ON PRESSURE APPROXIMATION VIA PROJECTION METHODS FOR NONSTATIONARY INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. Projection methods are an efficient tool to approximate strong solutions of the incompressible Navier-Stokes equations. As a major deficiency, these methods often suffer from reduced accuracy for pressure iterates caused by nonphysical boundary data, going along with suboptimal error estimates for pressure iterates. We verify a rigorous bound for arising boundary layers in Chorin’s scheme under realistic regularity assumptions. In a second step, the new Chorin-Penalty method is proposed, where optimal rate of convergence for pressure iterates is shown.

1. Introduction

Given an open bounded Lipschitz domain \( \Omega \subset \mathbb{R}^d \), for \( d = 2, 3 \), and a time \( T > 0 \), we consider the time-dependent Navier-Stokes equations for incompressible, viscous \((\nu > 0)\) Newtonian fluids,

\[
\begin{align*}
\mathbf{u}_t - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega_T := (0, T) \times \Omega, \\
\text{div } \mathbf{u} &= 0 \quad \text{in } \Omega_T, \\
\mathbf{u} &= 0 \quad \text{on } \partial \Omega_T := (0, T) \times \partial \Omega, \\
\mathbf{u}(0, \cdot) &= \mathbf{u}_0 \quad \text{in } \Omega.
\end{align*}
\]

Here, \( \mathbf{u} : \Omega_T \to \mathbb{R}^d \) denotes the velocity field, \( p : \Omega_T \to \mathbb{R} \) the scalar pressure of vanishing mean value, i.e., \( \int_{\Omega} p(\cdot, x) \, dx = 0 \), and a given force \( \mathbf{f} : \Omega_T \to \mathbb{R}^d \) is driving the fluid flow, with initial velocity field \( \mathbf{u}_0 : \Omega \to \mathbb{R}^d \).

To construct and analyze numerical schemes for (1.1)–(1.4), we benefit from well-known analytical results for the given problem, which we recall here to the convenience of the reader. For this purpose, we introduce some notation: let \( L^p(\Omega) \), \( H^r(\Omega) \), and \( H^r_0(\Omega) \), for \( r \in \mathbb{N} \) be usual Lebesgue and Sobolev spaces, which are endowed with standard scalar products and induced norms \( \| \cdot \|_{H^r} \).

We recall that \( H^{-1}(\Omega) = [H^1_0(\Omega)]^d \). Let \( L^p_0(\Omega) \subset L^p(\Omega) \) be the space of functions, whose elements have vanishing integrals. Spaces of vector-valued functions will be indicated with boldface letters, e.g., \( H^1_0(\Omega) = [H^1_0(\Omega)]^d \), for \( d = 2, 3 \). We make frequent use of the spaces

\[
\begin{align*}
\mathbf{J}_0(\Omega) &= \{ \mathbf{v} \in L^2(\Omega) : \text{div } \mathbf{v} = 0 \text{ in } \Omega, \langle \mathbf{v}, \mathbf{n} \rangle = 0 \text{ on } \partial \Omega \}, \\
\mathbf{J}_1(\Omega) &= \{ \mathbf{v} \in H^1_0(\Omega) : \text{div } \mathbf{v} = 0 \text{ in } \Omega \},
\end{align*}
\]

where \( \langle \cdot, \cdot \rangle \) denotes the standard scalar product in \( \mathbb{R}^d \), and \( \mathbf{n}(x) \in S^{d-1} \) is the unit vector field pointing outside \( \Omega \). For a Banach space \( X \), let \( L^p(0, T; X) \), and \( W^{m,p}(0, T; X) \) denote standard Bochner spaces.

Let us recall the concept of weak solutions to (1.1)–(1.4), for \( \mathbf{u}_0 \in \mathbf{J}_0(\Omega) \), and \( \mathbf{f} \in L^2(0, T, \mathbf{J}_1^*(\Omega)) \) from [20, Chapter 3]: A function \( \mathbf{u} \in L^2(0, T; \mathbf{J}_1(\Omega)) \cap L^1(0, T; \mathbf{J}_1^*(\Omega)) \) is called weak solution of (1.1)–(1.4), if (1.1)–(1.2) hold in distributional sense, and boundary and initial data in (1.3), (1.4)
are attained. Moreover, we have the following further properties specific for dimension \(d = 2\) and \(d = 3\):

- \(d = 2\): weak solutions are unique, belong to \(W^{1,2}(0,T; H^1(\Omega))\), and hence to \(C([0,T]; L^2(\Omega))\), and satisfy for almost all \(t \in [0,T]\) the energy identity

\[
\frac{1}{2} \| u(t, \cdot) \|^2_{L^2} + \nu \int_0^t \| \nabla u(s, \cdot) \|^2_{L^2} \, ds = \frac{1}{2} \| u_0 \|^2_{L^2} + \int_0^t \left( \mu(s, \cdot), u(s, \cdot) \right)_{H^1} \, ds.
\]

In addition, provided \(u_0 \in J_1(\Omega)\) and \(f \in L^2(0,T; L^2(\Omega))\), weak solutions are strong, i.e.,

\[(u, p) \in L^2(0,T; H^2(\Omega) \cap C([0,T]; L^2(\Omega))) \times L^2(0,T; L^2(\Omega) \cap H^1(\Omega)).\]

- \(d = 3\): weak solutions belong to \(W^{1,4/3}(0,T; H^1(\Omega))\), are weakly continuous mappings from \([0,T] \times J_0(\Omega)\), and satisfy an inequality version of (1.5). They are locally strong, provided \(u_0 \in J_1(\Omega)\) and \(f \in L^2(0,T; L^2(\Omega))\).

In below, we always suppose that the data of problem (1.1)–(1.4) satisfy

(A1) (regularity of domain) The unique solution \(w \in J_1(\Omega)\) of the stationary, incompressible Stokes problem \(-\nu \Delta w + \nabla p = g\) in \(\Omega \subset \mathbb{R}^d\) is already in \(J_1(\Omega) \cap H^2(\Omega)\), provided \(g \in L^2(\Omega)\), and satisfies \(\|w\|_{H^2} \leq C \|g\|_{L^2}\).

(A2) (regularity of data) For any \(T > 0\), let \(u_0 \in J_1(\Omega) \cap H^2(\Omega)\), and \(f \in W^{2,\infty}(0,T; L^2(\Omega))\).

In order to approximate weak solutions of (1.1)–(1.4) by using a general Galerkin method, one proper temporal discretization strategy is the implicit Euler method, where iterates satisfy the (damped) discrete energy law \((M > 0)\), see e.g. [20, Chapter 3],

\[
\frac{1}{2} \| u^M \|^2_{L^2} + \frac{k^2}{2} \sum_{m=1}^M \| d_t u^m \|^2_{L^2} + \nu k \sum_{m=1}^M \| \nabla u^m \|^2_{L^2} = \frac{1}{2} \| u_0 \|^2_{L^2} + k \sum_{m=1}^M \left( f(t_m, \cdot), u^m \right)_{J_1^* \times J_1}.
\]

Here, we denote \(d_t \phi^{m+1} := \frac{1}{k} \{ \phi^{m+1} - \phi^m \}\), where \(k = t_{m+1} - t_m > 0\) is the time-step. For given \(f \in L^2(0,T; J_0(\Omega))\), the uniform bound (1.6) is then the key to conclude (subsequence) convergence of iterates against weak solutions \(u : \Omega_T \to \mathbb{R}^d\) of (1.1)–(1.4), for \(k \to 0\).

The practical disadvantage of implicit discretization strategies is the significant computational effort implied from the necessity to solve coupled nonlinear algebraic problems to determine (Galerkin approximations) \((u^m, p^m)\) at every time-step given by \(1 \leq m \leq M\). As a consequence, splitting algorithms were developed to reduce complexity of actual computations; among them, and one of the first, is Chorin’s projection method [2, 3, 19], where iterates for velocity field and pressure are independently obtained at every time-step. In below, let \(f^m := f(t_m, \cdot)\), and suppose that \(u^0 \in J_1(\Omega)\) is given.

**Algorithm A.** 1. Let \(m \geq 0\). Given \(u^m \in J_0(\Omega)\), find \(\tilde{u}^{m+1} \in H^1_0(\Omega)\) that satisfies

\[
\frac{1}{k} \{ \tilde{u}^{m+1} - u^m \} - \nu \Delta \tilde{u}^{m+1} + (u^m, \nabla) \tilde{u}^{m+1} = f^{m+1} \quad \text{in } \Omega.
\]

2. Given \(\tilde{u}^{m+1} \in H^1_0(\Omega)\), compute \((u^{m+1}, p^{m+1}) \in J_0(\Omega) \times [L^2(\Omega) \cap H^1(\Omega)]\) from

\[
\frac{1}{k} \{ u^{m+1} - \tilde{u}^{m+1} \} + \nabla p^{m+1} = 0, \quad \text{div} u^{m+1} = 0 \quad \text{in } \Omega,
\]

\[
\left( u^{m+1}, n \right) = 0 \quad \text{on } \partial \Omega.
\]

The latter step can be reformulated as a problem for the pressure function only,

\[
-\Delta p^{m+1} = -\frac{1}{k} \text{div} \tilde{u}^{m+1} \quad \text{in } \Omega, \quad \partial_n p^{m+1} = 0 \quad \text{on } \partial \Omega.
\]

\[
-\Delta p^{m+1} = -\frac{1}{k} \text{div} \tilde{u}^{m+1} \quad \text{in } \Omega, \quad \partial_n p^{m+1} = 0 \quad \text{on } \partial \Omega.
\]
Hence, each step consists of (1.7), (1.10), and the algebraic update (1.8) to obtain \((u^{m+1}, p^{m+1})\).

