then by inequality (4.2) we infer that \( |\Phi^*(\omega)|_h \leq |M^j(\omega)|_h \). Moreover, since the function \( F_j : V_h \to V_h \) is continuous, the set \( \Lambda (M^j, z) := \{ \Phi \in V_h : G_j (\Phi, M^j, z) = 0 \} \) is closed for all \( (M^j, z) \in V_h \times \mathcal{K} \). Hence we proved that the map
\[ \Lambda : V_h \times \mathcal{K} \ni (M^j, z) \mapsto \Lambda (M^j, z) \in \mathcal{P}(V_h), \]
where \( \mathcal{P}(V_h) \) denotes the set of all subsets of \( V_h \), is well defined and the values of \( \Lambda \) lie in closed and bounded subsets of \( V_h \). Moreover, since the function \( G_j \) is continuous, the set
\[ \text{graph}(\Lambda) = \{ (\Phi, M^j, z) \in V_h \times V_h \times \mathcal{K} : G_j (\Phi, M^j, z) = 0 \} \]
is closed. Hence, by applying Theorem 3.1 from [4] (which is a generalization of the Kuratowski and Ryll-Nardzewski celebrated Theorem about selectors [37]), we infer that there exists a universally and Borel measurable map \( \kappa_j : V_h \times \mathcal{K} \to V_h \) which is a selector of \( \Lambda \).

We define the sequence \( M^j(\omega), j = 0, \cdots, J \) by the following inductive formula
\[ M^{j+1}(\omega) = 2\kappa_j (M^j(\omega), \Delta_j W(\omega)) - M^j(\omega) \quad (\omega \in \Omega, \ j \in \{0, \cdots, J-1\}). \]
(4.5)

Now, in order to prove property (iv) we use the Induction and so can assume that \( j \in \{0, \cdots, J-1\} \) and the map \( M^j : \Omega \to W^{1,2}(D, \mathbb{R}^3) \) is \( F_{\ell_j} \)-measurable. Since the function \( \Delta_j W : \Omega \to \mathcal{K} \) is \( F_{\ell_{j+1}} \)-measurable, the claim follows from (4.5) and the measurability properties of \( \kappa_j \).
Step 2: Proof of assertion (i). By choosing \( \Phi = M^{j+1/2}(\cdot, x_\ell) \varphi_\ell \in V_h \) for \( \ell \in L \), where \( \varphi_\ell \) is the nodal basis function attached to \( x_\ell \), in Algorithm 1.2, the properties of the vector product yield the assertion (i).

Step 3: Proof of assertion (ii). We choose \( \Phi = -\tilde{\Delta}_h M^{j+1} \in V_h \) as a test function in (1.6) to find that
\[
(M^{j+1} - M^j, -\tilde{\Delta}_h M^{j+1})_h + \alpha k \left( M^{j+1/2} \times [M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}], -\tilde{\Delta}_h M^{j+1} \right)_h = \sum_{l=1}^{\infty} \sqrt{q_l} \Delta_j \beta^l \left( M^{j+1/2} \times e_l, -\tilde{\Delta}_h M^{j+1} \right)_h.
\]
Using the classical formula
\[
2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2,
\]
and the definition (2.9) of the operator \(-\tilde{\Delta}_h\), the first term above becomes
\[
(M^{j+1} - M^j, -\tilde{\Delta}_h M^{j+1})_h = \frac{1}{2} \left( |\nabla M^{j+1/2}|^2_{L^2} - |\nabla M^j|_{L^2}^2 + |\nabla (M^{j+1} - M^j)|^2_{L^2} \right).
\]
By the definition (2.5) of the scalar product \((\cdot, \cdot)_h\), and a classical formula on vector products in \(\mathbb{R}^3\), we get
\[
\left( M^{j+1/2} \times [M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}], -\tilde{\Delta}_h M^{j+1} \right)_h = |M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}|^2_{L^2}.
\]
Since \((a \times b, b) = 0\) for \(a, b \in \mathbb{R}^3\), the third term in (4.6) is equal to zero. Putting these identities together then yields
\[
\frac{1}{2} \left( |\nabla M^{j+1/2}|^2_{L^2} - |\nabla M^j|_{L^2}^2 + |\nabla (M^{j+1} - M^j)|^2_{L^2} \right) + \alpha k |M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}|^2_{L^2} = -\sum_{l=1}^{\infty} \left( M^{j+1/2} \times e_l, \tilde{\Delta}_h M^{j+1} \right)_h \sqrt{q_l} \Delta_j \beta^l.
\]
We proceed independently with the last term. When we want to apply the definition (2.9) of \(-\tilde{\Delta}_h M^{j+1}\) we encounter a difficulty since \(M^{j+1/2} \times e_l\) does not in general belong to the space \(V_h\).
\[
(M^{j+1/2} \times e_l, \tilde{\Delta}_h M^{j+1})_h = \left( T_h [M^{j+1/2} \times e_l], \tilde{\Delta}_h M^{j+1} \right)_h = -\left( (\nabla T_h [M^{j+1/2} \times e_l], \nabla M^{j+1} \right)_h
\]
\[
= \sum_{K \in T_h} \left( (\text{Id} - T_h) [M^{j+1/2} \times e_l], \nabla M^{j+1} \right)_L^{2(K)} \right)
\]
\[
= \frac{1}{2} (\nabla M^j \times e_l, \nabla M^{j+1}) - (M^{j+1/2} \times \nabla e_l, \nabla M^{j+1})
\]
where in the last equality we use the fact that \(\frac{1}{2} (\nabla M^{j+1} \times e_l, \nabla M^{j+1})\) is equal to zero.
We use the algebraic identity $\frac{1}{2}(a + b) = b + \frac{1}{2}(a - b)$ to restate each term in the sum as

\[
\left( \nabla \{ (\text{Id} - \mathcal{I}_h)[M^{j+1/2} \times e_l] \}, \nabla [M^{j+1} - M^j] \right) \|_{L^2(K)}
\]

(4.9) \hspace{1cm} + \left( \nabla \{ (\text{Id} - \mathcal{I}_h)[M^j \times e_l] \}, \nabla M^j \right) \|_{L^2(K)}

+ \frac{1}{2} \left( \nabla \{ (\text{Id} - \mathcal{I}_h)[(M^{j+1} - M^j) \times e_l] \}, \nabla M^j \right) \|_{L^2(K)} = I_K^{jl} + II_K^{jl} + III_K^{jl}.

For the first term, we use standard interpolation theory, and $\nabla^2 M^{j+1/2} \|_K = 0$ to conclude

\[
|I_K^{jl}| \leq C_h |\nabla^2[M^{j+1/2} \times e_l]|_{L^2(K)} |\nabla [M^{j+1} - M^j]|_{L^2(K)}
\]

\[
\leq C_h \left[ |M^{j+1/2} \times \nabla^2 e_l|_{L^2(K)} + |\nabla M^{j+1/2} \times \nabla e_l|_{L^2(K)} \right] |\nabla [M^{j+1} - M^j]|_{L^2(K)}
\]

\[
\leq C_h \left( 1 + |\nabla^2 M^{j+1/2}|_{L^2} \right) |\nabla [M^{j+1} - M^j]|_{L^2}.
\]

Since $|M^{j+1}|_{L^\infty} \leq 1$, and by inequality (2.16) $|e_l|_{W^{1,\infty}(D,\mathbb{R}^3)}|W^{2,2}(D,\mathbb{R}^3)} \leq 1$, we obtain the following upper bound for the sum in (4.8),

\[
\sum_{K \in \mathcal{T}_h} |I_K^{jl}| \leq C_h \sum_{K \in \mathcal{T}_h} \left[ 1 + |\nabla M^{j+1/2}|_{L^2(K)} \right] |\nabla [M^{j+1} - M^j]|_{L^2(K)}
\]

\[
\leq C_h \left( \sum_{K \in \mathcal{T}_h} \left[ 1 + |\nabla M^{j+1/2}|_{L^2(K)} \right]^2 \right)^{1/2} \left( \sum_{K \in \mathcal{T}_h} |\nabla [M^{j+1} - M^j]|_{L^2(K)}^2 \right)^{1/2}
\]

\[
\leq C_h \left( 1 + |\nabla^2 M^{j+1/2}|_{L^2} \right) |\nabla [M^{j+1} - M^j]|_{L^2}.
\]

Since the last term in (4.7) is the scalar product with a stochastic increment, we may bound the product of the leading term on the right-hand side of (4.8) with $\Delta_j \mathbf{W}$ by

\[
\sum_{l=1}^{\infty} \sqrt{q_l} \sum_{K \in \mathcal{T}_h} |I_K^{jl}| |\Delta_j \beta^l| \leq \frac{1}{16} |\nabla [M^{j+1} - M^j]|_{L^2}^2 + C_h^2 \left( 1 + |\nabla M^{j+1/2}|_{L^2}^2 \right) \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2.
\]

By the inverse estimate $|\nabla M^{j+1/2}|_{L^2} \leq C h^{-1} |M^{j+1/2}|_{L^2}$, in combination with assertion (i), we may hence conclude

\[
\sum_{l=1}^{\infty} \sqrt{q_l} \sum_{K \in \mathcal{T}_h} |I_K^{jl}| |\Delta_j \beta^l| \leq \frac{1}{16} |\nabla [M^{j+1} - M^j]|_{L^2}^2 + C \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2.
\]

For the second term in (4.9) holds $\mathbb{E} \left[ \sum_{l=1}^{\infty} \sqrt{q_l} \sum_{K \in \mathcal{T}_h} II_K^{jl} |\Delta_j \beta^l| \right] = 0$. The third term in (4.9) may be similarly controlled as the first one,

\[
|III_K^{jl}| \leq C_h |\nabla^2 [(M^{j+1} - M^j) \times e_l]|_{L^2(K)} |\nabla M^j|_{L^2(K)}
\]

(4.10) \hspace{1cm} + |(M^{j+1} - M^j) \times \nabla^2 e_l|_{L^2(K)} + |\nabla (M^{j+1} - M^j) \times \nabla e_l|_{L^2(K)} |\nabla M^j|_{L^2(K)}.$
We need to control $|M^{j+1} - M^j|_{L^2_h}$; for this purpose, choose $\Phi = M^{j+1} - M^j$ in (1.6). After absorbing terms, we arrive at

$$
\frac{1}{2}|M^{j+1} - M^j|_{L^2_h}^2 \leq Ck^2(\alpha^2|M^{j+1/2}|_{L^\infty} + 1)|M^{j+1/2} \times \Delta_h M^{j+1}|_{L^2_h}^2
$$

(4.11)

$$
+ \frac{1}{2} \left( \sum_{l=1}^{\infty} \sqrt{q_l}|M^{j+1/2} \times e_l|\Delta_j \beta^l| \right)^2.
$$

We may now come back to (4.10) and sum over all $K \in \mathcal{T}_h$.

$$
\sum_{l=1}^{\infty} \sqrt{q_l} \sum_{K \in \mathcal{T}_h} |III_K^l| \Delta_j \beta^l| \leq Ch \sum_{l=1}^{\infty} \sqrt{q_l} \left( \sum_{K \in \mathcal{T}_h} |M^{j+1} - M^j|_{L^2(K)}^2 + |\nabla [M^{j+1} - M^j]|_{L^2(K)}^2 \right)^{1/2}
$$

$$
\times \left( \sum_{K \in \mathcal{T}_h} |\nabla M^j|_{L^2(K)}^2 \right)^{1/2} |\Delta_j \beta^l|
$$

$$
\leq Ch \left( |M^{j+1} - M^j|_{L^2} + |\nabla [M^{j+1} - M^j]|_{L^2} \right) \left| \nabla M^j \right|_{L^2} \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|
$$

$$
\leq \frac{1}{16} |\nabla [M^{j+1} - M^j]|_{L^2}^2 + C \left( k^2 |M^{j+1/2} \times \Delta_h M^{j+1}|_{L^2_h}^2 + \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2 \right)
$$

$$
+C h^2 \left| \nabla M^j \right|_{L^2}^2 \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2,
$$

thanks to (4.11). Now, summing up over $j = 0, \cdots, m - 1$, taking supremum over $m \leq J - 1$, and using an inverse estimate leads to

$$
\mathbb{E} \left[ \sup_{m \leq J - 1} \sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_l} \Delta_j \beta^l \sum_{K \in \mathcal{T}_h} III_K^l \right]
$$

(4.12)

$$
\leq \frac{1}{8} \mathbb{E} \left[ \sum_{j=0}^{J-1} |\nabla (M^{j+1} - M^j)|_{L^2}^2 \right] + C k^2 \mathbb{E} \left[ \sum_{j=0}^{J-1} |M^{j+1/2} \times \Delta_h M^{j+1}|_{L^2_h}^2 \right] + C t \text{Tr} Q^2 \Delta h^2.
$$

For the second term on the right-hand side of (4.8), we use the fact that $\langle a \times b, a \rangle = 0$, and hence obtain

$$
\sum_{l=1}^{\infty} \sqrt{q_l} \left( \frac{1}{2} \left| \left( \nabla M^j \times e_l, \nabla [M^{j+1} - M^j] \right) + \left( \nabla M^j \times e_l, \nabla M^j \right) \right| \Delta_j \beta^l \right)
$$

(4.13)

$$
\leq \frac{1}{16} |\nabla [M^{j+1} - M^j]|_{L^2}^2 + C |\nabla M^j|_{L^2}^2 \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2.
$$
We employ the algebraic identity \( \frac{1}{2}(a + b) = \frac{1}{2}(a - b) + b \) to restate the last term in (4.8) in the way

\[
I^{jl} := \left(M^{j+1/2} \times \nabla e_{t}, \nabla M^{j+1}\right)
= \frac{1}{2}\left([M^{j+1} - M^{j}] \times \nabla e_{t}, \nabla M^{j+1}\right) + (M^{j} \times \nabla e_{t}, \nabla M^{j+1})
\]

(4.14) \quad = \frac{1}{2}\left([M^{j+1} - M^{j}] \times \nabla e_{t}, \nabla[M^{j+1} - M^{j}]\right) + \frac{1}{2}\left([M^{j+1} - M^{j}] \times \nabla e_{t}, \nabla M^{j}\right)
+ (M^{j} \times \nabla e_{t}, \nabla[M^{j+1} - M^{j}]) + (M^{j} \times \nabla e_{t}, \nabla M^{j})
\quad j = 0, \ldots, J - 1, l \in \mathbb{N}.