In order to understand error effects inherent due to temporal discretization, and operator splitting in Chorin’s scheme, we shift the index in (1.8) back, and add the equation to (1.7); together with (1.10), we obtain

\[ d_t \tilde{u}^{m+1} - \nu \Delta \tilde{u}^{m+1} + (u^m \cdot \nabla) \tilde{u}^{m+1} + \nabla p^m = f^{m+1} \quad \text{in } \Omega, \tag{1.11} \]

\[ \text{div } \tilde{u}^{m+1} - k \Delta p^{m+1} = 0 \quad \text{in } \Omega, \tag{1.12} \]

\[ \partial_t p^{m+1} = 0 \quad \text{on } \partial \Omega. \tag{1.13} \]

We make the following crucial observations implied from Chorin’s decoupling strategy, for every \(0 \leq m \leq M\):

(i) The velocity field \(\tilde{u}^{m+1} : \Omega \to \mathbb{R}^d\) is not divergence-free any more, but satisfies a ‘quasi-compressibility equation’ (1.12), with a penalization parameter equal to the time-step, and a penalization term that requires \(p^{m+1} \in H^1(\Omega)\).

(ii) Iterates of the pressure satisfy a homogeneous Neumann boundary condition, which is in contrast to the pressure \(p : \Omega_T \to \mathbb{R}\) that satisfies (1.1)–(1.4).

(iii) The pressure iterate in (1.11) is used in an explicit fashion, which rules out an immediate discrete energy law, where test functions \(u^{m+1}\) and \(p^{m+1}\) in (1.11) and (1.12) are used.

As a consequence of the lack of a discrete energy law, we need not hope to construct weak solutions of (1.1)–(1.4) as proper limits of iterates from Chorin’s scheme (e.g., in the sense of weak subsequence convergence). Instead, given that strong solutions to (1.1)–(1.4) exist, we may exploit their improved regularity properties to establish convergence of iterates of Algorithm A at an optimal rate. As already mentioned, strong solutions exist globally in time in 2D, and local existence is known for \(d = 3\).

The convergence analysis of Algorithm A has a long history, which started with first studies by R. Temam [19], and continued with a series of interesting works of J. Shen, and W. E & J.G. Liu, see e.g. [17, 7], and [4, 5], where solutions to (1.1)–(1.4) were assumed to be smooth. Unfortunately, solutions to (1.1)–(1.4) suffer a breakdown of regularity for \(t \to 0\) even for smooth initial data, which is due to an incompatibility of (1.2) and the prescribed data [8], which restricts the applicability of these results. A major step towards getting optimal error estimates in the context of existing strong solutions has been done by R. Rannacher [16], where the different error effects in Chorin’s method as a semi-implicit pressure-stabilization method (1.11)–(1.13) were pointed out, leading to the following result which is first proved in [13, Theorem 6.1].

**Theorem 1.1.** Let \(\{\tilde{u}^m, p^m\}\) be the solution of Chorin’s method (1.7)-(1.9), and let \((u, p)\) be a strong solution of (1.1)-(1.4) up to \(t_M = T\). Suppose that

\[ \|u^0 - u_0\|_{L^2} + \sqrt{k} \|u^0 - u_0\|_{H^1} \leq Ck. \]

For sufficiently small time-steps \(k \leq k_0(T)\), there exists a constant \(C = C(T) > 0\), such that

(a) \[ \max_{1 \leq m \leq M} \left(\|u(t_m, \cdot) - \tilde{u}^m\|_{L^2} + \tau_m \|p(t_m, \cdot) - p^m\|_{H^{-1}}\right) \leq Ck, \]

(b) \[ \max_{1 \leq m \leq M} \left(\|u(t_m, \cdot) - \tilde{u}^m\|_{H^1} + \sqrt{\tau_m} \|p(t_m, \cdot) - p^m\|_{L^2}\right) \leq C \sqrt{k}, \]

where \(\tau_m = \min\{1, t_m\}\).

Corresponding results hold for \(\{u^m\}_{m=1}^M \subset J_0(\Omega)\) from Step 2 in Algorithm A, thanks to well-known stability properties of the Helmholtz projection \(P_{J_0} : L^2(\Omega) \to J_0(\Omega)\); cf. [20].

The proof of Theorem 1.1 in [13] is split into three steps: in a first step, optimal error estimates for the implicit Euler discretization from [8] in the presence of strong solutions of (1.1)–(1.4) are
recalled to control time-discretization effects. In a second step, a modified version of (1.11)–(1.13)
is studied, where the pressure iterate \( p^m \) in (1.11) is shifted to \( p^{m+1} \). We remark that this pressure-
stabilization method is of its own interest, since it allows for more finite element pairings [9, 1],
otherwise the discrete LBB condition restricts stable finite element pairings. Optimal error
estimates which control perturbation effects due to (1.12)–(1.13) are the key results of the analysis,
and provide \( k \)-independent a-priori bounds for velocity and pressure iterates in strong norms. The
latter bounds are then necessary for an optimal error estimate between this auxiliary problem, and
(1.11)–(1.13) in the last step, which closes the proof.

**Remark 1.1.** An extension of this result to a fully discrete (LLB-stable) finite element discretization
of Algorithm A is easily possible in the proof in [13]. Moreover, the stabilization effect in Algorithm A allows for equal order finite elements which violate the discrete Ladyshenskaja-Babuska-Brezzi condition for choices \( k \geq C h^2 \) [9]; see [13] for further details.

These \( L^2(\Omega) \)-error bounds show optimal convergence behavior for velocity fields computed from
Algorithm A, and only suboptimal convergence behavior for pressure iterates, which become optimal
in the negative norm \( H^{-1}(\Omega) = [H_0^1(\Omega)]^* \). This observation reflects observed boundary layers in the
computed iterate pressures, which are caused by the non-physical boundary condition in (1.10). In [16], it is conjectured that the thickness of the boundary layer is of order \( \mathcal{O}(\sqrt{k} \log(k)) \), and first order of convergence holds on compact subdomains \( \Omega_d \subset \Omega \), where \( \text{dist}(\Omega_d, \partial \Omega) \geq \delta \), for \( \delta = \sqrt{k} \log(k) \). The conjecture in [16] is based on a heuristic argument to control the error
due to pressure stabilization in the case of the stationary Stokes problem: as has been pointed out above, the perturbation of the incompressibility constraint, and prescription of nonphysical
boundary data for the pressure in Algorithm A are accounted for by considering a fully implicit
version of (1.11)–(1.13), where the key is again to study the following stationary problem, with
\( \varepsilon = k \): Find \( (u^\varepsilon, p^\varepsilon) \in [H_0^1(\Omega) \cap H^2(\Omega)] \times [L^2(\Omega) \cap H^1(\Omega)] \) such that
\begin{align}
-\nu \Delta u^\varepsilon + \nabla p^\varepsilon &= f, & \text{div} u^\varepsilon - \varepsilon \Delta p^\varepsilon &= 0 & \text{in } \Omega, \\
\partial_n p^\varepsilon &= 0 & \text{on } \partial \Omega. 
\end{align}

The following equations control errors \( e := u - u^\varepsilon \) and \( \eta = p - p^\varepsilon \), where \( (u, p) \) is strong solution of the stationary incompressible Stokes problem,
\begin{align}
-\nu \Delta e + \nabla \eta &= 0, & \text{div } e - \varepsilon \Delta \eta &= -\varepsilon \Delta p & \text{in } \Omega, \\
\partial_n \eta &= \partial_n p & \text{on } \partial \Omega. 
\end{align}

Thanks to \( e = \frac{1}{\nu} \nabla \Delta^{-1} \nabla \eta \), the second identity in (1.16) may be replaced as an equation only for the
pressure error \( \eta : \Omega \to \mathbb{R} \), which involves a pseudo-differential operator of order zero, such that
\begin{align}
\frac{1}{\nu} \nabla \Delta^{-1} \nabla \eta - \varepsilon \Delta \eta &= -\varepsilon \Delta p & \text{in } \Omega, & \partial_n \eta &= \partial_n p & \text{on } \partial \Omega.
\end{align}

Here, we denote \( \psi := \Delta^{-1} w \) as solution of \( \Delta \psi = w \) in \( \Omega \), and \( \psi = 0 \) on \( \partial \Omega \). In [16], the
operator \( \text{div} \Delta^{-1} \nabla \) is replaced by the identity operator, and the above control of boundary layers are then derived. Our first result in this work is a rigorous derivation of arising boundary layers
caused by Chorin’s method for existing strong solutions. We use the following notation, with
\( 0 < \delta < \frac{1}{2} \text{diam}(\Omega) \),
\[ \Omega_{\delta} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > \delta \} \subset \Omega \subset \mathbb{R}^d. \]

**Theorem 1.2.** Suppose that \( (u, p) \) is strong solution of (1.1)–(1.4) up to time \( t_M = T \), with
\( f \in C([0, T]; L^2(\Omega)) \), \( r \geq 1 \), and \( \Omega \subset \mathbb{R}^d \) of class \( C^{2,\alpha} \), for \( 0 < \alpha < 1 \). Let \( \{ (u^m, p^m) \}_{m=1}^M \)
be iterates from Algorithm A. Let \( \Omega_{\delta} \subset \mathbb{R}^d \), for \( d = 2, 3 \). For sufficiently small \( k \leq k_0(T) \), and
$r^{-1} + (r')^{-1} = 1$, there holds

$$\max_{1 \leq m \leq M} \left[ r_m \| p(t_m, \cdot) - p^m \|_{L^2(\Omega_S)} \right] \leq C \sqrt{k} \left[ \sqrt{k} + \left( \frac{\sqrt{k}}{2r'} \right)^{\frac{3}{2r'}} \| f \|_{L^\infty(L^{2r})} + \exp\left( - \frac{\delta}{\sqrt{k}} \right) \right].$$

The result is verified in Section 2, where the key observation is a corresponding bound for errors $\eta \in W^{1,2r}(\Omega)$ which solve (1.18); cf.

\[ [20, \text{Prop. 2.2}] \]

several substeps, and starts with initial data Chorin-Uzawa scheme from this deficiency. In [13], it is necessary to develop projection methods of comparable computational effort that are exempted of almost chemical hydrodynamics, or magnetohydrodynamics) that cannot be avoided in general [15]. Hence, the pressure or the velocity gradient close to the boundary are needed; undesirable consequences are incurred at the back of Chorin’s projection method. Let $\beta, \eta, r$ be given.

**Remark 1.2.** 1. The decay property has first been studied by W. E and J.G. Liu in [4] via asymptotic analysis for a restricted model problem, where first order of convergence is established on compact subdomains.

2. Regularity of $\Omega \subset \mathbb{R}^d$ is required to use $L^p$-theory for strong solutions of the stationary incompressible Stokes problem; cf. [20, Prop. 2.2].

3. This result evidences a boundary layer of order $\delta = \sqrt{k} | \log(k) |$, with improved rate of convergence of almost $\frac{3}{4}$ on subdomains $\Omega_\kappa$, $\kappa \geq \delta$.

Apparent boundary layers of the projection method cannot be accepted when accurate data for the pressure or the velocity gradient close to the boundary are needed; undesirable consequences include pollution effects to involved quantities in more complex fluid flow problems (e.g., physical-chemical hydrodynamics, or magnetohydrodynamics) that cannot be avoided in general [15]. Hence, it is necessary to develop projection methods of comparable computational effort that are exempted from this deficiency. In [13], the Chorin-Uzawa scheme ($\beta = 0$) is proposed that avoids this drawback of Chorin’s projection method. Let $\beta \geq 0$. The method again splits each iteration step into several substeps, and starts with initial data $(u^0, \tilde{u}^0, p^0, \tilde{p}^0) \in J_1(\Omega) \times H^1_0(\Omega) \times [L^2_0(\Omega)]^2$.

**Algorithm B.** 1. For $0 \leq m \leq M$, let $(u^m, \tilde{u}^m, p^m, \tilde{p}^m) \in J_0(\Omega) \times H^1_0(\Omega) \times [L^2_0(\Omega)]^2$ be given. Find $\tilde{u}^{m+1} \in H^1_0(\Omega)$ such that

$$\frac{1}{\kappa} \{ \tilde{u}^{m+1} - u^m \} - \beta \nabla \text{div } d_t \tilde{u}^{m+1} - \nu \Delta \tilde{u}^{m+1} + (u^m \cdot \nabla) \tilde{u}^{m+1} + \nabla \{ p^m - \tilde{p}^m \} = f^{m+1} \quad \text{in } \Omega. \tag{1.19}$$

2. Find $(u^{m+1}, \tilde{p}^{m+1}) \in J_0(\Omega) \times L^2_0(\Omega)$ that solves

$$\frac{1}{\kappa} \{ u^{m+1} - \tilde{u}^{m+1} \} + \nabla \tilde{p}^{m+1} = 0, \quad \text{div } u^{m+1} = 0 \quad \text{in } \Omega, \tag{1.20}$$

$$\langle u^{m+1}, n \rangle = 0 \quad \text{on } \partial \Omega. \tag{1.21}$$

3. Determine $p^{m+1} \in L^2_0(\Omega)$ from

$$p^{m+1} = p^m - \alpha \text{div } \tilde{u}^{m+1} \quad \text{in } \Omega, \quad 0 < \alpha < 1. \tag{1.22}$$

Again, (1.20) may be reformulated as a Poisson problem for the pressure $\tilde{p}^{m+1} : \Omega \rightarrow \mathbb{R}$. Step 1 in Algorithm B leads to a coupled computation of components of the velocity field, which is due to the second term in (1.19), and which is the price we pay to better enforce the incompressibility constraint. However, other advantages of projection methods are still valid, since velocity and pressure iterates are computed independently.
By eliminating $\tilde{p}^{m+1}$ from the scheme, we easily obtain the following reformulation of the Chorin-Uzawa method as a semi-explicit ‘artificial compressibility method’ [20, 16, 13],

\begin{equation}
    d_t \tilde{u}^{m+1} - \beta \nabla \text{div} d_t u^{m+1} - \nu \Delta \tilde{u}^{m+1} + \nabla p^m = f^{m+1} \quad \text{in } \Omega,
\end{equation}

(1.23)\[ d_t \tilde{u}^{m+1} - \beta \nabla \text{div} d_t u^{m+1} - \nu \Delta \tilde{u}^{m+1} + \nabla p^m = f^{m+1} \quad \text{in } \Omega, \]

(1.24)\[ \text{div} \tilde{u}^{m+1} + \frac{k}{\alpha} d_t p^{m+1} = 0 \quad \text{in } \Omega, \]

(1.25)\[ \tilde{u}^{m+1} = \mathbf{0} \quad \text{on } \partial \Omega. \]

Since no unphysical boundary conditions are involved any more, and motivated by numerical experiments in [13], we conjecture accurate approximations of the pressure up to the boundary $\partial \Omega$. In fact, the following result is taken from [13, Theorem 8.2].

**Theorem 1.3.** Suppose that $(u, p)$ is strong solution of (1.1)–(1.4). For $0 < t_{m_1} = \mathcal{O}(1)$, let initial data $\tilde{p}^{m_1} = 0$, and $(u^{m_1}, \tilde{u}^{m_1}, p^{m_1}) \in J_0(\Omega) \times H_0^1(\Omega) \times L_0^2(\Omega)$ are such that

\begin{equation}
    \|u^{m_1} - u(t_{m_1}, \cdot)\|_{L^2} + \|\tilde{u}^{m_1} - u(t_{m_1}, \cdot)\|_{L^2} + \sqrt{k} \|p^{m_1} - p(t_{m_1}, \cdot)\|_{L^2} \leq Ck.
\end{equation}

Then, iterates $\{(\tilde{u}^m, p^m)\}_{m=m_1+1}^M$ solving (1.26)–(1.22), for $\beta = 0$ satisfy for $k \leq k_0(T)$,

\begin{equation}
    \max_{m_1+1 \leq m \leq M} \left[ \|\tilde{u}^m - u(t_m, \cdot)\|_{L^2} + \sqrt{k} \|p^m - p(t_m, \cdot)\|_{L^2} \right] + \left( k \sum_{m = m_1+1}^M \|\tilde{u}^m - u(t_m, \cdot)\|_{L^2} \right)^{\frac{1}{2}} \leq C \left( 1 + \log \frac{1}{k} k \right).
\end{equation}

The Chorin-Uzawa scheme suffers from the need of accurate initial data for the pressure function and additional regularity requirements for strong solutions of (1.1)–(1.4), and computational experiments are reported in [13] where rates of convergence deteriorate if one of the requirements is violated; see also the computational experiments reported in Section 5. However, both drawbacks can be avoided, if stretched time-grids $m \mapsto k_m = \min\{mk_0^2, k_0\}$ are used throughout the calculation that refine near the origin to attribute a singular weight to iterates as $t \to 0$. Obviously, this strategy asymptotically requires same computational costs; for further details on the revised Chorin-Uzawa scheme, we refer to [13, Chapter 10].

Despite of this improvement over the original Chorin-Uzawa method, and improved rates of convergence for gradients of computed velocity fields $\{\nabla u^m\}$, no improved error statements for pressure iterates over those of Theorem 1.3 are known so far. Our goal here is to construct a scheme that does so; for this purpose, we come back to Algorithm B, with $\beta > 1$, and change (1.22) to

\begin{equation}
    p^{m+1} = -\frac{1}{k} \text{div} \tilde{u}^{m+1}.
\end{equation}

In the sequel, we refer to (1.19)-(1.21), (1.28) as the Chorin-Penalty scheme. The following result will be shown in Section 3, which verifies optimal order of convergence for iterates $\{p^m\} \subset L_0^2(\Omega)$ of Algorithm B. As will be shown in Section 3.3, choices $\beta > 1$ are necessary to effectively account for the decoupling in the Chorin-Penalty scheme.

**Theorem 1.4.** Suppose that initial data $(u^0, \tilde{u}^0, p^0, \tilde{p}^0) \in H_0^1(\Omega) \times [L_0^2(\Omega)]^2$ satisfy

\begin{equation}
    \|u^0 - u_0\|_{L^2} + \|\tilde{u}^0 - u_0\|_{L^2} \leq Ck, \quad \tilde{p}^0 = p^0 = 0.
\end{equation}

Let $\{(\tilde{u}^m, p^m)\}_{m=1}^M$ solve (1.19)–(1.21), (1.28), for $\beta > 1$, and let $(u, p)$ be strong solution to (1.1)–(1.4). There exists $C = C(\beta, T) > 0$, such that

\begin{equation}
    \max_{1 \leq m \leq M} \left[ \|\tilde{u}^m - u(t_m, \cdot)\|_{L^2} + \sqrt{\tau_m} \|\tilde{u}^m - u(t_m, \cdot)\|_{H^1} + \tau_m \|p^m - p(t_m, \cdot)\|_{L^2} \right] \leq Ck.
\end{equation}
Table 1. Comparison of different first order projection methods: additional regularity requirements for strong solutions of (1.1)–(1.4) needed for convergence analysis are displayed in the first column. The subsequent two columns display convergence rates for different quantities, reflecting presence/absence of boundary layers. The last column indicates the corresponding quasi-compressibility method (QCM).