Then, by (2.7),

\[
\sum_{l=1}^{\infty} \sqrt{q_{l}} I^{jl} \Delta_{j} \beta^{l} \leq \frac{1}{16} \left| \nabla [M^{j+1} - M^{j}] \right|^{2}_{L^{2}} + \frac{1}{16} \left| M^{j+1} - M^{j} \right|^{2}_{h} + \sum_{l=1}^{\infty} \sqrt{q_{l}} \left( M^{j} \times \nabla e_{t}, \nabla M^{j} \right) \Delta_{j} \beta^{l}
+ C \int_{1}^{\infty} \sqrt{q_{l}} \left| \Delta_{l} \beta^{l} \right|^{2}_{L^{2}} \left( \left| M^{j+1} - M^{j} \right|^{2}_{L^{2}} + \left| M^{j} \right|^{2}_{L^{2}} + \left| \nabla M^{j} \right|^{2}_{L^{2}} \right).
\]

(4.15)

We employ (4.11) to control \( \frac{1}{16} \left| M^{j+1} - M^{j} \right|^{2}_{h} \). Summing up over \( j = 0, \ldots, m - 1, m \leq J \) in (4.15) we use Lemma 4.1(i) to find

\[
\sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_{l}} I^{jl} \Delta_{j} \beta^{l} \leq \frac{1}{16} \left[ \sum_{j=0}^{m-1} \left| \nabla[M^{j+1} - M^{j}] \right|^{2}_{L^{2}} \right]
+ \frac{Ck}{16} \left[ \sum_{j=0}^{m-1} \left| M^{j+1/2} \times \Delta_{h} M^{j+1} \right|^{2}_{h} \right]
+ C \left[ \sum_{j=0}^{m-1} \left| \sum_{l=1}^{\infty} \sqrt{q_{l}} \left| \Delta_{l} \beta^{l} \right|^{2}_{L^{2}} \right] \left( 6 + \left| \nabla M^{j} \right|^{2}_{L^{2}} \right) \right]
+ \left[ \sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_{l}} \left( M^{j} \times \nabla e_{t}, \nabla M^{j} \right) \Delta_{j} \beta^{l} \right]
\]

Taking supremum over \( m \leq J \) and expectation we deduce that

(4.16) \quad \mathbb{E} \left[ \sup_{m \leq J} \sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_{l}} I^{jl} \Delta_{j} \beta^{l} \right] \leq \frac{1}{16} \mathbb{E} \left[ \sum_{j=0}^{J-1} \left| \nabla[M^{j+1} - M^{j}] \right|^{2}_{L^{2}} \right]
+ \frac{Ck}{16} \mathbb{E} \left[ \sum_{j=0}^{J-1} \left| M^{j+1/2} \times \Delta_{h} M^{j+1} \right|^{2}_{h} \right]
+ C \mathbb{E} \left[ \sum_{j=0}^{J-1} \left| \sum_{l=1}^{\infty} \sqrt{q_{l}} \left| \Delta_{l} \beta^{l} \right|^{2}_{L^{2}} \right] \left( 6 + \left| \nabla M^{j} \right|^{2}_{L^{2}} \right) \right]
+ C \mathbb{E} \left[ \sup_{m \leq J} \sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_{l}} \left( M^{j} \times \nabla e_{t}, \nabla M^{j} \right) \Delta_{j} \beta^{l} \right].
For the last term, we use Cauchy-Schwarz and Burkholder-Davis-Gundy inequalities (see [9, Theorem 2.4]) to estimate it as follows,

\[
\mathbb{E} \left[ \sup_{m \leq J} \sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_l} \langle M^j \times \nabla e_l, \nabla M^j \rangle \Delta_j \beta^l \right] \leq \left( \mathbb{E} \left[ \sup_{m \leq J} \sum_{j=0}^{m-1} \sum_{l=1}^{\infty} \sqrt{q_l} \langle M^j \times \nabla e_l, \nabla M^j \rangle^2 \right] \right)^{\frac{1}{2}} \\
\leq C \left( \mathbb{E} \left[ k \sum_{j=0}^{J-1} \sum_{l=1}^{\infty} q_l \langle M^j \times \nabla e_l, \nabla M^j \rangle^2 \right] \right)^{\frac{1}{2}} \\
\leq C \left( \text{Tr} Q \mathbb{E} \left[ k \sum_{j=0}^{J-1} |\nabla M^j|^2 \right] \right)^{\frac{1}{2}} \\
(4.17) \\
\leq \frac{k}{16} \mathbb{E} \left[ \sum_{j=0}^{J-1} |\nabla M^j|^2 \right] + C \text{Tr} Q.
\]

By summing over iteration steps \( j = 0, \ldots, m - 1 \) in (4.7), then taking the supremum over \( m \leq J \) and the expectation, using (4.11), (4.12), (4.13), (4.16), (4.17), and absorbing terms for \( k > 0 \) sufficiently small yields

\[
\frac{1}{2} \mathbb{E} \left[ \sup_{m \leq J} |\nabla M^m|^2 \right] - |\nabla M^0|^2 + \frac{1}{4} \mathbb{E} \left[ \sum_{j=0}^{J-1} |\nabla[M^{j+1} - M^j]|^2 \right] \\
+ \frac{\alpha}{2} \mathbb{E} \left[ k \sum_{j=0}^{J-1} |M^{j+1/2} \times \tilde{\Delta}_h M^{j+1/2}|^2 \right] \\
\leq C \mathbb{E} \left[ \sum_{j=0}^{J-1} \left( |\nabla M^j|^2 + 1 \right) \right] \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2 .
\]

Because of

\[
\mathbb{E} \left[ |\nabla M^j|^2 \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2 \right] = \mathbb{E} \left[ |\nabla M^j|^2 \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2 |F_{t_j} \right] \\
= \mathbb{E} \left[ |\nabla M^j|^2 \mathbb{E} \left[ \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta^l|^2 |F_{t_j} \right] \right] \\
\leq C k \text{Tr} Q^{\frac{3}{2}} \mathbb{E} \left[ |\nabla M^j|^2 \right],
\]

the discrete Gronwall’s inequality then leads to

\[
\frac{1}{2} \mathbb{E} \left[ \sup_{m \leq J} |\nabla M^m|^2 \right] + \frac{1}{4} \mathbb{E} \left[ \sum_{j=0}^{J-1} |\nabla[M^{j+1} - M^j]|^2 \right] \\
+ \frac{\alpha}{2} \mathbb{E} \left[ k \sum_{j=0}^{J-1} |M^{j+1/2} \times \tilde{\Delta}_h M^{j+1/2}|^2 \right] \\
\leq \left( \mathbb{E} \left[ |\nabla M^0|^2 \right] + C \text{Tr} Q^{\frac{3}{2}} \right) \exp \left( C \{1 + \text{Tr} Q^{\frac{3}{2}} \} t_J \right) .
\]

(4.18)

which is assertion (ii). This completes Step 3.
Step 4: Proof of assertion (iii). Assertion (iii) consists of three inequalities, where the first follows from (4.11), (4.18), and the third results from (4.18). To show the second inequality, we multiply (4.11) by \(|M_j|_2^2\)

\[
\frac{1}{2}|M_j|_2^4 \leq |M_j|^2 \mathcal{L}_k^2 (\alpha^2|M_j|_2^2 + 1)|M_j|_2^2 \times \Delta h M_j^2
\]

\[
+ \frac{1}{2}|M_j|_2^2 \left( \sum_{l=1}^{\infty} \sqrt{q_l} |M_j|_2^2 \times e_l \right)^2 .
\]

Because of part (i) in Lemma 4.1, we obtain \(|M_j|_2^2 \leq 4\) and together with the Young inequality we find

\[
\frac{1}{2}|M_j|_2^4 \leq Ck^2 (\alpha^2|M_j|_2^2 + 1)|M_j|_2^2 \times \Delta h M_j^2
\]

\[
+ \delta |M_j|_2^4 + C \left( \sum_{l=1}^{\infty} \sqrt{q_l} |M_j|_2^2 \times e_l \right) \leq \Delta h M_j^2
\]

with \(\delta < \frac{1}{2}\).

Thanks to \(\mathbb{E}[\Delta_j \beta^4] \leq Ck^2\), assumption \((S_2)\), and (4.18), summation over all indices \(j = 0, \ldots, J - 1\) then implies (iii).

This completes the proof of Lemma 4.1. □

For the following, we put \(\Phi = f_i \varphi_i\) (1 \(\leq i \leq n\) in (1.6), where \(f_i \in \mathbb{R}^n\) is the standard vector. By using \(\frac{1}{2}(a + b) = b + \frac{1}{2}(a - b)\), we conclude that (1.6) takes the form

\[
M_j^{j+1} = M_j^j + k \mathcal{I}_h \left[ \alpha M_j^{j+\frac{1}{2}} \times (M_j^{j+\frac{1}{2}} \times \Delta h M_j^{j+1}) - M_j^{j+\frac{1}{2}} \times \Delta h M_j^{j+1} \right]
\]

\[
= \sum_{l=1}^{\infty} \sqrt{q_l} \Delta_j \beta^l \mathcal{I}_h \left[ M_j^j \times e_l \right]
\]

\[
= \sum_{l=1}^{\infty} \sqrt{q_l} \Delta_j \beta^l \mathcal{I}_h \left[ M_j^j \times e_l \right] + \sum_{l=1}^{\infty} \frac{k}{2} q_l \mathcal{I}_h \left[ M_j^{j+\frac{1}{2}} \times e_l \right] + A^j_i
\]

(4.19)

where we use (1.6) to set

\[
A^j_i := \frac{1}{2} \sum_{l=1}^{\infty} q_l (|\Delta_j \beta^l|^2 - k) \mathcal{I}_h \left[ M_j^{j+\frac{1}{2}} \times e_l \right] \times e_l
\]

(4.20)

\[
- \frac{k}{2} \sum_{l=1}^{\infty} \sqrt{q_l} \Delta_j \beta^l \mathcal{I}_h \left[ \left( \alpha M_j^{j+\frac{1}{2}} \times (M_j^{j+\frac{1}{2}} \times \Delta h M_j^{j+1}) - M_j^{j+\frac{1}{2}} \times \Delta h M_j^{j+1} \right) \times e_l \right]
\]

\[
+ \frac{1}{2} \sum_{m_1 \neq m_2} \sqrt{q_{m_1} q_{m_2}} (\Delta_j \beta_{m_1} \Delta_j \beta_{m_2}) \mathcal{I}_h \left[ M_j^{j+\frac{1}{2}} \times e_{m_1} \right] \times e_{m_2}
\]

\[
= : \frac{1}{2} A^j_1 - kA^j_2 + \frac{1}{2} A^j_4 (j = 0, \ldots, J - 1).
\]

We show the following auxiliary result.
Lemma 4.2. In the framework above the following limits holds

\[ \mathbb{E}\left[ \left| \sum_{j=0}^{J-1} A_j^i \right|_{L^2} \right] \to 0, \]
\[ \mathbb{E}\left[ |A_j^i|_{L^\infty} \right] \to 0, \]
\[ \mathbb{E}\left[ k \sum_{j=0}^{J-1} |A_j^2|_{L^2} \right] \to 0, \]
\[ \mathbb{E}\left[ \left| \sum_{j=0}^{J-1} A_j^4 \right|_{L^2} \right] \to 0 \quad (k \to 0). \]

(4.21)

Proof. Step 1. The sequence \( \{A_j^1\}_{j=0}^{J-1} \).

Consider the sequence \( \{A_j^3\}_{j=0}^{J-1} \), where

\[ A_j^3 = \frac{1}{2} \sum_{l=1}^{\infty} q_l (|\Delta_j \beta|^2 - k) I_h \left[ (M_j^2 \times e_l) \times e_l \right] \quad (j = 0, \ldots, J - 1). \]

Let us show that

\[ \mathbb{E}\left[ \left| \sum_{j=0}^{J-1} A_j^3 \right|_{L^2} \right] \to 0. \]

Since \( \mathbb{E}\left[ \left| \sum_{j=0}^{J-1} A_j^3 \right|_{L^2} \right] \leq \sqrt{\mathbb{E}\left[ \left( \sum_{j=0}^{J-1} A_j^3 \right)^2 \right]} \) it is sufficient to show that

\[ \lim_{k \to 0} \mathbb{E}\left[ \left| \sum_{j=0}^{J-1} A_j^3 \right|_{L^2}^2 \right] = 0. \]

Since \( A_j^3 \) is \( \mathcal{F}_{t_m} \)-measurable for \( l < m \), and \( A_j^m \) is \( \mathcal{F}_{t_m} \)-independent and satisfies \( \mathbb{E}[A_j^m] = 0 \), we infer that the finite sequence \( \{A_j^3\}_{j=0}^{J-1} \) is an \( L^2 \)-valued martingale difference and therefore

\[ \mathbb{E}\left[ \left( \sum_{j=0}^{J-1} A_j^3 \right)^2 \right] = \sum_{j=0}^{J-1} \mathbb{E}\left[ |A_j^3|_{L^2}^2 \right] \leq C k^2 J = kCT \to 0 \quad (k \to 0). \]

Thus, in order to prove the first claim, it is sufficient to prove that

\[ \mathbb{E}\left[ \left| \sum_{j=0}^{J-1} A_j^4 \right|_{L^2} \right] \to 0 \quad (k \to 0). \]

Note that

\[ A_j^4 - A_j^3 = \frac{1}{2} \sum_{l=1}^{\infty} q_l (|\Delta_j \beta|^2 - k) I_h \left[ (M_j^+ - M_j^2) \cdot e_l \right] \times e_l \]
\[ = \frac{1}{4} \sum_{l=1}^{\infty} q_l (|\Delta_j \beta|^2 - k) I_h \left[ (M_j^+ - M_j^2) \times e_l \right] \times e_l. \]
Consequently, we deduce that

\[ (4.22) \quad \mathbb{E} \left[ \sum_{j=0}^{J-1} |A_j^1 - A_3^1|_{L^2} \right] \leq \sum_{j=0}^{J-1} \frac{k}{4} \sum_{l=1}^{\infty} q_l \mathbb{E} \left[ \left| \frac{\Delta_j \beta_l}{k} - 1 \right| \left| (M^{j+1} - M^j) \times e_l \right|_{L^2} \right] \]

\[ \leq \sum_{j=0}^{J-1} \frac{k}{4} \sum_{l=1}^{\infty} q_l \left( \mathbb{E} \left[ \left| \frac{\Delta_j \beta_l}{k} - 1 \right|^2 \right] \right)^{1/2} \left( \mathbb{E} \left[ |M^{j+1} - M^j|_{L^2}^2 \right] \right)^{1/2} \]

\[ \leq \sum_{j=0}^{J-1} \frac{k}{2} \sum_{l=1}^{\infty} q_l \left( \mathbb{E} \left[ |M^{j+1} - M^j|_{L^2}^2 \right] \right)^{1/2}, \]

because of Cauchy-Schwarz inequality. We use Lemma 4.1, (iii) and inequality (2.7) to further conclude

\[ \ldots \leq \frac{k}{2} \text{Tr} Q \sqrt{J \sum_{j=0}^{J-1} \left( \mathbb{E} \left[ |M^{j+1} - M^j|_{L^2}^2 \right] \right)^{1/2}^2} \]

\[ \leq \frac{k}{2} \text{Tr} Q \sqrt{J \sum_{j=0}^{J-1} \mathbb{E} \left[ |M^{j+1} - M^j|_{L^2}^2 \right]} \]

\[ \leq \frac{k}{2} \text{Tr} Q \sqrt{J C_T} = \text{Tr} Q \sqrt{\frac{kTC_T}{2}} \to 0. \]

Because of Lemma 4.1(i) and the identity

\[ (4.23) \quad \mathbb{E} \left[ (|\Delta_j \beta_l|^2 - k)^2 \right] = 2k^2 \]

we deduce for \( j = 0, \ldots, J - 1 \) the second identity in (4.21),

\[ \mathbb{E} \left[ |A_j^1|_{L^\infty} \right] = \mathbb{E} \left[ \sum_{l=1}^{\infty} q_l (|\Delta_j \beta_l|^2 - k) \mathcal{I}_h [(M^{j+\frac{1}{2}} \times e_l) \times e_l] \right]_{L^\infty} \]

\[ \leq \sum_{l=1}^{\infty} q_l \left( \mathbb{E} \left[ (|\Delta_j \beta_l|^2 - k)^2 \right] \right)^{1/2} \left( \mathbb{E} \left[ \mathcal{I}_h [(M^{j+\frac{1}{2}} \times e_l) \times e_l]^2 \right] \right)^{1/2} \]

\[ \leq Ck \text{ Tr } Q. \]
Step 2: The sequence \( \{A^j_2\}_{j=0}^{J-1} \). Similarly to the above we have for \( j = 0, \ldots, J - 1 \),

\[
\mathbb{E}[|A^j_2|_{L^2}] \leq \alpha \sum_{l=1}^{\infty} \sqrt{q_l} \mathbb{E}
\left[|\Delta_j \beta^l| \left| \mathcal{I}_h [M^{j+\frac{1}{2}} \times (\tilde{\Delta}_h M^{j+1})] \right|_{L^2} \right] \\
+ \sum_{l=1}^{\infty} \sqrt{q_l} \mathbb{E}
\left[|\Delta_j \beta^l| \left| \mathcal{I}_h [M^{j+\frac{1}{2}} \times \tilde{\Delta}_h M^{j+1}] \right|_{L^2} \right]
\]

\[(4.25) \leq (\alpha + 1) \sum_{l=1}^{\infty} \sqrt{q_l} \left( \mathbb{E}[|\Delta_j \beta^l|^2] \right)^{1/2} \left( \mathbb{E}[|\mathcal{I}_h [M^{j+\frac{1}{2}} \times \tilde{\Delta}_h M^{j+1}]|_{L^2}^2] \right)^{1/2}
\]

\[
\leq C(\alpha + 1) \text{Tr} Q \left( k \mathbb{E}[|M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}|_h^2] \right)^{1/2}.
\]

Hence,

\[
\mathbb{E} \left[ k \sum_{j=0}^{J-1} |A^j_2|_{L^2} \right] = k \sum_{j=0}^{J-1} \mathbb{E}[|A^j_2|_{L^2}]
\]

\[(4.26) \leq C \text{Tr} Q k(\alpha + 1) \sum_{j=0}^{J-1} \left( k \mathbb{E}[|M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}|_h^2] \right)^{1/2}
\]

\[
\leq C \text{Tr} Q k(\alpha + 1) J^{1/2} \left( k \sum_{j=0}^{J-1} \mathbb{E}[|M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}|_h^2] \right)^{1/2}
\]

\[
= C \text{Tr} Q k^{1/2} T^{1/2} (\alpha + 1) \left( k \sum_{j=0}^{J-1} \mathbb{E}[|M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}|_h^2] \right)^{1/2}.
\]

Therefore, thanks to (2.7), and assertion (ii) of Lemma 4.1, this implies

\[
\mathbb{E} \left[ k \sum_{j=0}^{J-1} |A^j_2|_{L^2} \right] \to 0 \quad (k \to 0).
\]

Step 3: The sequence \( \{A^j_j\}_{j=0}^{J-1} \). The proof is similar to the Step 1 and left to the reader.

As a consequence of the three steps the assertion of Lemma 4.2 is proved. \( \square \)

The following result is a sharpened version of Lemma 4.1, (iii)_{1.2}; its proof uses the reformulation (4.19) of (1.6), and Lemma 4.1, (iii)_{3}, in particular.

**Lemma 4.3.** For every \( \delta > 0 \), there exists a constant \( C_\delta > 0 \) such that for every \( \ell \geq 1 \), the following inequality holds

\[
\mathbb{E} \left[ k \sum_{j=0}^{J-\ell} \left( t_\ell M^{j+\ell} - M^j_h \right)^2 + t_\delta^j \left( M^{j+\ell} - M^j_h \right) \right] \leq C_\delta t_\delta^2.
\]
Proof. Step 1: First inequality. Summation in (4.19), then testing with \( \Phi = M^{j+\ell} - M^j \), and using Lemma 4.1 (i) leads to

\[
\frac{1}{2}|M^{j+\ell} - M^j|_h^2 \leq \left( k \sum_{r=1}^{\ell-1} (\alpha + 1) |M^{j+r+1/2} \times \Delta_h M^{j+r+1}|_h^2 \right)^2
\]

(4.27)

\[
+ \left( k \sum_{r=1}^{\ell-1} \infty q_r (M^{j+r+1/2} \times e) \times e|_h \right)^2 + I + II,
\]

where

\[
I = C \left| \sum_{i=j}^{j+\ell-1} \infty \sqrt{q_i} I_h [M^i \times e] \Delta_i \beta |_h^2,
\]

(4.28)

\[
II = C \left( \sum_{r=1}^{\ell-1} |A^{i+r}|_h \right)^2 + C \left( \sum_{r=1}^{\ell-1} |A^{4j+r}|_h \right)^2 + C \sum_{r=1}^{\ell-1} |A^{2j+r}|_h.
\]

(4.29)

We will separately deal with the terms \( I \) and \( II \).

Since \( M^j \) is \( F_{t_j} \)-measurable and \( \Delta_j \mathcal{W} \) is \( F_{t_j} \)-independent, by using Lemma 4.1 (i), and properties of conditional expectation, we infer that

\[
\mathbb{E}[I] = \mathbb{E} \left[ \left| \sum_{i=j}^{j+\ell-1} \infty \sqrt{q_i} I_h [M^i \times e] \Delta_i \beta |_h^2 \right]^2 \right] \leq \sum_{i=j}^{j+\ell-1} \mathbb{E} \left[ |I_h [M^i]|_h^2 \right] \mathbb{E} \left[ |\Delta_i |^2 \right] \leq Ct_\ell.
\]

By (4.23), (4.24), and Lemma 4.1 (i), the first part of \( II \) is bounded as follows,

\[
(4.30) \quad \mathbb{E} \left[ \left( \sum_{r=1}^{\ell-1} |A^{i+r}|_h \right)^2 \right] + \mathbb{E} \left[ \left( \sum_{r=1}^{\ell-1} |A^{4j+r}|_h \right)^2 \right] \leq \ell \mathbb{E} \left[ \sum_{r=1}^{\ell-1} |A^i|_h^2 \right] + \ell \mathbb{E} \left[ \sum_{r=1}^{\ell-1} |A^{4j+r}|_h^2 \right] \leq Ct^2_\ell.
\]

Similar to (4.26), the remaining part of \( II \) may be controlled by

\[
\mathbb{E} \left[ \sum_{r=1}^{\ell-1} |A^{2j+r}|_h \right] \leq Ck^{1/2} \sqrt{t_\ell}.
\]

We may use these bounds in (4.27): summing up over \( j = 1, \ldots, J - \ell \), multiplying by \( k \), and taking expectations then yields to

(4.31) \quad \mathbb{E} \left[ k \sum_{j=1}^{J-\ell} |M^{j+\ell} - M^j|_h^2 \right] \leq Ct_J [(\alpha^2 + 1)t_\ell + 1]t_\ell.$
Step 2: Second inequality. Multiply (4.27) with \( |M^{i+\ell} - M^j|_h^2 \) and use Lemma 4.1(i), together with Young’s inequality to find

\[
\frac{1}{2^{\frac{3}{2}}} |M^{i+\ell} - M^j|_h^4 \leq 4 \left( k \sum_{r=1}^{\ell-1} (\alpha + 1) |M^{i+r+1/2} \times \tilde{\Delta}_h M^{i+r+1}|_h^2 \right)^2 \\
+ 4 \left( k \sum_{r=1}^{\ell-1} \sum_{l=1}^{\infty} q_l |(M^{i+r+1/2} \times e_{il}) \times e_{il}|_h^2 \right)^2 + I' + II' + III',
\]

(4.32)

where

\[
I' = \left| \sum_{i=j}^{j+\ell-1} \mathcal{I}_h \left[ \sum_{l=1}^{\infty} \sqrt{q_l} M^i \times e_{il} \right] \Delta_i \beta_l^4 \right|_h^4,
\]
\[
II' = C \left( \sum_{r=1}^{\ell} |A_{i+r}^j|_h \right)^2 + C \left( \sum_{r=1}^{\ell} |A_{i+r}^j|_h \right)^2,
\]
\[
III' = C_\sigma \left( \sum_{r=1}^{\ell} |A_{i+r}^j|_h \right)^\sigma \text{ for every } 1 \leq \sigma < 2.
\]

Again, we independently consider terms \( I' \) through \( III' \).

Note that by Lemma 4.1(i) there exists a constant \( C > 0 \) such that for all \( i = 0, \ldots, J-1 \),

\[
|\mathcal{I}_h [M^i \times e_{il}]^2 | \leq C.
\]

By using the discrete Burkholder’s inequality [13], we may then infer that

\[
\mathbb{E}[I'] = \mathbb{E} \left[ \left| \sum_{i=j}^{j+\ell-1} \sum_{l=1}^{\infty} \sqrt{q_l} \mathcal{I}_h [M^i \times e_{il}] \Delta_i \beta_l^4 \right|_h^4 \right] \\
\leq C \mathbb{E} \left[ \left( \sum_{i=j}^{j+\ell-1} \sum_{l=1}^{\infty} \sqrt{q_l} \mathbb{E} \left[ \mathcal{I}_h (M^i \times e_{il}) \Delta_i \beta_l^4 \right]_h^2 \right)^2 \right] \\
= C \mathbb{E} \left[ \left( \sum_{i=j}^{j+\ell-1} \sum_{l=1}^{\infty} q_l \mathbb{E} \left[ \mathcal{I}_h (M^i \times e_{il})_h^2 \right] (t_{i+1} - t_i) \right)^2 \right] \leq C Tr Q k t_\ell.
\]

Because of (4.30), it only remains to prove the following estimate,

\[
\mathbb{E} \left[ \left( k \sum_{r=1}^{\ell} \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta_l^\sigma| |M^{j+\frac{1}{2}} \times \tilde{\Delta}_h M^{j+1}|_h^\sigma \right)^\sigma \right] \\
\leq k^\sigma \mathbb{E} \left[ \left( \sum_{r=1}^{\ell} \left( \sum_{l=1}^{\infty} \sqrt{q_l} |\Delta_j \beta_l^\sigma| \right) |M^{j+\frac{1}{2}} \times \tilde{\Delta}_h M^{j+1}|_h^\sigma \right)^\sigma \right] \left( \sum_{r=1}^{\ell} \right)^{\sigma-1} \\
= k^{\frac{\sigma}{2}} t_\ell^{\sigma-1} k^{\frac{\sigma}{2}} A.
\]
We use Young’s inequality, inequality (2.15) and Lemma 4.1 (ii) to estimate from above $k^{2/5}A$ in the following way,

$$k^{2/5}A \leq \left( \mathbb{E}\left[ \sum_{i=1}^{\infty} \left( \sum_{l=1}^{\ell} \sqrt{q_l} \left| \Delta_j \beta_l^i \right| \right)^{2-\sigma} \right] \right)^{\frac{2-\sigma}{\sigma}} \left( \mathbb{E}\left[ k \sum_{i=1}^{\ell} \left\| \mathcal{M}^{i+1/2} \times \Delta_h \mathcal{M}^{i+1} \right\|_h^2 \right] \right)^{\frac{\sigma}{2}} \leq C_\sigma \text{Tr} Q \left( k^{\frac{2-\sigma}{\sigma}} \right) \leq C_\sigma \text{Tr} Q t^{\frac{2-\sigma}{\sigma}} k^{\sigma-1}.$$ 

Putting things together in (4.32), and using inequality $k^{2/5}t^{\sigma}_\ell \leq t^{\sigma}_\ell$ then leads to

$$\mathbb{E}\left[ \sum_{j=0}^{J-\ell} \left| \mathcal{M}^{j+\ell} - \mathcal{M}^j \right|_h^4 \right] \leq C(\alpha^2 + 1)t^{\sigma}_\ell + C_\sigma \text{Tr} Q t^{\sigma}_\ell \quad (1 \leq \sigma < 2).$$

The results (4.31) and (4.33) yield the assertion of the lemma.