<table>
<thead>
<tr>
<th>Method</th>
<th>additional requirements</th>
<th>$\left( k \sum_{m=1}^{M} | \tilde{u}^m - u(t_m, \cdot) |_{H^1}^2 \right)^{1/2}$</th>
<th>$| p^M - p(t_M, \cdot) |_{L^2}$</th>
<th>Related QCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chorin</td>
<td>no</td>
<td>$\leq C \sqrt{k}$</td>
<td>$\leq C \frac{1}{\sqrt{\tau_M}} \sqrt{k}$</td>
<td>pressure stabilization</td>
</tr>
<tr>
<td>Chorin-Uzawa</td>
<td>yes</td>
<td>$\leq C k$</td>
<td>$\leq C (1 +</td>
<td>\log k</td>
</tr>
<tr>
<td>Chorin-Penalty</td>
<td>no</td>
<td>$\leq C k$</td>
<td>$\leq C \frac{1}{\tau_M} k$</td>
<td>penalty</td>
</tr>
</tbody>
</table>

Remark 1.3. 1. The Chorin-Penalty method can be reformulated as a semi-explicit penalty method, see (1.28). This stationary quasi-compressibility method has been analyzed in [13]. 2. The choice $\beta > 1$ is essential for the analysis of the Chorin-Penalty scheme. 3. No additional regularity requirements for strong solutions of (1.1)–(1.4) are needed, and $p^0 \equiv \bar{p}^0 \equiv 0$ is convenient.

Chorin’s original method is by no means the only existing projection method to solve (1.1)–(1.4): over the last four decades, many different further projection schemes have been developed, which use modified splitting strategies (e.g., the Gauge method, see e.g. [5, 11, 12], modified boundary conditions for pressure iterates), or variants which use higher order temporal discretization, in combination with different projection (or quasi-compressibility) strategies. All these schemes share the goal to circumvent the drawbacks of Chorin’s original projection scheme, and we refer to [7] for a review of the current state of the art. However, most numerical analyses of these schemes are based on the assumption that solutions of (1.1)–(1.4) are smooth, which leaves unclear whether these results apply to strong solutions which are known to exist. Hence, for general applicability, it is our goal in this work to validate the results discussed above on solid analytical grounds.

The remainder of this work is organized as follows: In Section 2, we quantify arising boundary layers due to pressure-stabilization methods for the stationary Stokes problem in the context of strong solutions, see Theorem 2.1. Theorem 1.4 then follows from this result. The main results and properties for the Chorin-Penalty method (i.e., Algorithm B, for $\beta > 0$) are given and compared with other methods discussed above in Table 1; in Section 3, we validate optimal convergence for pressure iterates to strong solutions of (1.1)–(1.4), as stated in Theorem 1.4. Comparative computational studies for Chorin-, Chorin-Uzawa-, and Chorin-Penalty schemes are reported in Section 4. A conclusion is given in Section 5.

2. Boundary layers in Chorin’s projection method

As is already discussed in the introduction, Chorin’s projection method, i.e, Algorithm A, suffers from marked boundary layers for the pressure error, which are bounded in Theorem 1.2. In this section, we verify this theorem by first studying a corresponding effect for strong solutions of the stationary, stabilized Stokes problem: For given $f \in L^2(\Omega)$, and $\varepsilon > 0$, find solutions $(u^\varepsilon, p^\varepsilon) \in$
\begin{align}
\{ \mathbf{H}_0(\Omega) \cap \mathbf{H}^2(\Omega) \} \times \{ L_0^2(\Omega) \cap H^1(\Omega) \} \text{ of } (\nu > 0)
\end{align}

(2.1) \quad -\nu \Delta \mathbf{u}^\varepsilon + \nabla p^\varepsilon = \mathbf{f}, \quad \text{div } \mathbf{u}^\varepsilon - \varepsilon \Delta p^\varepsilon = 0 \quad \text{in } \Omega,
(2.2) \quad \partial_n p^\varepsilon = 0 \quad \text{on } \partial \Omega.

This problem is a perturbation of the incompressible Stokes equation, where strong solutions
\((\mathbf{u}, p) \in \{ \mathbf{J}_1(\Omega) \cap \mathbf{H}^2(\Omega) \} \times \{ L_0^2(\Omega) \cap H^1(\Omega) \}\) solve \(-\nu \Delta \mathbf{u} + \nabla p = \mathbf{f}\) in \(\Omega\). In below, we consider compactly contained subsets \(\Omega_\delta\) of \(\Omega\),
\begin{align}
\Omega_\delta = \{ \mathbf{x} \in \Omega : \text{dist}(\mathbf{x}, \partial \Omega) > \delta \}, \quad \text{ for } 0 < \delta < \frac{1}{2} \text{diam}(\Omega).
\end{align}

**Theorem 2.1.** Let \(\Omega \subset \mathbb{R}^d\), for \(d \geq 2\), and \(\mathbf{f} \in \mathbf{L}^2(\Omega)\), such that \(\text{div } \mathbf{f} \in \mathbf{L}^2(\Omega)\). Suppose that
\((\mathbf{u}, p)\) is strong solution of the stationary, incompressible Stokes equation in \(\Omega \subset \mathbb{R}^d\), such that \(p \in W^{1,2r}(\Omega), r \geq 1\), and \((\mathbf{u}^\varepsilon, p^\varepsilon)\) is strong solution of (2.1)–(2.2). There holds
\begin{align}
\| p - p^\varepsilon \|^2_{L^2(\Omega_\delta)} \leq C \sqrt{\nu} \left[ \sqrt{\varepsilon \nu} \| \Delta p \|_{L^2} + \left( \frac{\sqrt{\varepsilon \nu}}{2r} \right)^{\frac{1}{2r}} \| \nabla p \|^2_{L^2} + \exp\left( -\frac{\delta}{\sqrt{\varepsilon \nu}} \right) \| \nabla p \|_{L^2} \right].
\end{align}

**Proof.** The equations for the error \((\mathbf{e}, \eta) := (\mathbf{u} - \mathbf{u}^\varepsilon, p - p^\varepsilon)\) are
\begin{align}
-\nu \Delta \mathbf{e} + \nabla \eta = 0, \quad \text{div } \mathbf{e} - \varepsilon \Delta \eta = -\varepsilon \Delta p \quad \text{in } \Omega,
\partial_n \eta = \partial_n p \quad \text{on } \partial \Omega.
\end{align}

Let \(\sigma(\mathbf{x}) = \exp\left( -\frac{\delta}{\sqrt{\varepsilon \nu}} \right) \text{min} \left[ \exp\left( \frac{d(\mathbf{x})}{\sqrt{\varepsilon \nu}} \right), \exp\left( \frac{\delta}{\sqrt{\varepsilon \nu}} \right) \right], \) where \(d(\mathbf{x}) = \text{dist}(\mathbf{x}, \partial \Omega)\).

We employ a duality argument: Find \((\mathbf{w}, q) \in \{ \mathbf{H}_0(\Omega) \cap \mathbf{H}^2(\Omega) \} \times \{ L_0^2(\Omega) \cap H^1(\Omega) \}\), such that
\begin{align}
-\nu \Delta \mathbf{w} + \nabla q = 0, \quad \text{div } \mathbf{w} - \varepsilon \Delta q = \sigma \eta \quad \text{in } \Omega,
\partial_n q = 0 \quad \text{on } \partial \Omega.
\end{align}

The following stability bound for solutions of (2.4)–(2.5) is easy to verify,
\begin{align}
\frac{1}{\nu} \| q \|^2_{L^2} + \nu \| \nabla \mathbf{w} \|^2_{L^2} + \varepsilon \| \nabla q \|^2_{L^2} \leq C \nu \| \sigma \eta \|^2_{L^2}.
\end{align}

By testing (2.4) with \((\mathbf{e}, \eta)\), and (2.3) with \((\mathbf{w}, q)\), we find
\begin{align}
\| \sqrt{\sigma} \eta \|^2_{L^2} = \varepsilon \langle \nabla q, \nabla \eta \rangle + \langle \text{div } \mathbf{w}, \eta \rangle - \langle \text{div } \mathbf{e}, q \rangle - \varepsilon \langle \nabla \eta, \nabla q \rangle + \varepsilon \langle \nabla q, \nabla p \rangle.
\end{align}

Since \(\exp\left( -\frac{d(\mathbf{x})}{\sqrt{\varepsilon \nu}} \right) = 1\) on \(\partial \Omega\), we further conclude for \(\tilde{\sigma}(\mathbf{x}) = \max \left[ \exp\left( -\frac{d(\mathbf{x})}{\sqrt{\varepsilon \nu}} \right), \exp\left( -\frac{\delta}{\sqrt{\varepsilon \nu}} \right) \right], \)
\begin{align}
\varepsilon \langle \partial_n p, \tilde{\sigma} q \rangle_{\partial \Omega} = \varepsilon \langle \Delta p, \tilde{\sigma} q \rangle + \varepsilon \langle \nabla p, \nabla (\tilde{\sigma} q) \rangle.
\end{align}

There remains to bound the last term in (2.8). By \(|\nabla \tilde{\sigma}| \leq \frac{\tilde{\sigma}}{\sqrt{\varepsilon \nu}}\), we conclude
\begin{align}
\varepsilon \langle \nabla p, \tilde{\sigma} \nabla q \rangle + \varepsilon \langle \nabla p, q \nabla \tilde{\sigma} \rangle
\leq C \left[ \frac{\varepsilon \nu}{4} \| \tilde{\sigma} \nabla p \|^2_{L^2(\Omega)} + \frac{\varepsilon}{4} \| \nabla p \|_{L^2(\Omega)} + \frac{\varepsilon}{4} \| q \|_{L^2(\Omega)} \right] + \frac{\varepsilon}{4} \| \nabla q \|^2_{L^2(\Omega)} + \frac{1}{\nu} \| q \|^2_{L^2(\Omega)}.
\end{align}
where $\mathcal{B} := \text{supp} |\nabla \sigma|$, which has $d$-dimensional Lebesgue measure $\mathcal{L}^d(\mathcal{B}) = O(\delta)$. Hence,

$$
\int_{\mathcal{B}} \exp \left( -\frac{d(x)}{\sqrt{\varepsilon v}} \right) \, dx = O(\sqrt{\varepsilon v}),
$$

and we may conclude from (2.7), (2.6) as follows, for $r, r' \geq 1$, such that $\frac{1}{r} + \frac{1}{r'} = 1$,

$$
\| \sqrt{\varepsilon} \eta \|_{L^2} \leq C \varepsilon v \| \Delta p \|_{L^2} + C \sqrt{\varepsilon v} \left[ \exp \left( -\frac{\delta}{\sqrt{\varepsilon v}} \right) \| \nabla p \|_{L^2(\Omega \setminus B)} + \| \exp \left( -\frac{d(x)}{\sqrt{\varepsilon v}} \right) \nabla p \|_{L^2(B)} \right]
$$

$$
\leq C \varepsilon v \| \Delta p \|_{L^2} + C \sqrt{\varepsilon v} \left[ \exp \left( -\frac{\delta}{\sqrt{\varepsilon v}} \right) \| \nabla p \|_{L^2} + \| \nabla p \|_{2r'} \left( \int_{B} \exp \left( -\frac{2r'd(x)}{2r^2} \right) \, dx \right)^{\frac{1}{2r'}} \right]
$$

$$
\leq C \varepsilon v \| \Delta p \|_{L^2} + C \sqrt{\varepsilon v} \left[ \exp \left( -\frac{\delta}{\sqrt{\varepsilon v}} \right) \| \nabla p \|_{L^2} + \left( \frac{\sqrt{\varepsilon v}}{2r'} \right)^{\frac{1}{2r'}} \| \nabla p \|_{L^{2r'}} \right].
$$

Thanks to $\| p \|_{L^2(\Omega_\delta)} \leq \| \sigma p \|_{L^2}$, this proves the assertion of the Lemma.