**Remark 4.1.** The authors would like to thank Anne de Bouard for pointing out the necessity of using the selector theorem in the proof of part (iv) of Lemma 4.1. Our proof of that part is modeled on the proof from [18].

We may now use the bounds from Lemmas 4.1 and 4.3 for (increments of) the $V_h$-valued iterates $\{\mathcal{M}^j\}_{j=1}^J$ that solve Algorithm 1.2 to construct proper limiting functions for $k, h \to 0$ which are possible candidates for weak martingale solutions of (1.5). For this purpose, we use piecewise affine interpolation in time of those iterates.

**Definition 4.1.** Let $h > 0$ and $I_k = \{t_j\}_{j \in \{0, \ldots, J\}}$ be a net of fineness $k > 0$ covering $[0, T]$. For $x \in D$ and $t \in [t_j, t_{j+1})$ define

\begin{align}
(4.34) & \quad \mathcal{M}_{k,h}(t, x) := \frac{t-t_j}{k} \mathcal{M}^{j+1}(x) + \frac{t_{j+1}-t}{k} \mathcal{M}^j(x), \\
(4.35) & \quad \mathcal{M}_{k,h}^-(t, x) := \mathcal{M}^j(x), \quad \mathcal{M}_{k,h}^+(t, x) := \mathcal{M}^{j+1}(x), \quad \overline{\mathcal{M}_{k,h}}(t, x) := \mathcal{M}^{j+1/2}(t, x).
\end{align}

We may rewrite (1.6) in the form

\begin{align}
(4.36) & \quad (\mathcal{M}_{k,h}^+ - \mathcal{M}_{k,h}^-, \Phi)_h + \alpha k \left( \overline{\mathcal{M}_{k,h}} \times \overline{\mathcal{M}_{k,h}} \times \Delta_h \mathcal{M}_{k,h}^+, \Phi \right)_h \\
& \quad + k \left( \overline{\mathcal{M}_{k,h}} \times \Delta_h \mathcal{M}_{k,h}^+, \Phi \right)_h = (\mathcal{M}_{k,h} \times \Delta_j W, \Phi)_h \quad \forall \Phi \in V_h.
\end{align}

Because of Lemma 3.1, Lemma 4.3 controls fractional derivatives of $\mathcal{M}_{k,h} : D_T \to \mathbb{R}^3$. Since $N_{s_1,r} \subset W_{s_2,r}$ continuously if $s_1 > s_2$ ([46, Cor. 24]), we have the following result.

**Theorem 4.1.** Let $T > 0$. Then for any $\delta \in (0, 3]$ there exists a constant $C_{T,\delta} > 0$ such that the solution $\mathcal{M}_{k,h} : \Omega \times D_T \to \mathbb{R}^3$ of (4.36) satisfies for every $\alpha \in (0, \frac{1}{2})$ the following bound

\begin{align}
(4.37) & \quad \mathbb{E}\left[ \left\| \mathcal{M}_{k,h} \right\|_{\mathbb{W}^{\alpha,\frac{4}{\alpha}-\delta}(0,T;L^2)}^{4-\delta} \right] \leq C_{\delta,T}.
\end{align}

**Remark 4.2.** Since each function $\mathcal{M}_{k,h}$ is piecewise affine on intervals $[t_j, t_{j+1})$, $j = 0, \ldots, J-1$, with the nodes values belonging to $L^{\infty} \cap W^{1,2}$, it follows that $\mathbb{P}$-almost surely $\mathcal{M}_{k,h}(\omega, \cdot) \in W^{1,2}(0, T; L^{\infty} \cap W^{1,2})$.

However, we do not expect the estimate (4.37) to hold for the expectation of $W^{1,2}(0, T; L^{\infty} \cap W^{1,2})$-norm instead of the expectation of $W^{\alpha,\frac{4}{\alpha}-\delta}(0, T; L^2)$-norm.
5. Convergence of iterates from Algorithm 1.2

The bounds in Lemma 4.1 and Theorem 4.1 allow for the following compactness result.

**Lemma 5.1.** Let $1 \leq \kappa < \infty$, $\alpha \in (0, \frac{1}{2})$, and $1 \leq r < 4$. Then the sequence of laws \( \{L(\mathcal{M}_{k,h})\}_{k,h} \) is tight on the space \( L^2(0; T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*) \).

**Proof.** We prove tightness of the sequence of laws \( \{L(\mathcal{M}_{k,h})\}_{k,h} \) in the space \( L^2(0, T; L^2) \cap W^{\alpha,r}(0, T; (W^{1,2})^*) \), which is based on Lemma 4.1 (ii), and Theorem 4.1; together with \( |\mathcal{M}_{k,h}|_{L^\infty(0,T;L^\infty)} \leq 1 \) this implies the assertion.

We apply Lemma 3.3 with \( X_0 = L^2(D) \), which is compactly embedded into \( X = (W^{1,2})^*(D) \). Then, provided \( 0 < \alpha < \beta < 1 \), the embedding

\[
W^{\beta,r}(0, T; L^2) \subset W^{\alpha,r}(0, T; (W^{1,2})^*)
\]

is compact.

To validate the second part of the assertion, we use Lemma 3.2, with \( B_0 = W^{1,2}(D) \) compactly embedded into \( B \equiv B_1 = L^2(D) \). Hence, the embedding

\[
L^2(0, T; W^{1,2}) \cap W^{\alpha,r}(0, T; L^2) \hookrightarrow L^2(0, T; L^2)
\]

is compact. \( \square \)

**Remark 5.1.** We remark that in a corresponding result in [12] there is an additional assumption on \( \kappa \) to be strictly less than \( 6 \). Our result here is stronger since, opposite to [12], we have pointwise a-priori estimates on \( \mathcal{M}_{k,h} \), see part (i) in Lemma 4.1.

By Lemma 5.1 we can find a subsequence \( \{\mathcal{M}'_{k,h}\}_{k,h} \), denoted in the same way as the full sequence, such that the sequence of laws \( \{L(\mathcal{M}'_{k,h})\}_{k,h} \) converges weakly to a certain probability measure \( \mu \) on \( L^2(0, T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*) \). The following result is based on the general Skorokhod embedding theorem [31, p. 9], which allows to turn over to possibly another sequence \( \{\mathcal{M}_{k,h}\}_{k,h} \) with improved convergence properties.

**Proposition 5.1.** Let $1 \leq \kappa < \infty$, $\alpha \in (0, \frac{1}{2})$, and $1 < r < 4$ such that $\alpha > \frac{1}{r}$. There exist a probability space \( \mathcal{Q}' = (\Omega', \mathcal{F}', P') \), and

(i) a sequence \( \{\mathcal{M}'_{k,h}\}_{k,h} \) such that for all indices \( k, h \)

\[
\mathcal{M}'_{k,h} : \Omega' \to L^2(0, T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*)
\]

is a measurable map and

\[
L(\mathcal{M}_{k,h}) = L(\mathcal{M}'_{k,h}) \quad \text{on} \quad L^2(0, T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*).
\]

(ii) an \( L^2(0, T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*) \)-valued random variable \( m \) defined on \( \mathcal{Q}' \) such that

\[
L(m) = \mu \quad \text{on} \quad L^2(0, T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*)
\]

and \( P' \)-almost surely

\[
\mathcal{M}'_{k,h} \to m \quad \text{in} \quad L^2(0, T; L^\infty) \cap W^{\alpha,r}(0, T; (W^{1,2})^*) \quad (k, h \to 0).
\]

Let us formulate an important consequence of part (ii) of Proposition 5.1.
Corollary 5.1. If $\beta \in [0, \frac{1}{4}]$, then $\mathbb{P}'$-almost surely as $k, h \to 0$,

\begin{equation}
\mathcal{M}_{k,h}' \rightarrow m \quad \text{in } C^3([0,T];(W^{1,2})^*) .
\end{equation}

In particular, $\mathbb{P}'$-almost surely as $k, h \to 0$,

\begin{equation}
\mathcal{M}_{k,h}'(t, \cdot) \rightarrow m(t, \cdot) \quad \text{in } (W^{1,2})^*
\end{equation}

uniformly with respect to $t \in [0,T]$.

**Proof.** If $\beta < 1/4$ then we can find $\alpha < \frac{1}{2}$ and $r < 4$ such that $\alpha - \frac{1}{r} > \beta$. Then we apply the Sobolev embedding theorem. \qed

Let us also formulate a very important consequence of part (i) of Proposition 5.1.

Corollary 5.2. For each pair $k, h \to 0$, $\mathbb{P}'$-almost surely, $\mathcal{M}_{k,h}'$ is piecewise affine, globally continuous $V_{h,-}$-valued function, with nodes at the set $I_k = \{t_j : j = 0, \ldots, J\}$.

**Proof.** The vector space of piecewise affine $V_{h,-}$-valued functions with nodes at the set $I_k = \{t_j : j = 0, \ldots, J\}$ is finite dimensional and hence a closed subspace of $L^2(0,T;L^k) \cap W^{\alpha,r}(0,T;(W^{1,2})^*)$. Since the law of $\mathcal{M}_{k,h}$ is supported by this space also the law $\mathcal{M}_{k,h}'$ is supported by it, and hence the result follows. \qed

The sequence $\{\mathcal{M}_{k,h}'\}_{k,h}$ satisfies the same estimates as the original sequence $\{\mathcal{M}_{k,h}\}_{k,h}$; see Lemma 4.1 and Theorem 4.1. By a standard subsequence argument, together with Proposition 5.1, we may assume that as $k, h \to 0$ the following convergences holds:

\begin{equation}
\mathcal{M}_{k,h}' \rightharpoonup m \quad \text{in } L^2(\Omega'; L^\infty(0,T;W^{1,2})) ,
\end{equation}

\begin{equation}
\mathcal{M}_{k,h}' \rightarrow m \quad \text{in } L^2(\Omega', L^2(0,T;L^\kappa)) \quad (1 \leq \kappa < \infty) ,
\end{equation}

\begin{equation}
\mathcal{M}_{k,h}' \rightarrow m \quad \text{almost everywhere in } D_T, \mathbb{P}' - \text{almost surely}.
\end{equation}

Let us recall that $|\mathcal{M}_{k,h}(t, x)| = 1$ for all $x \in \mathcal{E}_h$ and for all $t \in [0,T]$. As a consequence, $\mathbb{T}_h[|\mathcal{M}_{k,h}(t, \cdot)|^2] = 1$ for all $t \in [0,T]$. Then, by standard results for nodal interpolation [7], for every $K \in \mathcal{T}_h$,

\begin{equation}
|\mathcal{M}_{k,h}^+|^2 - 1|_{L^2(K)} \leq C h |\nabla|\mathcal{M}_{k,h}^+|^2 - 1|_{L^2(K)}^2
\end{equation}

\begin{equation}
\leq C h |(\mathcal{M}_{k,h}^+)^T \nabla \mathcal{M}_{k,h}^+|^2_{L^2(K)} \leq C h |\nabla \mathcal{M}_{k,h}^+|^2_{L^2(K)}.
\end{equation}

Therefore, by Lemma 4.1 (ii),

\begin{equation}
|\mathcal{M}_{k,h}'|^2 \rightarrow 1 \quad \text{in } L^2(\Omega', L^\infty(0,T;L^2)) \quad (k, h \to 0) .
\end{equation}

Since $\alpha > \frac{1}{2}$, by the Sobolev embedding theorem, $W^{\alpha,r}(0,T;(W^{1,2})^*) \subset C([0,T];(W^{1,2})^*)$. Hence in order to prove that $\mathbb{P}'$-almost sure

\begin{equation}
(m_0, \varphi) = \lim_{k,h \to 0} (\mathcal{M}_{k,h}'(0, \cdot), \varphi) \quad \forall \varphi \in C^\infty(D) ,
\end{equation}

it is sufficient to assume $\mathcal{M}_{k,h}'(0, \cdot) \to m_0$ in $L^2(\Omega', L^2)$ for $h \to 0$.

Next, we identify limits of deterministic integrals in Algorithm 1.2.
We employ (5.6), (2.8) to easily conclude that

\[
\lim_{k,h \to 0} \mathbb{E}' \left[ \int_0^T \sum_{l=1}^\infty q_l \left( (M_{k,h}^\prime \times e_l) \times e_l, \varphi \right)_h \right. \\
\left. = \mathbb{E}' \left[ \int_0^T \sum_{l=1}^\infty q_l \left( (m \times e_l) \times e_l, \varphi \right)_h \right] \quad \forall \varphi \in \mathbf{W}^{1,2}(D). \right]
\]

Because of part (ii) in Lemma 4.1, we can also assume

\[
\begin{align*}
\mathcal{I}_h \left[ M_{k,h}^\prime \times \Delta_h(M_{k,h}^\prime) \right] &\rightarrow \mathbf{Y} \quad \text{in} \ L^2(\Omega; L^2(0,T; L^2)), \\
\mathcal{I}_h \left[ M_{k,h}^\prime \times \left( \bar{M}_{k,h} \times \Delta_h(M_{k,h}^\prime) \right) \right] &\rightarrow \mathbf{Z} \quad \text{in} \ L^2(\Omega; L^2(0,T; L^2)).
\end{align*}
\]

The following two results identify corresponding processes \(\mathbf{Y}, \mathbf{Z}\). Property (iii)\(_3\) of Lemma 4.1 turns out useful to prove the following result.

**Lemma 5.2.** Let \(\langle \cdot, \cdot \rangle_1 = (\cdot, \cdot)\), and \(\langle \cdot, \cdot \rangle_2 = (\cdot, \cdot)_h\). For \(T > 0\), let \(\{M_{k,h}^\prime\}_{k,h}\) be the sequence from Proposition 5.1. Then, for \(r = 1, 2\), for every \(\varphi \in \mathbb{C}_0^2(\Omega)\), and every \(t \in [0,T]\),

\[
\begin{align*}
\text{(i)} & \quad \lim_{k,h \to 0} \mathbb{E}' \left[ \int_0^t \left( \mathcal{I}_h \left[ M_{k,h}^\prime \times \Delta_h(M_{k,h}^\prime) \right], \varphi \right)_r \; ds \right] = \mathbb{E}' \left[ \int_0^t \left( \mathcal{I}_h \left[ M_{k,h}^\prime \times \left( \bar{M}_{k,h} \times \Delta_h(M_{k,h}^\prime) \right) \right], \varphi \right)_r \; ds \right] \\
& \quad = \mathbb{E}' \left[ \int_0^t \left( m \times \Delta m, \varphi \right)_r \; ds \right],
\end{align*}
\]

\[
\begin{align*}
\text{(ii)} & \quad \lim_{k,h \to 0} \mathbb{E}' \left[ \int_0^t \left( \mathcal{I}_h \left[ M_{k,h}^\prime \times \left( \bar{M}_{k,h} \times \Delta_h(M_{k,h}^\prime) \right) \right], \varphi \right)_r \; ds \right] \\
& \quad = \mathbb{E}' \left[ \int_0^t \left( m \times (m \times \Delta m), \varphi \right)_r \; ds \right] \\
& \quad = \mathbb{E}' \left[ \int_0^t \left( m \times \Delta m, \varphi \right)_r \; ds \right].
\end{align*}
\]

**Proof.** Step 1: Assertion (i). Consider \(r = 2\), and fix \(\varphi \in \mathbb{C}_0^2(\Omega)\). By Corollary 5.2 we infer that the sequence \(\{M_{k,h}^\prime\}_{k,h}\) is \(C([0,T]; V_h)\)-valued.

Using the definition (2.5), equality \(\langle a \times b, c \rangle = -\langle b, a \times c \rangle\) which holds for all \(a, b, c \in \mathbb{R}^3\), and (2.9) we obtain

\[
\left( \bar{M}_{k,h} \times \Delta_h(M_{k,h}^\prime), \varphi \right)_h = -\left( \bar{M}_{k,h} \times \varphi, \Delta_h(M_{k,h}^\prime) \right)_h \\
= \sum_{i=1}^n \left( \frac{\partial}{\partial x_i} \mathcal{I}_h \left[ \varphi \times M_{k,h}' \right], \frac{\partial}{\partial x_i} \left( M_{k,h}' \right) \right) .
\]

In order to control effects due to interpolation, we benefit from the use of piecewise finite elements, such that \(D^2 \bar{M}_{k,h} |_K = \mathbf{0}\) for all \(K \in T_h\). Consequently, by using standard interpolation estimates [7] for every \(K \in T_h\), and putting things together again,

\[
\left| \left( \frac{\partial}{\partial x_i} (M_{k,h}')^+, \frac{\partial}{\partial x_i} \left[ \mathcal{I}_h - \text{Id} \right] \left[ \varphi \times \bar{M}_{k,h} \right] \right) \right| \leq C h \left| \frac{\partial}{\partial x_i} M_{k,h}' \right|_{L^2} \left( \left| \frac{\partial}{\partial x_i} \bar{M}_{k,h} \right|_{L^2} \left| \frac{\partial}{\partial x_i} \varphi \right|_{L^2} + \left| \varphi \right|_{W^{2,2}} \right),
\]
and after integration over the interval \((0, t)\) and then taking the expectation, this term tends to zero for \(k, h \to 0\).

Hence we need to show that the following difference converges to zero.

\[
\mathbb{E}' \left[ \left| \left( \frac{\partial \varphi}{\partial x_i} \times m, \frac{\partial m}{\partial x_i} \right) - \left( \frac{\partial}{\partial x_i} [\varphi \times \mathcal{M}'_{k,h}], \frac{\partial (\mathcal{M}'_{k,h})^+}{\partial x_i} \right) \right| \right] \\
\leq \mathbb{E}' \left[ \left| \left( \frac{\partial}{\partial x_i} [\varphi \times (m - \mathcal{M}'_{k,h})], \frac{\partial m}{\partial x_i} \right) \right| \right] \\
+ \mathbb{E}' \left[ \left| \left( \frac{\partial}{\partial x_i} [\varphi \times \mathcal{M}'_{k,h}], \frac{\partial}{\partial x_i} [m - (\mathcal{M}'_{k,h})^+] \right) \right| \right] := I + II.
\]

We proceed independently with terms \(I, II\).

By (5.6), \(\mathbb{P}'\)-almost surely \(|\frac{\partial m(\omega, \cdot)}{\partial x_i}|\in L^1(D_T)\). Hence by invoking the Lebesgue dominated convergence theorem we infer that

\[
I \leq \mathbb{E}' \left[ \int_0^T \left| \left( \frac{\partial \varphi}{\partial x_i} \times (m - \mathcal{M}'_{k,h}) + \varphi \times \frac{\partial}{\partial x_i} (m - \mathcal{M}'_{k,h}), \frac{\partial m}{\partial x_i} \right) \right| \, ds \right] \to 0 \quad (k, h \to 0),
\]

while (5.6) is employed for the second part.

For the leading term in \(II\) we easily obtain from (5.5) and Lemma 4.1 (i) that

\[
\mathbb{E}' \left[ \int_0^T \left| \left( \frac{\partial \varphi}{\partial x_i} \times \mathcal{M}'_{k,h}, \frac{\partial}{\partial x_i} [m - (\mathcal{M}'_{k,h})^+] \right) \right| \, ds \right] \to 0 \quad (k, h \to 0).
\]

The crucial term in \(II\) then comes from \(\varphi \times \frac{\partial \mathcal{M}'_{k,h}}{\partial x_i}, \frac{\partial}{\partial x_i} [m - (\mathcal{M}'_{k,h})^+]\). By using the identity

\[
(5.13) \quad \mathcal{M}'_{k,h} = [(\mathcal{M}'_{k,h})^+ - m] + m + \frac{1}{2}[(\mathcal{M}'_{k,h})^- - (\mathcal{M}'_{k,h})^+],
\]

the term related to the first difference on the right-hand side vanishes, thanks to \(\langle a \times b, a \rangle = 0\), and (5.5). For the one related to the second contribution in (5.13), we may conclude

\[
\mathbb{E}' \left[ \int_0^T \left| (\varphi \times \frac{\partial m}{\partial x_i}, \frac{\partial}{\partial x_i} [(\mathcal{M}'_{k,h})^+ - m]) \right| \, ds \right] \to 0 \quad (k, h \to 0),
\]

because of (5.5), and Lemma 4.1 (iii)3. There remains to show convergence for \(k, h \to 0\) of

\[
\frac{1}{2} \mathbb{E}' \left[ \int_0^T \left| (\varphi \times \frac{\partial}{\partial x_i} [(\mathcal{M}'_{k,h})^- - (\mathcal{M}'_{k,h})^+]), \frac{\partial}{\partial x_i} [m - (\mathcal{M}'_{k,h})^+] \right| \, ds \right].
\]

By Lemma 4.1 (ii)1, (iii)3, and (5.5), we obtain the upper bound

\[
\leq \frac{1}{2} |\varphi|_{C^2} \left( \mathbb{E}' \left[ \int_0^T |\nabla [m - (\mathcal{M}'_{k,h})^+]|^2 \, ds \right] \right)^{1/2} \\
\times \left( \mathbb{E}' \left[ \int_0^T |\nabla [(\mathcal{M}'_{k,h})^+ - (\mathcal{M}'_{k,h})^-]|^2 \, ds \right] \right)^{1/2} \to 0 \quad (k, h \to 0).
\]

Therefore it remains to show that \((r = 1)\)

\[
(5.14) \quad \mathcal{I}_h \mathcal{M}'_{k,h} \times \Delta_h (\mathcal{M}'_{k,h})^+ \to m \times \Delta m \quad \text{in} \ L^2(\Omega' \cap L^2(0, t; L^2)).
\]
For this purpose, we compute

\[
\begin{align*}
(I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] ; \varphi) \\
= & \left[ (I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , \varphi) - (I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , \varphi) \right] \\
& + (M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r) , \varphi) =: A_1 + A_2.
\end{align*}
\]

Since

\[
|A_1| \leq C h |I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)]|_{L^2} \|
\nabla \varphi\|_{L^2}
\]

and

\[
|A_2| \leq C h |M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)|_{L^2} \|
\nabla \varphi\|_{L^2},
\]

in view of (2.8) we conclude that

\[
\lim_{k,h \to 0} E'[\int_0^t (I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , \varphi) ds] = E'[\int_0^t (m \times \Delta m, \varphi) ds].
\]

This concludes the proof of assertion (i) in Lemma 5.2.

\textbf{Step 2: Assertion (ii).} Consider \( r = 2 \), and fix \( \varphi \in C_0^2(D) \). We use the reformulation

\[
\begin{align*}
(M_{k,h}^r \times (M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)) , \varphi)
& = - (I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , I_h[M_{k,h}^r \times \varphi]) \\
& + \left[ (I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , I_h[M_{k,h}^r \times \varphi]) \\
& - (I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , I_h[M_{k,h}^r \times \varphi]) \right] \\
& =: B_1 + B_2.
\end{align*}
\]

Because of (2.7) and (2.8), and the \( W^{1,2} \)-stability property of the Lagrange interpolation operator (see e.g. [7]) we may continue as follows,

\[
\begin{align*}
|B_2| & \leq C h |I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)]|_{L^2} \|
\nabla \varphi\|_{L^2} \\
& \leq C h |M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)|_{L^2} \|
\nabla \varphi\|_{L^2} + \|
\varphi\|_{L^\infty} \|
\nabla M_{k,h}^r\|_{L^2} \to 0 \quad (k, h \to 0).
\end{align*}
\]

Moreover,

\[
B_1 = \left( I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , [I_h - \text{Id}][M_{k,h}^r \times \varphi] \right) \\
+ \left( I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , M_{k,h}^r \times \varphi \right) \\
= B_{1,1} - \left( M_{k,h}^r \times I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] , \varphi \right).
\]

Therefore, as in (5.16), we infer that

\[
|B_{1,1}| \leq C h |I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)]|_{L^2} \|
\nabla [M_{k,h}^r \times \varphi]\|_{L^2} \to 0 \quad (k, h \to 0).
\]

Let now \( r = 1 \). Because of (5.14), and (5.6), we then have for \( k, h \to 0 \) that

\[
M_{k,h}^r \times I_h[M_{k,h}^r \times \tilde{\Delta}_h(M_{k,h}^r)] \to m \times (m \times \Delta m) \quad \text{in } L^2(\Omega; L^2(0; T; L^2)).
\]

This concludes the proof of assertion (ii) in Lemma 5.2. \( \square \)
6. Existence of a solution to the stochastic LLGE’s

The aim of this section is to prove that the process \( m \) constructed in Section 5 is a weak martingale solution of problem (1.5). For this we will make use of the modification of the Skorokhod embedding theorem about which we learnt in [51]. This version is formulated and proved in the monograph [50], see Theorem 1.10.4 and the Addendum 1.10.5 therein. According to this result the new probability space \( \mathbb{P}' \equiv (\Omega', \mathcal{F}', \mathbb{P}') \) from Proposition 5.1, together with a family of measurable maps \( \phi_{k,h} : \Omega' \to \Omega \), can be constructed so that for all pairs \((k, h)\),

\[
\mathbb{P} = \mathbb{P}' \circ \phi_{k,h},
\]

\[
\mathcal{M}_{k,h}' = \mathcal{M}_{k,h} \circ \phi_{k,h}.
\]

Let us first restate the following well-known results.

**Proposition 6.1.** Assume that \((\Omega, \mathcal{F}, \mathbb{P})\) and \((\Omega', \mathcal{F}', \mathbb{P}')\) are probability spaces, and \( \phi : \Omega' \to \Omega \) be a measurable map. For a Polish space \( X \), if \( \xi : \Omega \to X \) is a random variable and \( \xi' = \xi \circ \phi \), then

\[
\sigma(\xi') = \phi^{-1}(\sigma(\xi)),
\]

where \( \sigma(\xi') \) is the \( \sigma \)-field generated by the random variable \( \xi \). Here, we denote \( \phi^{-1}(\mathcal{A}) := \{ \phi^{-1}(A) : A \in \mathcal{A} \} \) for any family \( \mathcal{A} \) of subsets of \( \Omega \).

Moreover, assume that \( \xi = \{ \xi_t : t \geq 0 \} : \mathbb{R}_+ \times \Omega \to X \) is an \( X \)-valued stochastic process on \((\Omega, \mathcal{F}, \mathbb{P})\) and \( \xi' = \{ \xi'_t : t \geq 0 \} : \mathbb{R}_+ \times \Omega' \to X \) is an \( X \)-valued stochastic process on \((\Omega', \mathcal{F}', \mathbb{P}')\) such that

\[
\xi' = \xi \circ (\text{id} \circ \phi),
\]

where \( \text{id} : \mathbb{R}_+ \to \mathbb{R}_+ \) is the identity map on \( \mathbb{R}_+ \) and \( \text{id} \circ \phi : \mathbb{R}_+ \times \Omega' \ni (s, \omega') \mapsto (s, \phi(\omega')) \in \mathbb{R}_+ \times \Omega \). Then for any set \( S \subset \mathbb{R}_+ \),

\[
\sigma(\{ \xi'_s : s \in S \}) = \phi^{-1}\left( \sigma(\{ \xi_s : s \in S \}) \right).
\]

If \( \mathcal{A} \) and \( \mathcal{B} \) are two \( \mathbb{P} \)-independent \( \sigma \)-fields of \( \mathcal{F} \), then the \( \sigma \)-fields \( \phi^{-1}(\mathcal{A}) \) and \( \phi^{-1}(\mathcal{B}) \) are \( \mathbb{P}' \)-independent \( \sigma \)-fields of \( \mathcal{F}' \).