Let $1 \leq r < \infty$. By [20, Prop. 2.2], solutions of the incompressible Stokes equation with $f \in L^{2r}(\Omega)$ are strong, and satisfy $(u, p) \in W^{2,2r}(\Omega) \times W^{1,2r}(\Omega)$, provided the open bounded set $\Omega \subset \mathbb{R}^d$ is of class $C^{2,\alpha}$, for $0 < \alpha < 1$. Then, for large values $r \to \infty$, the above lemma motivates an $L^2(\Omega_\delta)$-error decay behavior for the pressure which is almost order $\frac{1}{2}$ in the interior, and deteriorates to order $\frac{1}{4}$ if errors on a boundary layer of width $\delta = \sqrt{\varepsilon v} |\log(\sigma v)|$ are included.

We use Theorem 2.1 to show Theorem 1.2. As has already be pointed out in the introduction, an error analysis for Algorithm A to optimally bound time-discretization, perturbation, and decoupling error effects is split into three steps; the most critical step to optimally bound arising errors is the one where the pressure stabilization effect is accounted for. As a consequence, we consider the following auxiliary problem: Let $u_0^0 = u_0$ be given. For every $0 \leq m \leq M$, find $(u^{m+1}_k, p^{m+1}) \in \{H^1_0(\Omega) \cap H^2(\Omega)\} \times \{L^2_0(\Omega) \cap H^1(\Omega)\}$ such that

$$
d_t u^{m+1}_k - \nu \Delta u^{m+1}_k + (P_j u^{m+1}_k \cdot \nabla) u^{m+1}_k + \nabla p^{m+1}_k = f^{m+1} \quad \text{in } \Omega,
$$

$$
\text{div } u^{m+1}_k - k \Delta p^{m+1}_k = 0 \quad \text{in } \Omega,
$$

$$
\partial_n p^{m+1}_k = 0 \quad \text{on } \partial \Omega.
$$

The following uniform estimates for solutions $\{(v^{m+1}, \pi^{m+1})\}_{m=0}^M \subset \{L^2_0(\Omega) \cap H^2(\Omega)\} \times \{L^2_0(\Omega) \cap H^1(\Omega)\}$ of (2.9) will be useful below; we refer to [16, 13] for a proof.

**Lemma 2.1.** Let (A1), (A2) be valid. Then, iterates $\{(v^{m+1}, \pi^{m+1})\}_{m=0}^M \subset \{L^2_0(\Omega) \cap H^2(\Omega)\} \times \{L^2_0(\Omega) \cap H^1(\Omega)\}$ of (2.9) satisfy

i) the following uniform bounds for values $i \in \{0, 1, 2\}$, and $r \in \{1, 2, 3\}$,

$$
\max_{0 \leq m \leq M} \left[ \frac{r+4}{r+m} \| d_r^i v^m \|_{W^{1,2}} \right] + \left( k \sum_{m=r}^{M} \frac{2(r+1)i}{r+m} \| d_r^i v^m \|_{W^{2,1}} \right)^{1/2} \leq C,
$$

where $[x]_+ := \max\{x, 0\}$. For values $i \in \{0, 1\}$, and $r \in \{1, 2, 3\}$, there holds

$$
\max_{m+1 \leq m \leq M} \left[ \frac{r+4}{r+m} \| d_r^i \pi^m \|_{W^{1,2}} \right] + \left( k \sum_{m=r}^{M} \frac{2(r+1)i}{r+m} \| d_r^i \pi^m \|_{W^{2,1}} \right)^{1/2} \leq C.
$$

ii) the following error estimates, for $(u, p)$ a strong solution of (1.1)–(1.4),

$$
\max_{1 \leq m \leq M} \left[ \| u(t_m, \cdot) - v^m \|_{L^2} + \sqrt{\tau_m} \left[ \| p(t_m, \cdot) - \pi^m \|_{H^{-1}} + \| u(t_m, \cdot) - v^m \|_{H^1} + \tau_m \| p(t_m, \cdot) - \pi^m \|_{L^2} \right] \leq C k \right.
$$


We are now in a position to sketch the proof of Theorem 1.2.

Proof. Consider the following quasi-stationary auxiliary problem, for every $0 \leq m \leq M$: Find $(U^{m+1}_k, \Pi^{m+1}_k) \in [H^1_0(\Omega) \cap H^2(\Omega)] \times [L^2_0(\Omega) \cap H^1(\Omega)]$ such that
\begin{align}
-\Delta U^{m+1}_k + \nabla \Pi^{m+1}_k &= F^{m+1} \quad \text{in } \Omega, \tag{2.12} \\
\text{div } U^{m+1}_k - k\Delta \Pi^{m+1}_k &= 0 \quad \text{in } \Omega, \tag{2.13} \\
\partial_n \Pi^{m+1}_k &= 0 \quad \text{in } \partial \Omega, \tag{2.14}
\end{align}
for $F^{m+1} := f^{m+1} + d_t v^{m+1} - (v^m \cdot \nabla) v^{m+1}$, and where $\{ (v^{m+1}, \pi^{m+1}) \}_{m=0}^M \subset [J_1(\Omega) \cap H^2(\Omega)] \times [L^2_0(\Omega) \cap H^1(\Omega)]$ solves (2.9). In order to apply Theorem 2.1, we need $F^{m+1} \in L^{2r}(\Omega)$ to apply [20, Prop. 2.2]. Thanks to the first result in Lemma 2.1, and Sobolev’s inequality for $d = 2, 3$, we easily obtain uniform bounds for $\tau_{m+1} \| F^{m+1} \|_{L^2_r}$, with $1 \leq r < \infty$, and hence the right-hand side of $(r \geq 1)$
\[ \| \pi^m - \Pi_k^m \|_{L^2(\Omega)} \leq C \sqrt{k\nu} \left[ \sqrt{k\nu} \| \Delta \pi^m \|_{L^2} + \left( \frac{\sqrt{k\nu}}{2r} \right)^2 \| \pi^m \|_{W^{1,2r}} + \exp \left( -\frac{\delta}{\sqrt{k\nu}} \right) \| \pi^m \|_{W^{1,2}} \right] \]
can be controlled uniformly if a time-weight $\tau_m$ is used. — Next, we employ the bound
\[ \max_{1 \leq m \leq M+1} \| \Pi_k^m - p_k^m \|_{L^2(\Omega)} \leq C k, \]
which is already known from [13], and easily follows from the fact that both, $p_k^{m+1}$ and $\Pi_k^{m+1}$ satisfy (2.13)--(2.14).

3. Algorithm B: The Chorin-Penalty projection method

In order to analyze the Chorin-Penalty scheme (1.19)-(1.20), (1.28), we use its reformulation as a semi-explicit penalty method,
\begin{align}
&d_t \left( \tilde{u}^{m+1}_k - \beta \nabla \text{div } \tilde{u}^{m+1}_k \right) - \nu \Delta \tilde{u}^{m+1}_k + (P_{\text{div}} \tilde{u}^{m+1}_k \cdot \nabla) \tilde{u}^{m+1}_k + \nabla P_{\text{div}}^{m+1} = f^{m+1} \quad \text{in } \Omega, \tag{3.1} \\
&\text{div } \tilde{u}^{m+1}_k + k P_{\text{div}}^{m+1} = 0 \quad \text{in } \Omega. \tag{3.2}
\end{align}
The main part of the subsequent analysis focuses on the fully implicit modification of (3.1),
\begin{align}
&d_t \left( \tilde{u}^{m+1}_k - \beta \nabla \text{div } \tilde{u}^{m+1}_k \right) - \nu \Delta \tilde{u}^{m+1}_k + (P_{\text{div}} \tilde{u}^{m+1}_k \cdot \nabla) \tilde{u}^{m+1}_k + \nabla Q^{m+1} = f^{m+1} \quad \text{in } \Omega, \tag{3.3} \\
\text{and strong solutions } \{ (\tilde{u}^{m+1}_k, q^{m+1}_k) \}_{m=0}^M \subset \left[ H^1_0(\Omega) \cap H^2(\Omega) \right] \times \left[ L^2_0(\Omega) \cap H^1(\Omega) \right], \text{ with } \tilde{u}^0_k = u_0. \tag{3.4}
\end{align}
In the sequel, we use the sequence $\{ (v^{m+1}, \pi^{m+1}) \}_{m=0}^M \subset [J_1(\Omega) \cap H^2(\Omega)] \times [L^2_0(\Omega) \cap H^1(\Omega)]$, with $v^0 = u_0$, which solves (2.9). We independently bound errors in (3.3), (3.2) which are due to perturbation of the incompressibility constraint in the linear case, which are amplified by the nonlinear term, and those due to decoupling in the Algorithm B.