The following result is taken from [51].

**Proposition 6.2.** Assume that \((\Omega, \mathcal{F}, \mathbb{P})\) and \((\Omega', \mathcal{F}', \mathbb{P}')\) are probability spaces, and \( \phi : \Omega' \to \Omega \) be a measurable map such that \( \mathbb{P} = \mathbb{P}' \circ \phi \).

(i) If \( W \) is a \( \mathcal{K} \)-valued \( \mathbb{Q} \)-Wiener process on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\),

where \( \mathbb{F} = \{ \mathcal{F}_t \}_{t \in [0, T]} \), then a process \( W' \) defined by

\[
W' := W \circ (\text{id} \circ \phi)
\]

is a \( \mathcal{K} \)-valued \( \mathbb{Q} \)-Wiener process on filtered probability space \((\Omega', \mathcal{F}', \mathbb{F}', \mathbb{P}')\), where \( \mathbb{F}' = \{ \mathcal{F}'_t \}_{t \geq 0} \) and \( \mathcal{F}'_t := \phi^{-1}(\mathcal{F}_t) \). In particular, the laws of the processes \( W \) and \( W' \) on the space \( C(\mathbb{R}_+, \mathcal{K}) \) are equal.

(ii) If in addition \( \xi' \) and \( \xi \) are processes with trajectories in the Skorokhod space \( \mathbb{D}_E \), where \( E \) is a Polish space, satisfying equality (6.3), then the laws of the processes \( (\xi, W) \) and \( (\xi', W') \) on the space \( \mathbb{D}_E \times C(\mathbb{R}_+, \mathcal{K}) \) are equal.

**Proof.** Just apply Proposition 6.1.

\[\square\]
Proposition 6.3. Assume that $X$ is a separable Hilbert space. In the framework of Proposition 6.2, if $\xi = \{\xi_t \colon t \geq 0\}$ is a progressively measurable $X$-valued process on $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P}^\prime)$, and $\xi' = \{\xi'_t \colon t \geq 0\}$ is a progressively measurable process on $(\Omega', \mathcal{F}', \mathbb{P}', \mathbb{P}')$ such that (6.3) holds and
\begin{equation}
\mathbb{E}\left[\int_0^T |\xi(s)|^2 \, ds\right] < \infty, \quad \text{for } T > 0.
\end{equation}
Then the process $\xi'$ satisfies the corresponding version of (6.5), i.e.
\begin{equation}
\mathbb{E}'\left[\int_0^T |\xi'(s)|^2 \, ds\right] < \infty, \quad \text{for } T > 0,
\end{equation}
and for each $t \geq 0$, $\mathbb{P}'$-almost surely,
\begin{equation}
\int_0^t \xi'(s) \, d\mathbb{W}'(s) = \left[ \int_0^t \xi(s) \, d\mathbb{W}(s) \right] \circ \phi.
\end{equation}

Proof. Consider first a random step process $\xi$ that satisfies (6.5). Then, also the process $\xi'$ that enjoys (6.3) is a random step process. Moreover, $\xi'$ satisfies (6.6) and by the equalities (6.4) and (6.3), the equality (6.7) follows easily. The general case follows by approximation since each progressively measurable process $\xi$ satisfying (6.5) can be approximated in the sense of (6.5) by random step processes. \hfill \square

In the above framework we define a $C_0(\mathbb{R}_+, \mathcal{K})$-valued random variables $W'_{k,h}$ by
\begin{equation}
W'_{k,h} := W \circ \phi_{k,h}.
\end{equation}
It follows from Proposition 6.2 that each process $W'_{k,h}$ is a Brownian motion.

Finally we define the following $\mathbb{F}'$-martingale on the probability space $(\Omega', \mathcal{F}', \mathbb{P}')$,
\begin{equation}
X'_{k,h}(t, \cdot) := \int_0^t \mathcal{I}_h[(\mathcal{M}^{-}_{k,h})'(s, \cdot) \times dW'_{k,h}(s)] \quad \forall t \in [0, T].
\end{equation}

In a similar fashion we define the following $\mathbb{F}$-martingale on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$,
\begin{equation}
X_{k,h}(t, \cdot) := \int_0^t \mathcal{I}_h[\mathcal{M}^{-}_{k,h}(s, \cdot) \times dW_{k,h}(s)] \quad \forall \in [0, T].
\end{equation}
The quadratic variation of this martingale is given by, for $\psi_1, \psi_2 \in \mathcal{W}^{1,2}$,
\begin{equation}
\langle Q_{k,h}(t)\psi_1, \psi_2 \rangle := \int_0^t \sum_{l=1}^\infty q_l \langle \mathcal{I}_h[\mathcal{M}^{-}_{k,h}(s, \cdot) \times e_l], \psi_1 \rangle \langle \mathcal{I}_h[\mathcal{M}^{-}_{k,h}(s, \cdot) \times e_l], \psi_2 \rangle \, ds
\end{equation}
\begin{equation}
\forall t \in [0, T].
\end{equation}

We need the following technical result.

Lemma 6.1. Assume $a, b \in \mathbb{R}$, $a < b$ and $\gamma : [a, b] \to \mathbb{R}$ is a Hölder continuous function with exponent $\alpha \in (0, \frac{1}{2})$, i.e., there exists some finite $C > 0$ such that
\begin{equation}
|\gamma(t_2) - \gamma(t_1)| \leq C|t_2 - t_1|^\alpha \quad \forall t_1, t_2 \in [a, b].
\end{equation}
Define a function
\begin{equation}
Z(t) := \frac{b - t}{b - a} (\gamma(t) - \gamma(a)) - \frac{t - a}{b - a} (\gamma(b) - \gamma(t)) \quad \forall t \in [a, b].
\end{equation}
Therefore for 

\[ |Z(t)| \leq \frac{C(b-a)^{\alpha}}{2^\alpha} \quad \forall t \in [a, b], \]

\[ |Z(t) - Z(s)| \leq 2C|t - s|^{\alpha} \quad \forall s, t \in [a, b]. \]

**Proof.** The proof of the first part is based on direct calculations and of the following two easy identities.

\[
\sup_{t \in [a,b]} \frac{(t-a)(b-t)}{b-a} = \frac{b-a}{4},
\]

\[
\sup_{x \in [0,1]} (1-x)^{1-\alpha} + x^{1-\alpha} = 2^{\alpha}.
\]

To prove the second part we observe that

\[
(b-a)Z(t) := (b-t)(\gamma(t) - \gamma(a)) - (t-a)(\gamma(b) - \gamma(t))
\]

\[
= (b-a)\gamma(t) + t(\gamma(a) - \gamma(b)) + a\gamma(b) - b\gamma(a) \quad \forall t \in [a, b].
\]

Therefore for \( s, t \in [a, b] \),

\[
|(b-a)(Z(t) - Z(s))| = |(b-a)(\gamma(t) - \gamma(s)) + (t-s)(\gamma(a) - \gamma(b))|
\]

\[
\leq |b-a| |\gamma(t) - \gamma(s)| + |t-s| |\gamma(a) - \gamma(b)|.
\]

Hence, by the assumptions on \( \gamma \),

\[
|Z(t) - Z(s)| \leq 2C|t - s|^{\alpha} \quad \forall s, t \in [a, b]
\]

as claimed. \( \square \)

The above lemma can be used to prove the following approximation result.

**Proposition 6.4.** Suppose that \( \gamma : [0,T] \to \mathbb{R} \) is a Hölder continuous function with exponent \( \alpha \in (0, \frac{1}{2}) \), i.e., there exists some finite \( C > 0 \) such that

\[
|\gamma(t_2) - \gamma(t_1)| \leq C|t_2 - t_1|^\alpha \quad \forall t_1, t_2 \in [0,T].
\]

Let \( I_k := \{ 0 < \cdots < t_j < \cdots < t_J = T \} \) be an equidistant partition of \([0,T]\) of mesh-size \( k > 0 \). Let \( \gamma_k : [0,T] \to \mathbb{R} \) be a piecewise affine approximation of the function \( \gamma \), i.e. \( \gamma_k(t_j) = \gamma(t_j) \) for \( j = 0, \ldots, J \), and \( \gamma_k \) is affine on every segment \([t_{j-1},t_j)\), \( j = 0, \ldots, J \). Then

\[
\sup_{t \in [0,T]} |\gamma_k(t) - \gamma(t)| \leq \frac{Ck^\alpha}{2^\alpha},
\]

and

\[
\sup_{t \in [0,T]} |\gamma_k^2(t) - \gamma^2(t)| \leq Ck^\alpha 2^{1-\alpha} \|\gamma\|_{L^\infty}.
\]

**Proof.** Since

\[
\sup_{t \in [0,T]} |\gamma_k(t) - \gamma(t)| = \max_{1 \leq j \leq J} \sup_{t \in [t_{j-1},t_j)} |\gamma_k(t) - \gamma(t)|,
\]

the first result is a direct consequence of Lemma 6.1. Since \( \|\gamma_k\|_{L^\infty} \leq \|\gamma\|_{L^\infty} \), the second result is then a direct consequence of the first one. \( \square \)
It is in (4.34) that we defined the process $\mathcal{M}_{k,h}$ by piecewise affine extension of the random variables $\{M^j\}_{j=0}$. Another natural definition would be to use integrals. To be precise, one can define a process $\tilde{\mathcal{M}}_{k,h}$ by the following formula,

$$
\tilde{\mathcal{M}}_{k,h}(t, \cdot) := M^j + \int_{t_j}^{t} f^j_{k,h}(s) \, ds + \int_{t_j}^{t} g^j_{k,h}(s) \, dW(s)
$$

(6.10)

$$
= M^j + f^j_{k,h}(t - t_j) + \sum_{l=1}^{\infty} \sqrt{q_l} g^j_{k,h}(\beta^l(t) - \beta^l(t_j)) \quad \forall \, t \in [t_j, t_{j+1}],
$$

where using the definition (4.20),

$$
f^j_{k,h} = -\mathcal{I}_h \left[ \alpha M^{j+1/2} \times (M^{j+1/2} \times \tilde{\Delta}_h M^{j+1}) + M^{j+1/2} \times \tilde{\Delta}_h M^{j+1} \right] + \frac{1}{2} \sum_{l=1}^{\infty} q_l \mathcal{I}_h \left[ (M^{j+1/2} \times e_l) \times e_l \right] + \frac{1}{k} \left[ \frac{1}{2} A_1 - k A_2 + A_4 \right],
$$

(6.11)

$$
g^j_{k,h} = \sum_{l=1}^{\infty} g^j_{k,h, l} \, e_l, \quad g^j_{k,h, l} = \mathcal{I}_h \left[ M^j \times e_l \right].
$$

(6.12)

We will also use the following notation,

$$
f_{k,h}(s, \cdot) := \sum_{j=0}^{J-1} 1_{[t_j, t_{j+1})}(s) f^j_{k,h} \quad \forall \, s \in [0, T],
$$

(6.13)

$$
g_{k,h}(s, \cdot) := \sum_{j=0}^{J-1} 1_{[t_j, t_{j+1})}(s) g^j_{k,h} \quad \forall \, s \in [0, T].
$$

(6.14)

Note that in view of (4.19), and (4.34)

$$
\lim_{t \nearrow t_{j+1}} \tilde{\mathcal{M}}_{k,h}(t, \cdot) = M^{j+1} = \mathcal{M}_{k,h}(t_{j+1}, \cdot) \quad (i \in \{0, \ldots, J\} - 1).
$$

(6.15)

Hence, $\tilde{\mathcal{M}}_{k,h}$ is a continuous function on $[0, T]$. From (6.10) we find out that for $t \in [0, T]$,

$$
\tilde{\mathcal{M}}_{k,h}(t, \cdot) = M_0 + \int_0^t \sum_{l=1}^{\infty} \sqrt{q_l} \mathcal{I}_h \left[ \tilde{\mathcal{M}}_{k,h}^{-}(s, \cdot) \times e_l \right] \, d\beta^l(s) + \int_0^t f_{k,h}(s, \cdot) \, ds
$$

(6.16)

$$
= M_0 + X_{k,h}(t, \cdot)
$$

$$
+ \int_0^t F_{k,h}(s; W^-, W^+; \tilde{\mathcal{M}}_{k,h}^{-}, \tilde{\mathcal{M}}_{k,h}^+) \, ds,
$$

where $\mathcal{I}_h$ is the indicator function of the interval $[t_j, t_{j+1})$. The function $f_{k,h}(s, \cdot)$ is defined as

$$
f_{k,h}(s, \cdot) := \sum_{j=0}^{J-1} 1_{[t_j, t_{j+1})}(s) f^j_{k,h} \quad \forall \, s \in [0, T].
$$
where, with $M^{1/2} = \frac{1}{2}(M^0 + M^1)$,
\[
F_{k,h}(s; W_0, W_1; M^0, M^1) := -\mathcal{I}_h \left[ \alpha M^{1/2} \times (M^{1/2} \times \Delta_h M^1) + M^{1/2} \times \Delta_h M^1 \right]
\]
(6.17) 
\[+ \frac{1}{2} \sum_{l=1}^{\infty} \sqrt{q_l} \mathcal{I}_h \left[ \left( M^{1/2} \times e_i \right) \times e_i \right]
\]
\[+ \frac{1}{2} \sum_{l=1}^{\infty} q_l \beta_0 [\beta_1 - \beta_0] - k \mathcal{I}_h [M^{1/2} \times e_i]
\]
\[= \sum_{l=1}^{\infty} \sqrt{q_l} \beta_1 \mathcal{I}_h \left[ (\alpha M^{1/2} \times (M^{1/2} \times \Delta_h M^1) + M^{1/2} \times \Delta_h M^1) \times e_i \right]
\]
\[+ \frac{1}{2} \sum_{m_1 \neq m_2} \sqrt{q_{m_1} q_{m_2}} \left( \Delta_j \beta_{m_1} \Delta_j \beta_{m_2} \right) \mathcal{I}_h \left[ (M^{1/2} \times e_{m_1}) \times e_{m_2} \right].
\]

We use (4.34), (4.36), and (4.19) to obtain the following useful identity for $\mathcal{M}_{k,h}$ at times $t \in [t_j, t_{j+1})$,
\[
(\mathcal{M}_{k,h}(t, \cdot) - \mathcal{M}_{k,h}(t, \cdot), \Phi)_h
\]
(6.18) 
\[+ \alpha (t - t_j) \left[ (\mathcal{M}_{k,h}(t, \cdot) \times [\mathcal{M}_{k,h}(t, \cdot) \times \Delta_h \mathcal{M}_{k,h}^+(t, \cdot), \Phi)_h
\]
\[+ (\mathcal{M}_{k,h}(t, \cdot) \times \Delta_h \mathcal{M}_{k,h}^+(t, \cdot), \Phi)_h \right]
\]
\[= \frac{t - t_j}{k} \left( \mathcal{M}_{k,h}(t, \cdot) \times \Delta_j W, \Phi \right)_h
\]
\[= \frac{t - t_j}{k} \left[ (\mathcal{M}_{k,h}(t, \cdot) \times \Delta_j W, \Phi \right)_h + \frac{1}{2} \sum_{l=1}^{\infty} q_l \left( [\mathcal{M}_{k,h}(t, \cdot) \times e_i] \times e_i, \Phi \right)_h + (\mathcal{A}^j, \Phi)_h \right],
\]
where $\{\mathcal{A}^j\}_j$ is defined in (4.20).