3.1. Perturbation analysis for the penalized formulation (3.3), (3.2), Part I: The linear case. Let $w^0_k = u_0$. For every $0 \leq m \leq M$, let $(w^{m+1}_k, t^{m+1}_k) \in [H^1_0(\Omega) \cap H^2(\Omega)] \times [L^2_0(\Omega) \cap H^1(\Omega)]$ be the strong solution of
\begin{align}
&d_t \left( w^{m+1}_k - \beta \nabla \text{div } w^{m+1}_k \right) - \nu \Delta w^{m+1}_k + \nabla t^{m+1}_k = F^{m+1} \quad \text{in } \Omega, \tag{3.4} \\
&\text{div } w^{m+1}_k + k t^{m+1}_k = 0 \quad \text{in } \Omega, \tag{3.5}
\end{align}
where $F^{m+1} = f^{m+1} - (v^m \cdot \nabla) v^{m+1}$, for every $0 \leq m \leq M$. By Lemma 2.1, we have
\[ \max_{0 \leq m \leq M} \left[ \| F^{m+1} \|_{L^2} + \| d_t v^{m+1} \|_{L^2} \right] \leq C. \]
Problem (3.4)–(3.5) is a semi-discretization in time of a penalized version of the nonstationary, incompressible Stokes equations. In order to gain optimal rates of convergence with respect to $k > 0$ towards strong solutions of the nonstationary, incompressible Stokes equations, we need to study the following auxiliary problem first: For every $0 \leq m \leq M$, let $(W_k^{m+1}, B_k^{m+1}) \in [H^1_0(\Omega) \cap H^2(\Omega)] \times [L_2^0(\Omega) \cap H^1(\Omega)]$ be the strong solution of
\begin{align}
-\nu&\Delta W_k^{m+1} + \nabla B_k^{m+1} = F^{m+1} - d_t v^{m+1} & \quad & \text{in } \Omega, \\
\text{div } W_k^{m+1} + k B_k^{m+1} = 0 & \quad & \text{in } \Omega.
\end{align}

The following convergence properties have been shown in [13],
\begin{align}
\|W_k^{m+1} - v^{m+1}\|_{H^1} + \|B_k^{m+1} - \pi^{m+1}\|_{L^2} \leq C_k \|\pi^{m+1}\|_{L^2},
\end{align}
for every $0 \leq m \leq M$. By linearity of the problem (3.6)–(3.7), and Lemma 2.1, we easily obtain for $r \in \{1, 2, 3\}$,
\begin{align}
\max_{r \leq m \leq M} \tau_{m+1}^{r-1/2} \left[ \|d_t' (W_k^m - v^m)\|_{H^1} + \|d_t' (B_k^m - \pi^m)\|_{L^2} \right] \\
+ \left( k \sum_{m=r}^M \tau_{m-1}^2 \|d_t' (W_k^m - v^m)\|_{H^1}^2 \right)^{1/2} \leq C_k.
\end{align}

In the next step, we bound errors $(e_k^{m+1}, \eta_k^{m+1}) := (w_k^{m+1} - W_k^{m+1}, \tau_k^{m+1} - B_k^{m+1})$ between solutions of (3.3), (3.2), and (3.6)–(3.7). We have the following identities $(1 \leq m \leq M + 1)$,
\begin{align}
d_t (e_k^m - \beta \nabla \text{div } e_k^m) - \nu \Delta e_k^m + \nabla \eta_k^m = (\text{Id} - \beta \nabla \text{div }) d_t (v^m - W_k^m) & \quad \text{in } \Omega, \\
\text{div } e_k^m + k \eta_k^m = 0 & \quad \text{in } \Omega.
\end{align}
with $e_k^0 = 0$ on $\partial \Omega$, and $e_k^0 = 0$ on $\Omega$. We define $W_k^0 := u_0$. By testing (3.10)-(3.11) with $(e_k^{m+1}, \eta_k^{m+1})$, we arrive at
\begin{align}
\frac{1}{2} \max_{1 \leq m \leq M} \left[ \|e_k^m\|_{L^2}^2 + \beta \|\text{div } e_k^m\|_{L^2}^2 \right] + k^2 \sum_{m=1}^M \left[ \|d_t e_k^m\|_{L^2}^2 + \beta \|\text{div } d_t e_k^m\|_{L^2}^2 \right] \\
+k \sum_{m=1}^M \left[ \nu \|\nabla e_k^m\|_{L^2}^2 + k \|\eta_k^m\|_{L^2}^2 \right] \leq C k^2.
\end{align}

In the next step, we want to verify error bounds for the velocity gradient in $L^\infty(0, t_M; L^2)$. For this purpose, we make $r$ times ‘discrete derivatives’ (3.10), test with $\tau_{m+1} r \cdot d_t e_k^{m+1}$, $r \in \{1, 2\}$, and use (3.12) to find
\begin{align}
\max_{r \leq m \leq M} \tau_{m+1}^{r-1} \left[ \|d_t e_k^m\|_{L^2}^2 + \beta \|\text{div } d_t e_k^m\|_{L^2}^2 \right] + k \sum_{m=1}^M \tau_{m+1}^{r-1} \left[ \nu \|\nabla d_t e_k^m\|_{L^2}^2 + k \|\eta_k^m\|_{L^2}^2 \right] \\
\leq C k^2 + C k \sum_{m=1}^M \tau_{m+1}^{r-1} \left[ \nu \|\text{div } d_t e_k^m\|_{L^2}^2 + k \|\eta_k^m\|_{L^2}^2 \right].
\end{align}

A similar argument, and (3.12) lead to $(r \geq 1)$
\begin{align}
k \sum_{m=r}^M \tau_{m+1}^{r-1} \left[ \|d_t e_k^m\|_{L^2}^2 + \beta \|\text{div } d_t e_k^m\|_{L^2}^2 \right] + \max_{r \leq m \leq M} \tau_{m+1}^{r-1} \left[ \nu \|\nabla d_t e_k^m\|_{L^2}^2 + k \|\eta_k^m\|_{L^2}^2 \right] \\
\leq C k^2 + C k \sum_{m=r}^M \tau_{m+1}^{r-1} \left[ \nu \|\nabla d_t e_k^m\|_{L^2}^2 + k \|\eta_k^m\|_{L^2}^2 \right].
\end{align}
We can now combine (3.13), (3.14), and use (3.12) to verify the following bound, for \( r \in \{1, 2\} \),

\[
\max_{1 \leq m \leq M} \tau_m^{2r-1} \left[ \tau_m \left( \| d^r e_k^m \|^2_{L^2} + \beta \| \text{div} d^r e_k^m \|^2_{L^2} \right) + \nu \| \Delta d^r e_k^m \|^2_{L^2} + k \| d^r \gamma_k^m \|^2_{L^2} \right] \\
+k \sum_{m=r}^M \tau_m^{2r-1} \left[ \nu \tau_m \| \Delta d^r e_k^m \|^2_{L^2} + \beta \| \text{div} d^r e_k^m \|^2_{L^2} + k \tau_m \| d^r \gamma_k^m \|^2_{L^2} \right] \leq C k^2.
\]

(3.15)

Then, Lemma 2.1, (3.8), (3.9), and a stability result for the \( \text{div} \)-operator for (3.10) yield

\[
\max_{0 \leq m \leq M} \left[ \| u(t_m, \cdot) - w_k^m \|_{L^2} + \sqrt{\tau_m} \| u(t_m, \cdot) - w_k^m \|_{H^1} + \tau_m \| p(t_m, \cdot) - b_k^m \|_{L^2} \right] \leq C k,
\]

with the latter result being a consequence of a stability result for the divergence operator.

3.2. Perturbation analysis for the penalized formulation (3.3), (3.2), Part II: Extension to the nonlinear case. Because of (3.16), there remains to bound errors \((\xi^m, \chi^m) := (\tilde{u}_k^m - w_k^m, q_k^m - b_k^m)\) to know the error between strong solutions of (1.1)–(1.4), and (3.3), (3.2). We subtract the equations (3.4)–(3.5) of the linear auxiliary problem from corresponding ones (3.3)–(3.2). For every \( 1 \leq m \leq M + 1 \), there holds

\[
(\partial_t - \Delta) \tilde{\xi}_k^m = \nu \chi^m \text{div} \xi^m, \quad \text{in} \quad \Omega,
\]

(3.17)

\[
(\partial_t - \Delta) \tilde{\chi}_k^m = 0, \quad \text{in} \quad \Omega.
\]

(3.18)

We compute

\[
-(v^{m-1} \cdot \nabla) v^m + (P_{J_0} u_k^{m-1} - \nabla) u^m = (P_{J_0} \xi^m - \nabla) v^m - (P_{J_0} u_k^{m-1} - \nabla) u_k^m \quad \text{in} \quad \Omega,
\]

\[
-(P_{J_0} u_k^{m-1} - \nabla) (v^m - w_k^m) - (P_{J_0} (v^{m-1} - w_k^{m-1}) - \nabla) v^m.
\]