From Proposition 6.4 we have the following fundamental result.

**Lemma 6.2.** For each $\beta \in (0, \frac{1}{2})$ there is a random variables $K_1^1$ and $K_2^2$ such that for all pairs $(k, h),$
\[
\sup_{t \in [0, T]} |\mathcal{M}_{k,h}(t, \cdot) - \mathcal{M}_{k,h}(t, \cdot)|_{L^\infty} \leq K_1^1 k^\beta,
\]
(6.19) 
\[
\sup_{t \in [0, T]} |\mathcal{M}_{k,h}(t, \cdot) - \mathcal{M}_{k,h}(t, \cdot)|_{W^{1,2}} \leq K_2^2 k^\beta.
\]
(6.20) 
Moreover, $\mathbb{E}[|K_1^1|^r] < \infty$ for each $r \leq 2$ and $\mathbb{E}[|K_1^1|^r] < \infty$ for each $r < 4$.

**Proof.** We use (6.18) and (6.16) to deduce the following identity,
\[
\mathcal{M}_{k,h}(t, \cdot) - \mathcal{M}_{k,h}(t, \cdot) = \sum_{l=1}^{\infty} \sqrt{q_l} g_{k,h}^{1l} \mathcal{Z}_l(t) \quad \forall t \in [t_j, t_{j+1}),
\]
(6.21)
where \( g_{k,h}^j = \mathcal{I}_h \left[ M^j \times e_j \right] \), and
\[
Z_{jl}^k(t) = \beta^l(t) - \beta^l(t_j) - \frac{t-t_j}{t_{j+1} - t_j} \left( \beta^l(t_{j+1}) - \beta^l(t_j) \right) \\
= \frac{t_{j+1} - t}{k} \left( \beta^l(t) - \beta^l(t_j) \right) - \frac{t-t_j}{k} \left( \beta^l(t_{j+1}) - \beta^l(t) \right).
\]

Because of parts (i) and (ii) of Lemma 4.1, there exists a constant \( C > 0 \) such that for all \( \omega \in \Omega \),
\[
\sup_{k,h, i \in \{0,\ldots,J\}} |g_{k,h}^i|_{L^{\infty} \cap W^{1,2}} \leq C.
\]

By applying Lemma 6.1, we infer that there exists a random variable \( K_\beta \) such that
\[
\sup_{t \in [0,T]} |\tilde{\mathcal{M}}_{k,h}(t, \cdot) - \mathcal{M}_{k,h}(t, \cdot)|_{L^{\infty} \cap W^{1,2}} = \sup_{t \in [0,T]} \sup_{i \in \{0,\ldots,J\}} |\tilde{\mathcal{M}}_{k,h}(t, \cdot) - \mathcal{M}_{k,h}(t, \cdot)|_{L^{\infty} \cap W^{1,2}} \leq K_\beta \sup_{0 \leq j \leq J} |t_{j+1} - t_j|^\beta = K_\beta k_\beta^\beta.
\]

Hence the result follows.

It is from (6.15) that continuity of \( \tilde{\mathcal{M}}_{k,h} \) is known for every finite \((k,h)\); the following result asserts Hölder continuity of it for every \( \alpha \in (0, \frac{1}{2}) \), which follows from (6.21), and the corresponding property of the Wiener process.

**Proposition 6.5.** If \( \alpha \in (0, \frac{1}{2}) \) and \( r \in (1,4) \), then for each pair \((k,h)\),
\[
\tilde{\mathcal{M}}_{k,h} \in W^{\alpha,r}([0,T], \mathbb{L}^2).
\]

**Proof.** Because of (6.21), we have
\[
\tilde{\mathcal{M}}_{k,h} - \mathcal{M}_{k,h} = \sum_{j=0}^{J-1} 1_{[t_j, t_{j+1})} \sum_{l=1}^\infty \sqrt{\gamma_l} Z_{jl}^k \cdot g_{k,h}^j = Z_{k,h}^j.
\]

Note that \( g_{k,h}^j \in \mathbb{W}^{1,2} \) for all \( j \). Since \( \mathbb{W} \) is Hölder continuous with constant \( \alpha \in (0, \frac{1}{2}) \) on every interval \([t_j, t_{j+1})\), we infer, for instance [11, Corollary 3.3], that \( \tilde{\mathcal{M}}_{k,h} - \mathcal{M}_{k,h} \) is Hölder continuous function for every \( \alpha \in (0, \frac{1}{2}) \) on the whole interval \([0,T]\). The result then follows since \( \mathbb{W}^{\alpha,r}([0,T], \mathbb{W}^{1,2}) \subset \mathbb{W}^{\alpha,r}([0,T], \mathbb{L}^2) \), and \( \mathcal{M}_{k,h} \in \mathbb{W}^{\alpha,r}([0,T], \mathbb{L}^2) \) by Theorem 4.1.

Define now a family of processes \( \{\tilde{\mathcal{M}}_{k,h}'\}_{k,h} \) by an analog of the formula (6.2), i.e.
\[
(6.22) \quad \tilde{\mathcal{M}}_{k,h}(t, \cdot) := \tilde{\mathcal{M}}_{k,h}(t, \cdot) \circ \phi_{k,h}, \quad \forall t \in [0,T].
\]

It follows then from (6.16), (6.8) and Proposition 6.3 that for every \( t \in [0,T] \),
\[
\tilde{\mathcal{M}}_{k,h}'(t, \cdot) = M_0 + \int_0^t f'_{k,h}(s, \cdot) \, ds + \int_0^t \mathcal{I}_h \left[ (\tilde{\mathcal{M}}_{k,h}')^+(s, \cdot) \times d\mathbb{W}_{k,h}'(s) \right] \]
\[
= M_0 + X_{k,h}'(t, \cdot) + \int_0^t F_{k,h}'(s; (\mathbb{W}_{k,h}')^-, (\mathbb{W}_{k,h}')^+, (\tilde{\mathcal{M}}_{k,h}')^-, (\tilde{\mathcal{M}}_{k,h}')^+) \, ds.
\]
According to Lemma 6.2, for each \( \alpha \in (0, \frac{1}{2}) \) there exist random variables \( C'_\alpha \) and \( C''_\alpha \) defined on the probability space \( \mathcal{P}' \) such that

\[
\sup_{t \in [0,T]} |\tilde{M}_{k,h}(t, \cdot) - M_{k,h}(t, \cdot)|_{L^\infty} \leq C'_\alpha k^\alpha,
\]

\[
\sup_{t \in [0,T]} |\tilde{M}'_{k,h}(t, \cdot) - M'_{k,h}(t, \cdot)|_{W^{1,2}} \leq C''_\alpha k^\alpha,
\]

where \( \mathbb{E}'[(C'_\alpha)^r] < \infty \) for each \( r < 4 \) and \( \mathbb{E}'[(C''_\alpha)^r] < \infty \) for each \( r \leq 2 \). We summarize further important properties of the family \( \{\tilde{M}'_{k,h}\}_{k,h} \).

**Corollary 6.1.** If \( \alpha \in (0, \frac{1}{2}) \), \( r \in (1,4) \) satisfy \( \alpha > \frac{1}{r} \) and \( \kappa \in [1, \infty) \), then

(i) for all indices \( k, h \)

\[
\tilde{M}'_{k,h} : \Omega' \to L^2(0,T; \mathbb{L}^\kappa) \cap W^{\alpha,r}(0,T; (\mathbb{W}^{1,2})^*),
\]

is a measurable map and

\[
\mathcal{L}(\tilde{M}_{k,h}) = \mathcal{L}(\tilde{M}'_{k,h}) \text{ on } L^2(0,T; \mathbb{L}^\kappa) \cap W^{\alpha,r}(0,T; (\mathbb{W}^{1,2})^*).
\]

(ii) \( \mathbb{P}' \)-almost surely

\[
\tilde{M}'_{k,h} \to m \quad \text{in} \quad L^2(0,T; \mathbb{L}^\kappa) \cap C([0,T]; (\mathbb{W}^{1,2})^*) \quad (k, h \to 0),
\]

**Proof.** By Proposition 6.5, \( \tilde{M}_{k,h} \in W^{\alpha,r}(0,T; (\mathbb{W}^{1,2})^*) \); hence, by the definition of \( \tilde{M}'_{k,h} \) the property (i) is satisfied.

Finally, property (ii) is a direct consequence of inequalities (6.24) and (6.25) and of Proposition 5.1.

\[\square\]

In the sequel, let \( \mathbb{F}'_{k,h} \) denote the natural filtration generated by the process \( \tilde{M}'_{k,h} \).

**Lemma 6.3.** For each pair \( (k, h) \), the process \( X'_{k,h} \) is a \( (\mathbb{W}^{1,2})^* \)-valued square integrable \( \mathbb{F}'_{k,h} \)-martingale. Moreover, the quadratic variation process of the martingale \( X'_{k,h} \) is equal to the process \( Q'_{k,h} \) defined for all \( \psi_1, \psi_2 \in \mathbb{W}^{1,2} \) by

\[
\langle Q'_{k,h}(t)\psi_1, \psi_2 \rangle := \sum_{l=1}^\infty q_l \int_0^t \langle \mathcal{I}_l[(\tilde{M}'_{k,h})^+ - e_l], \psi_1 \rangle \langle \mathcal{I}_l[(\tilde{M}'_{k,h})^- - e_l], \psi_2 \rangle \mathrm{d}s \quad \forall t \in [0,T].
\]

**Proof.** From equality (6.23) we have that the following representations of the processes \( X_{k,h} \) and \( X'_{k,h} \) holds for \( t \in [0,T] \),

\[
X_{k,h}(t, \cdot) = \tilde{M}_{k,h}(t, \cdot) - M_0 - \int_0^t F_{k,h}(s; \mathbb{W}^-, (\tilde{M}'_{k,h})^-, (\tilde{M}'_{k,h})^-) \mathrm{d}s,
\]

\[
X'_{k,h}(t, \cdot) = \tilde{M}'_{k,h}(t, \cdot) - M_0 - \int_0^t F_{k,h}(s; (\mathbb{W}'_{k,h})^-, (\mathbb{W}'_{k,h})^+; (\tilde{M}'_{k,h})^-, (\tilde{M}'_{k,h})^+) \mathrm{d}s.
\]
For all \( t \in [0, T] \), since by Propositions 5.1 and 6.2, (ii) the laws of the processes \((W, \tilde{M}_{k,h})\) and \((W'_{k,h}, \tilde{M}'_{k,h})\) are equal, by the measurability of the mapping \( F_{k,h} \) we infer that the laws of \( X_{k,h} \) and \( X'_{k,h} \) also coincide at every time \( t \in [0, T] \). Therefore, by repeating the proof of Lemma 5.1 from [12] we conclude that since \( X_{k,h} \) is a \( \mathbb{F}_{k,h} \)-martingale, and \( X'_{k,h} \) is a \( \mathbb{F}'_{k,h} \)-martingale.

The proof of the second part of the Lemma also follows the lines of the proof of Lemma 5.1 from [12] and is based on representations (6.29) and (6.30). Hence the proof is complete. \( \Box \)

Now we shall prove the following result.