This observation, the skew-symmetricity property \((P_{J_0} \phi \cdot \nabla) \psi, \psi) = 0 \) for \( \psi \in H^1_0(\Omega) \), and \( H^1 \)-stability of \( P_{J_0} \) lead to

\[
\frac{1}{2} \left( \tau_m \| \xi^m \|^2_{L^2} + \nu \| \Delta \xi^m \|^2_{L^2} + k \| \chi^m \|^2_{L^2} \right) + k \sum_{m=1}^M \| d_t \xi^m \|^2_{L^2} + \beta \| \text{div} d_t \xi^m \|^2_{L^2} \\
+ k \sum_{m=1}^M \left( \tau_m \| \Delta \xi^m \|^2_{L^2} + k \| \chi^m \|^2_{L^2} \right) \leq \| \xi_0^m \|^2_{H^1(\Omega)} + C k \sum_{m=1}^M \| \Delta v^m \|^2_{L^2} \left[ \| \xi^m \|^2_{L^2} + \| v^m - w_k^m \|^2_{L^2} \right] + \left[ \left( P_{J_0} u_k^{m-1} - \nabla \right) (v^m - w_k^m), \xi^m \right] \\
\leq C k^2 + k \sum_{m=1}^M \| \nabla \xi^m \|^2_{L^2} + k \sum_{m=1}^M \left[ \| \nabla \xi^m \|^2_{L^2} + \| \Delta (v^{m-1} - w_k^{m-1}) \|^2_{L^2} \right] \times \\
\times \| \nabla (v^m - w_k^m) \|^2_{L^2} \| \xi^m \|^2_{L^2} + \| v^m - w_k^m \|^2_{L^2} \] \\
\leq C k^2 + k \sum_{m=1}^M \| \chi^m \|^2_{L^2} + \| v^m - w_k^m \|^2_{L^2} \right] \leq C k^2.
\]

We use Lemma 2.1, i), and (3.16) to conclude with the discrete Gronwall’s inequality that

\[
\max_{1 \leq m \leq M} \left[ \| \xi^m \|^2_{L^2} + \beta \| \text{div} \xi^m \|^2_{L^2} \right] + k \sum_{m=1}^M \| d_t \xi^m \|^2_{L^2} + \beta \| \text{div} d_t \xi^m \|^2_{L^2} \\
+ k \sum_{m=1}^M \left[ \nu \| \Delta \xi^m \|^2_{L^2} + k \| \chi^m \|^2_{L^2} \right] \leq C k^2.
\]

(3.19)
Next, making ‘discrete time-derivatives’ in (3.17), (3.18) with respect to time and finally test the system with \( (\tau_{m+1}^2 d_t \xi_m^{m+1}, \tau_{m+1} d_t \chi_{m+1}) \),

\[
\max_{2 \leq m \leq M} \tau_m^2 \left[ \|d_t \xi^m\|^2_{L^2} + \beta \|\text{div} \; d_t \xi^m\|^2_{L^2} \right] + k^2 \sum_{m=2}^M \tau_m^2 \left[ \|d_t^2 \xi^m\|^2_{L^2} + \beta \|\text{div} \; d_t^2 \xi^m\|^2_{L^2} \right] \\
+ k \sum_{m=2}^M \tau_m^2 \left[ ? \right] \leq C \; k \sum_{m=2}^M \left[ \text{NLT}^{m+1}_A(d_t \xi^m) \right],
\]

(3.20)

where

\[
\text{NLT}^{m+1}_A = (P_{J_0} d_t \xi^{m+1} \cdot \nabla) v^{m+1} + (P_{J_0} \xi^{m+1} \cdot \nabla) d_t v^{m+1} + (P_{J_0} d_t \tilde{u}_k^m \cdot \nabla) \xi^{m+1} \\
+ (P_{J_0} \xi^{m-1} \cdot \nabla) d_t v^{m+1} - (P_{J_0} d_t \tilde{u}_k^m \cdot \nabla)(v^{m+1} - w_k^{m+1}) \\
- (P_{J_0} \tilde{u}_k^m \cdot \nabla)(v^{m+1} - w_k^{m+1}) \\
- (P_{J_0} d_t (v^m - w_k^m) \cdot \nabla)v^{m+1} - (P_{J_0} (v^{m-1} - w_k^{m-1}) \cdot \nabla)d_t v^{m+1}.
\]

The first two terms on the right hand side of (3.21) in (3.20) can be easily controlled, as well as the fourth term. (Note that \( d_t \tilde{u}_k^{m+1} \) can be bounded uniformly in \( \ell^\infty(0, t_{M+1}; L^2(\Omega)) \).) By Lemma 2.1, i), and (3.15), the fifth, seventh and eighth term on the right hand side of (3.21) can be handled in a standard way. To bound the third and sixth term on the right hand side of (3.21), we use the following reformulation,

\[
(P_{J_0} d_t \xi_k^m \cdot \nabla) \xi^{m+1} = (P_{J_0} d_t (\xi_k^m \cdot \nabla)) v^{m+1} + (P_{J_0} d_t v^m \cdot \nabla) \xi^{m+1}, \\
(P_{J_0} \xi_k^m \cdot \nabla) d_t (v^{m+1} - w_k^{m+1}) = (P_{J_0} \xi^m \cdot \nabla) d_t (v^{m+1} - w_k^{m+1}) \\
+ (P_{J_0} w_k^m \cdot \nabla) d_t (v^{m+1} - w_k^{m+1}).
\]

(3.22)

We use Lemma 2.1, i), (3.9), and (3.15) to obtain the uniform estimate

\[
\max_{1 \leq m \leq M} \left[ \|\nabla w_k^m\|_{L^2} + \sqrt{\tau_m} \|\nabla d_t w_k^m\|_{L^2} \right] \leq C.
\]

(3.23)

We conclude

\[
\max_{2 \leq m \leq M} \tau_m^2 \left[ \|d_t \xi^m\|^2_{L^2} + \beta \|\text{div} \; d_t \xi^m\|^2_{L^2} \right] + k^2 \sum_{m=2}^M \tau_m^2 \left[ \|d_t^2 \xi^m\|^2_{L^2} + \beta \|\text{div} \; d_t^2 \xi^m\|^2_{L^2} \right] \\
+ k \sum_{m=2}^M \tau_m^2 \left[ ? \right] \leq C \; k^2 \sum_{m=2}^M \tau_m \left[ \|d_t \xi^m\|^2_{L^2} + \|\text{div} \; d_t \xi^m\|^2_{L^2} \right],
\]

(3.24)
The last term in (3.24) can be controlled, if we test (3.17) by $d\xi^m$ and the ‘discrete time-derivative’ version of (3.18) by $\chi^m$,
\[
\|\dot{d}\xi^m\|_2^2 + \beta \frac{\nu}{2} \hat{d}_t \|\nabla \xi^m\|_2^2 + \nu k \frac{\nu}{2} \|\nabla \nabla d\xi^m\|_2^2 + \nu k \frac{\nu}{2} \|\nabla d\xi^m\|_2^2 + \nu k \frac{\nu}{2} \|\Delta d\xi^m\|_2^2 
\leq C \left[ \|\nabla \xi^m\|_2^2 + \|\nabla \xi^m\|_2^2 + \frac{\delta}{\delta \tau_m} \|\nabla \xi^m\|_2^2 \|\nabla \xi^m\|_2^2 + \|\nabla \xi^m - \xi^m\|_2^2 \right] 
\]
\[
+ \frac{C}{\delta} \|\nabla d\xi^m\|_2^2 + \frac{k}{\delta} \|\nabla d\xi^m\|_2^2 ,
\]
for $\delta, \kappa \geq 1$. If we choose $\kappa = \frac{1}{\sqrt{\delta}}$ and take $\delta$ sufficiently large, the last but one term can be absorbed on the left hand side, and we obtain
\[
k \sum_{m=1}^M \tau_m \left[ \|\nabla \xi^m\|_2^2 + \beta \|\nabla \nabla \xi^m\|_2^2 + \|\nabla \xi^m\|_2^2 \right] + \max_{1 \leq m \leq M} \tau_m \left[ \nu \|\nabla \xi^m\|_2^2 + \|\chi^m\|_2^2 \right] 
\leq Ck \sum_{m=1}^M \left[ \nu \|\nabla \xi^m\|_2^2 + \|\chi^m\|_2^2 + \nu \|\nabla \xi^m - \xi^m\|_2^2 + \frac{\nu}{\delta \tau_m} \|\nabla \xi^m\|_2^2 \right].
\]
As a consequence, (3.25) and (3.24) in combination with a stability result for the divergence operator give the desired bound
\[
\max_{1 \leq m \leq M} \left[ \|u(t_m, \cdot) - u^m_k\|_2^2 + \sqrt{\tau_m} \|u(t_m, \cdot) - u^m_k\|_{H^1} + \tau_m r(t_m) - q^m_k \|_2^2 \right] \leq C k .
\]
We finish this part with useful uniform bounds for $\left\{ \left( \hat{u}_k^{m+1}, q_k^{m+1} \right) \right\}_{m=0}^M$: make $r = 1$ ‘discrete time-derivatives’ in (3.3), (3.2), for $r \in \{1, 2\}$, and test with $\tau_m \hat{u}_k^{m+1}$; in a second step, we make $r \hat{d}_t u_k^{m+1}$. A simple calculation then yields
\[
\max_{r \leq m \leq M} \tau_m \left[ \|\hat{d}_t \hat{u}_k^m\|_2^2 + \beta \|\nabla \hat{d}_t \hat{u}_k^m\|_2^2 \right] + k \sum_{m=r}^M \tau_m \left[ \|\hat{d}_t \hat{u}_k^m\|_2^2 + \beta \|\nabla \hat{d}_t \hat{u}_k^m\|_2^2 \right] 
\leq C ,
\]
thanks to (3.24), (3.26). Also, we may use Lemma 2.1, i), (3.9), (3.13), and (3.24) together with (3.18) to find
\[
\max_{1 \leq m \leq M} \left[ \tau_m \|\hat{d}_t q_k^m\|_2^2 \right] + k \sum_{m=1}^M \tau_m \|\hat{d}_t q_k^m\|_2^2 \leq C ,
\]

3.3. Perturbation analysis for the Chorin-Penalty scheme (3.1)–(3.2): Transition from the penalty method to the projection method. In this section, let $\left\{ \left( \hat{u}_k^{m+1}, q_k^{m+1} \right) \right\}_{m=0}^M$ denote the solution of (3.3), (3.2), and $\left\{ \left( \hat{u}_k^{m+1}, p_k^{m+1} \right) \right\}_{m=0}^M$ solves (3.1)–(3.2) — where the latter is the reformulation of Algorithm B. The last step to verify Theorem 1.4 consists in bounding the error $(E^m, \Pi^m) := (\hat{u}_k^m - u_k^m, q_k^m - p_k^m)$, which satisfies for every $1 \leq m \leq M$,
\[
d_t (E^m - \beta \nabla \Delta E^m) - \nu \Delta E^m + \nabla \Pi^m = -k \nabla \hat{d}_t q_k^m - Q(E^m) \quad \text{in } \Omega ,
\]
\[
div E^m + k \Pi^m = 0 \quad \text{in } \Omega ,
\]
with \( Q(E^{m+1}) := (PJ_0E^m \cdot \nabla)u^{m+1}_k + (PJ_0[u^m_k - E^m] \cdot \nabla)E^{m+1} \). The choice \( \beta > 1 \) is sufficient to obtain the following stability result for iterates, which we obtain when we test with \((E^m, \Pi^m)\), and employ (3.28),

\[
\frac{1}{2} \max_{1 \leq m \leq M} \left[ \|E^m\|^2_{L^2} + \beta \|\text{div} E^m\|^2_{L^2} \right] + \frac{k^2}{2} \sum_{m=1}^{M} \left[ \|d_t E^m\|^2_{L^2} + \beta \|\text{div} d_t E^m\|^2_{L^2} \right]
\]

\[
\leq k^2 \sum_{m=1}^{M} \left[ \|\nabla u^{m+1}\|^2_{L^2} + \frac{1}{4} \|\Pi^m\|^2_{L^2} \right] + \frac{k}{\nu} \sum_{m=1}^{M} \|E^m\|^2_{L^2} \|\nabla u^{m+1}\|^4_{L^2}, \quad \delta > 4.
\]

Thanks to (3.30), and \( \beta > 1 \), the first term on the right hand side can be absorbed on the left hand side. The second term can be bounded by \( Ck^2 \) through (3.27). To control the error for the pressure, we ‘make time-derivatives’ (3.29), test with \( \tau_{m+1} d_t E^{m+1} \), and note (3.27), (3.28).

\[
\frac{1}{2} \max_{1 \leq m \leq M} \tau_m^2 \left[ \|d_t E^m\|^2_{L^2} + \beta \|\text{div} d_t E^m\|^2_{L^2} \right] + \frac{k^2}{2} \sum_{m=1}^{M} \tau_m^2 \left[ \|d_t^2 E^m\|^2_{L^2} + \frac{1}{3} \|\text{div} d_t^2 E^m\|^2_{L^2} \right]
\]

\[
\leq Ck^2 + Ck^4 \sum_{m=2}^{M} \tau_m^2 \|d_t^2 q_k^m\|^2_{L^2} + 2 k \sum_{m=2}^{M} \tau_m \|d_t E^m\|^2_{L^2} + k \sum_{m=2}^{M} \tau_m^2 |\text{NLT}^m|.
\]

We skip the detailed study of NL \( T_B \), since it does not involve further difficulties superior to those detailed in Subsection 3.2 at this place. The crucial term, however, is the last but one term on the right hand side of (3.32). For this purpose, we test (3.29) with \( \tau_{m+1} d_t E^{m+1} \). As it turns out again, we have to deal with \( Q(E^{m+1}) \) more effectively, according to Subsection 3.2. By (3.27), (3.28),

\[
k \sum_{m=1}^{M} \tau_m \left[ \|d_t E^m\|^2 + [\beta - 1] \|\text{div} d_t E^m\|^2 \right] + \frac{1}{2} \max_{1 \leq m \leq M} \tau_m \left[ \nu \|\nabla E^m\|^2 + k \|\Pi^m\|^2 \right]
\]

\[
+ \frac{k^2}{4} \sum_{m=1}^{M} \tau_m \left[ \|\nabla d_t^2 E^m\|^2 + k \|d_t \Pi^m\|^2 \right] \leq C_k k^2 + \frac{k}{\delta^3/2} \sum_{m=1}^{M} \tau_m^2 \|\nabla d_t E^m\|^2, \quad \delta > 1.
\]

If we choose \( \delta \) sufficiently large, we may use (3.33) to control the last but one term on the right hand side of (3.32). Together with (3.28)_1, a standard argumentation then establishes the bound \( \max_{0 \leq m \leq M} \tau_m \|\Pi^m\| \leq Ck \). Together with (3.31) through (3.33) and the results from Subsections 3.1 to 3.3, this settles the proof of Theorem 1.4.

### 4. Computational Experiments

We computationally compare Chorin’s method (Algorithm A) with its variants, i.e., Chorin-Uzawa method (Algorithm B, for \( \alpha \in (0, 1) \)), and Chorin-Penalty method ((1.19)–(1.21), (1.28), for \( \beta > 1 \)). Our main focuses are

(i) to compare possible boundary layers of \( m \mapsto [p(t_m, \cdot) - p^m] \in L^2_\theta(\Omega) \), including their evolution in time,
(ii) to compare possible transition layers caused by starting with initial pressure data $p^0$ for Chorin-Uzawa, and Chorin-Penalty scheme, where $\| p^0 - p(0, \cdot) \|_{L^2}$ is large, and

(iii) to study convergence behavior of computed pressures for Chorin-Uzawa, and Chorin-Penalty method, depending on different choices of $\alpha, \beta$.

**Example 4.1.** Let $\Omega = (0,1)^2 \subset \mathbb{R}^2$, and

$$u(x, y, t) = \left( \frac{x^2(1 - x)^2(2y - 6y^2 + 4y^3)}{-y^2(1 - y)^2(2x - 6x^2 + 4x^3)} \right), \quad p(x, y, t) = (x^2 + y^2 - \frac{2}{3})(1 + t^2),$$

be solutions of the evolutionary Stokes problem, i.e., $f: \Omega_T \to \mathbb{R}^2$ is computed from (1.1)–(1.4), where the nonlinear term in (1.1) is neglected. Let $T_h$ be an equidistant triangulation of $\Omega$ of mesh-size $h = 1/30$, and $k = 2^{j}/500$, $j = 0, 1, 2, \ldots$ an equidistant time-step for the time interval $[0, 1]$. The LBB-stable MINI-Stokes element is used for spatial discretization of the three projection methods.

Snapshots for the pressure error in Figure 1 show marked boundary layers for the pressure (first line), as opposed to almost uniform errors for Chorin-Uzawa (middle line), and Chorin-Penalty (last line); comparative plots in the last line for corresponding $L^2$-errors motivate that significant errors close to the boundary control the overall error in Chorin’s scheme. Corresponding profiles are obtained for plots of $m \mapsto \text{div} \tilde{u}^m$.

Both, Chorin-Uzawa ($\alpha \in (0, 1)$) and Chorin-Penalty ($\beta > 1$) involve additional parameters; its dependence is studied in Figure 2, which motivates values $\alpha \approx 1$, and $\beta \approx 1$.

Over the last decade, different projection schemes have been developed, and tested in academic examples, whose performance crucially relies on given initial functions $q \equiv p(0, \cdot)$; cf. [7] for further details. In fact, to compute pressure initial functions in general amounts to solving the following problem (for nonstationary Stokes)

$$\Delta q = \text{div} f(0, \cdot) \quad \text{in} \ \Omega, \quad \partial_n q = [f(0, \cdot) + \nu \Delta u_0] \cdot n \quad \text{on} \ \partial \Omega,$$

where optimal convergence for (finite element) approximations $q_h \approx q$ is not clear. In our case, choosing accurate data $p^0 \approx p(0, \cdot)$ for the Chorin-Uzawa method has been pointed out to be crucial in Theorem 1.3; in contrast, Chorin-Penalty is designed in order to avoid boundary layers for the pressure error in space, and also perform optimally for zero initial pressure data (Theorem 1.4). Figure 3 supports these theoretical results: we observe marked transition layers for Chorin-Uzawa in the case of ‘noncorrect’ initial pressures, while $L^2$-errors of the pressure in Chorin-Penalty are almost instantaneously reduced to spatial discretization errors.

5. Conclusion

In recent papers [11, 12], the authors stress the importance to construct and analyze practical projection methods under realistic regularity assumptions — which is also the guideline in this paper. Over the last decade, several projection methods are studied in the literature where (i) smooth solutions to (1.1)–(1.4), and (ii) accurate initial pressure data are assumed, leaving serious doubts on the applicability of these results to more realistic situations of incompatible data and limited solution’s regularity.

Projection methods are efficient methods to approximate strong solutions of the nonstationary incompressible Navier-Stokes equations; the most well-known example is Chorin’s method, which suffers from marked pressure error boundary layers. We give a first rigorous analysis of its structure in the case of existing strong solutions of (1.1)–(1.4) (Theorem 1.2). The new Chorin-Penalty method is proposed, and optimal (i.e., first order) rate of convergence for the pressure is proved (Theorem 1.4), which reflects uniform, optimal convergence behavior up to the boundary. Comparative computational studies illustrate that the Chorin-Penalty method is exempted from the
deficiencies of the Chorin method, in the way that no significant pressure errors arise close to the boundary.

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References


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Figure 1. Example 4.1: $L^2$ errors of $p(t_m, \cdot) - p^m$ at time $t_m = 1$, for $k = 1/500$ (1st line), $k = 1/1000$ (2nd line), and $k = 1/2000$ (3rd line), and evolution plot (4th line), for Chorin (left), Chorin-Uzawa (middle, $\alpha = 0, 9$), and Chorin-Penalty (right, $\beta = 1.1$).
Figure 2. Example 4.1: $L^2$-errors of pressure at $t = 1$ for (i) Chorin-Uzawa with exact initial pressure, for different $\alpha \in (0, 1)$ (left), and (ii) for Chorin-Penalty for different $\beta \in (1, 100)$ (right) ($k = 1/500$).

Figure 3. Example 4.1: Correct vs. non-correct initial pressures for Chorin-Uzawa ($\alpha = 0.9$), and comparison with Chorin-Penalty ($\beta = 1.1$) for $k = 1/500$ (left), $k = 1/1000$ (middle), $k = 1/2000$ (right).