**Lemma 6.4.** Let \( t \in [0, T] \), then for all \( \psi_1, \psi_2 \in W^{1,2} \),

\[
\langle Q'_{k,h}(t)\psi_1, \psi_2 \rangle \to \langle Q'(t)\psi_1, \psi_2 \rangle \quad (k, h \to 0),
\]

where, for \( t \in [0, T] \) and \( \psi_1, \psi_2 \in W^{1,2} \),

\[
\langle Q'(t)\psi_1, \psi_2 \rangle = \int_0^t \sum_{l=1}^\infty q_l(m(s, \cdot) \times e_l, \psi_1) \langle m(s, \cdot) \times e_l, \psi_2 \rangle \, ds.
\]

**Proof.** It is enough to apply convergence (6.27) from Corollary 6.1 in conjunction with Lebesgue dominated convergence theorem. \( \Box \)

Let \( m \) be the process constructed in the previous section as well as Corollary 6.1; note that both sequences \( \{M'_{k,h}\}_{k,h} \) and \( \{\tilde{M}'_{k,h}\}_{k,h} \) converge to the same process \( m \) in an appropriate sense. Recall that the convergence results in Lemma 5.2 identify limits of the deterministic integrals for the sequence \( \{M'_{k,h}\}_{k,h} \) for \( k, h \to 0 \); it is because of (6.15), and Definition 4.1 that

\[
\tilde{M}_{k,h} = M^{j+1/2} = \tilde{M}_{k,h}, \quad \text{and} \quad (M_{k,h})^+ = M^{j+1} = (\tilde{M}_{k,h})^+ \quad t \in [t_j, t_{j+1}),
\]

such that Lemma 5.2 applies to the sequence \( \{\tilde{M}_{k,h}\}_{k,h} \) as well, i.e., there holds for the processes \( Y, Z \) defined in (5.10), (5.11) that for \( k, h \to 0 \),

\[
\mathcal{I}_h[\tilde{M}_{k,h} \times \Delta_h(\tilde{M}'_{k,h})^+] \to Y \quad \text{in } L^2(\Omega'; L^2(0, T; L^2)),
\]

\[
\mathcal{I}_h[\tilde{M}_{k,h} \times (\tilde{M}'_{k,h} \times \Delta_h(\tilde{M}'_{k,h})^+)] \to Z \quad \text{in } L^2(\Omega'; L^2(0, T; L^2)).
\]

For \( t \in [0, T] \), we denote by \( \mathcal{F}_t \) the \( \sigma \)-field generated by the random variables \( m(s, \cdot) \), for \( s \leq t \). Denote by \( \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]} \) the natural augmentation of the filtration \( \mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]} \). Define an \( (W^{1,2})^* \)-valued process \( X' \) that is a natural candidate for the martingale part of the process \( m \) by the following formula

\[
X'(t) := m(t, \cdot) - m_0 - \int_0^t F(m(s, \cdot)) \, ds, \quad t \in [0, T],
\]

where

\[
F(m) := -\alpha m \times (m \times \Delta m) + m \times \Delta m + \frac{1}{2} \sum_{l=1}^\infty q_l[(m \times e_l) \times e_l].
\]
Note that by Lemma 5.2 we have with the above processes $Y$ and $Z$ that

$$F(m) := -\alpha Z(t, \cdot) + Y(t, \cdot) + \frac{1}{2} \sum_{l=1}^{\infty} q_l(m \times e_l) \times e_l.$$  

(6.37)

The two auxiliary Lemmata 6.5, 6.6 are needed below: together with the martingale representation theorem, they ensure the existence of a $K$-valued $Q$-Wiener process on some extended probability space, such that the process $X'$ is an Itô integral with respect to that new Wiener process of the process $m(s, \cdot) \times \cdot$ appearing in formula (6.32). It is important in both these lemmata that $F$ is the natural augmentation of the filtration generated by the process $m$.

**Lemma 6.5.** The process $X'$ defined by the formula (6.35) is a $(W^{1,2})^*$-valued square integrable $F$-martingale.

*Proof.* As in [52], in view of [19, p. 75] it is enough to show that the process $m$ is an $F$-martingale. Let us fix $t, s \in [0, T]$ such that $s \leq t$. We have to show that for any choice of times $0 \leq s_1 < s_2 < \ldots < s_n \leq s$ where $n \in \mathbb{N}$, and any bounded and continuous functions $h_i : (W^{1,2})^* \to \mathbb{R}$, $i = 1, \ldots, N$, and any $\varphi \in W^{1,2}(D)$ the following equality holds

$$E'[\left(\left(\left(\left(X'(t, \cdot) - X'(s, \cdot), \varphi \right) \prod_{i=1}^{N} h_i(m(s_i, \cdot))\right)\right)\right] = 0.$$  

(6.38)

Let us also fix such $n \in \mathbb{N}$, $0 \leq s_1 < s_2 < \ldots < s_n \leq s$, bounded and continuous functions $h_i : (W^{1,2})^* \to \mathbb{R}$, $i = 1, \ldots, N$ and $\varphi \in W^{1,2}(D)$.

First of all, by (6.35) and (6.37) we see that

$$E'[\left(\left(\left(\left(\left(X'(t, \cdot) - X'(s, \cdot), \varphi \right) \prod_{i=1}^{N} h_i(m(s_i, \cdot))\right)\right)\right)\right]] =$$

$$= E'[\left(\left(\left(\left(m(t, \cdot) - m(s, \cdot) - \int_{s}^{t} F(m(r, \cdot)) dr, \varphi \right) \prod_{i=1}^{N} h_i(m(s_i, \cdot))\right)\right)\right]$$

$$= E'[\left(\left(\left(\left(\left(m(t, \cdot) - m(s, \cdot) - \int_{s}^{t} F(m(r, \cdot)) dr - \alpha \int_{s}^{t} Z(r, \cdot) dr - Y(r, \cdot) dr \right.\right)\right)\right)\right]$$

$$- \frac{1}{2} \sum_{l=1}^{\infty} q_l(m(r, \cdot) \times e_l) \times e_l dr, \varphi \right) \prod_{i=1}^{N} h_i(m(s_i, \cdot))\right).$$

(6.39)

Next, since $L^2(\Omega'; L^2(0, T; L^2))$ is isomorphic to $L^2(0, T; \Omega'; L^2(L^2))$ from Lemma 5.2 and (5.3) (since $\alpha > \frac{1}{r}$) we infer that

$$E'[\left(\left(\left(\left(\left(m(t, \cdot) - m(s, \cdot), \varphi \right) \prod_{i=1}^{N} h_i(m(s_i, \cdot))\right)\right)\right)\right]] =$$

$$= \lim_{h,k \to 0} E'[\left(\left(\left(\left(\left(\tilde{M}_{k,h}(t, \cdot) - \tilde{M}_{k,h}(s, \cdot), \varphi \right) \prod_{i=1}^{N} h_i(m(s_i, \cdot))\right)\right)\right)\right].$$

(6.40)
Finally, by (5.9), we have
\[
\mathbb{E} \left[ \left\langle \int_{s}^{t} Z(r, \cdot) \, dr, \varphi \right\rangle \prod_{i=1}^{N} h_{i}(m(s_{i}, \cdot)) \right] = \lim_{h, k \to 0} \mathbb{E} \left[ \left( \int_{s}^{t} \mathcal{I}_{h} \left[ \mathcal{M}_{k,h}^{\prime} \times \overline{\mathcal{M}}_{k,h}^{\prime} \times \Delta_{h}(\overline{\mathcal{M}}_{k,h}^{\prime}+) \right] \, dr, \varphi \right) \prod_{i=1}^{N} h_{i}(m(s_{i}, \cdot)) \right],
\]

(6.42)
\[
\mathbb{E} \left[ \left\langle \int_{s}^{t} Y(r, \cdot) \, dr, \varphi \right\rangle \prod_{i=1}^{N} h_{i}(m(s_{i}, \cdot)) \right] = \lim_{h, k \to 0} \mathbb{E} \left[ \left( \int_{s}^{t} \mathcal{I}_{h} \left[ \mathcal{M}_{k,h}^{\prime} \times \overline{\mathcal{M}}_{k,h}^{\prime} \times \Delta_{h}(\overline{\mathcal{M}}_{k,h}^{\prime}+) \right] \, dr, \varphi \right) \prod_{i=1}^{N} h_{i}(m(s_{i}, \cdot)) \right].
\]

Finally, by (5.9), we have
\[
\mathbb{E} \left[ \left\langle \int_{s}^{t} \sum_{l=1}^{\infty} q_{l}(m(r, \cdot) \times e_{l}) \times e_{l} \, dr, \varphi \right\rangle \prod_{i=1}^{N} h_{i}(m(s_{i}, \cdot)) \right] = \lim_{h, k \to 0} \mathbb{E} \left[ \int_{s}^{t} \sum_{l=1}^{\infty} q_{l} \left( \left( \overline{\mathcal{M}}_{k,h}^{\prime} \times e_{l} \right) \times e_{l}, \varphi \right) \, dr \right] = \lim_{h, k \to 0} \mathbb{E} \left[ \int_{s}^{t} \sum_{l=1}^{\infty} q_{l} \left( \mathcal{I}_{h} \left[ \left( \overline{\mathcal{M}}_{k,h}^{\prime} \times e_{l} \right) \times e_{l}, \varphi \right] \right) \, dr \right].
\]

(6.43)
\[
\mathbb{E} \left[ \left\langle \int_{s}^{t} \sum_{l=1}^{\infty} q_{l}(m(r, \cdot) \times e_{l}) \times e_{l} \, dr, \varphi \right\rangle \prod_{i=1}^{N} h_{i}(m(s_{i}, \cdot)) \right] = \lim_{h, k \to 0} \mathbb{E} \left[ \int_{s}^{t} \sum_{l=1}^{\infty} q_{l} \left( \mathcal{I}_{h} \left[ \left( \overline{\mathcal{M}}_{k,h}^{\prime} \times e_{l} \right) \times e_{l}, \varphi \right] \right) \, dr \right].
\]

Now, we use (6.18), in combination with Lemma 4.2 to conclude (6.38).

Hence the proof of Lemma 6.5 is complete.

We apply Theorem 6.1 from the appendix to show the following second auxiliary lemma.

**Lemma 6.6.** The quadratic variation of the \((W^{1,2})^{\ast}\)-valued square integrable \(\mathbb{P}\)-martingale \(X^{\prime}\) defined by the formula (6.35) is equal to the process \(Q^{\prime}\) defined by the formula (6.32).

**Proof.** Use Theorem 6.1, where we put \(E = (W^{1,2})^{\ast}\) and \(H = L^{2}\), and consider processes \(\{\overline{\mathcal{M}}_{k,h}^{\prime}\}_{k,h}\) and \(\{X_{k,h}^{\prime}\}_{k,h}\) on the probability space \(\mathcal{Q}^{\prime}\). Because of part (i) of Lemma 4.1, there exists a constant \(C > 0\) such that for all \(\omega \in \Omega\), \(\sup_{k,h} \sup_{i \in \{0, \ldots, J\}} |g_{i,h}^{j}|_{L^{2}} \leq C\). Hence, in view of definition (6.28), the assumption (6.49) is satisfied. Finally, assumption (6.49) is satisfied in view of (6.22) and Theorem 4.1.

Now the argument is more or less standard, see for instance [12]. Without changing notation, we may now enlarge the probability space given by the Skorokhod Theorem (as well as the filtration \(\mathcal{F}^{\prime}\)) so that on the enlarged spaces there exists a \(\mathcal{K}\)-valued \(\mathbb{P}^{\prime}\)-adapted \(Q\)-Wiener process independent of \(M^{\prime}\). Hence, the probability space \((\Omega^{\prime}, \mathcal{F}^{\prime}, \mathbb{P}^{\prime}, \mathcal{F}^{\prime})\) and the \((W^{1,2})^{\ast}\)-valued continuous martingale \(X^{\prime}\) satisfy all the assumptions of the martingale representation theorem in the version proved in [43, Theorem 2]. Therefore, there exists a \(\mathcal{K}\)-valued \(\mathbb{P}^{\prime}\)-adapted \(Q\)-Wiener process \(W^{\prime}\) such that
\[
X^{\prime}(t, \cdot) = \int_{0}^{t} \left( m(r, \cdot) \times dW^{\prime}(r) \right), \quad t \in [0, T].
\]

(6.44)
This, together with (6.35) and (6.36) implies that

\begin{equation}
(6.45) \quad m(t, \cdot) := m_0 - \alpha \int_0^t m \times (m \times \Delta m) \, ds + \int_0^t m \times \Delta m \, ds \\
+ \frac{1}{2} \int_0^t \sum_{i=1}^\infty q_i \left( m \times e_i \right) \times e_i \, ds + \int_0^t (m(r, \cdot) \times dW'(r)),
\end{equation}

in \((W^{1,2})^*, \mathbb{P}'\)-almost surely, for all \(t \in [0, T]\).

The proof of Theorem 2.1 is now complete.

**APPENDIX**

The following result is taken from [12, Theorem C.2]. We changed the notation so that it coincides with the one used in the current paper.

**Theorem 6.1.** Assume that \(H\) and \(E\) are separable metric and respectively Hilbert spaces. Assume that \(\mathcal{P} = (\Omega, \mathcal{F}, \mathbb{P})\) is a probability space. Assume that for some \(T > 0\) and each \(s > 0\),

\[ M_s : [0, T] \times \Omega \to H, \]
\[ X_s : [0, T] \times \Omega \to E \]

are stochastic processes. Assume also that

\begin{align}
(6.46) & \quad M_s(t) \to M(t), \\
(6.47) & \quad X_s(t) \to X(t) \quad (s \to 0).
\end{align}

We denote by \(\mathbb{F}_s\), for \(s > 0\), the natural filtration on the probability space \(\mathcal{P}\) generated by the process \(M_s\). Similarly, we denote by \(\mathbb{F}\) the natural filtration on the probability space \(\mathcal{P}\) generated by the process \(M\). Finally, by \(\mathbb{F}'\) we denote the augmentation of the filtration \(\mathbb{F}\).

Assume that for each \(s > 0\), \(Q'_s = \langle X_s \rangle\) is the quadratic variation process of the \(\mathbb{F}_s\)-martingale \(X_s\). Assume that \(Q'\) is an \(\mathcal{S}_1(E)\)-valued process such that for every \(t \in [0, T]\), \(\mathbb{P}\)-a.s.

\begin{equation}
(6.48) \quad \langle Q'_s(t)x, y \rangle \to \langle Q'(t)x, y \rangle \quad \text{for all } x, y \in E.
\end{equation}

Assume also that for some \(r > 1\) and for every \(t \in [0, T]\),

\begin{align}
(6.49) & \quad \sup_{s > 0} \mathbb{E} \left[ |X_s(t)|^r_{L(E)} \right] < \infty, \\
(6.50) & \quad \sup_{s > 0} \mathbb{E} \left[ |Q'_s(t)|^r_{L(E)} \right] < \infty.
\end{align}

Then \(Q'\) is \(\mathbb{F}\)-progressively measurable and is equal to \(\langle X \rangle\), the quadratic variation process of the \(\mathbb{F}\)-martingale \(X\).
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