CONVERGENCE OF A FINITE ELEMENT BASED SPACE-TIME DISCRETIZATION IN ELASTODYNAMICS

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ABSTRACT. We study a finite element based space-time discretization of the elastodynamics equation
\[ u_t^\varepsilon - \text{div} (\nabla u^\varepsilon) - \varepsilon \Delta u_t^\varepsilon = 0 \]
for \( \varepsilon \geq 0 \), where \( \sigma = D\phi \), and \( \phi \) is a nonconvex function. Unconditional convergence
for regularization parameters \( \varepsilon > 0 \) of iterates to weak solutions, and conditional convergence
for the limiting problem \( \varepsilon = 0 \) of iterates towards generalized solutions is shown in a
general setting of data. Computational experiments are included to motivate formation
and propagation of 2D-microstructures for decreasing values of the regularization parameter.

1. INTRODUCTION

We show convergence of a fully discrete scheme to approximate generalized solutions
\( u^\varepsilon : \Omega_T \to \mathbb{R}^m \) of the nonlinear hyperbolic system
\[
(1.1) \quad u_t^{\varepsilon} - \text{div} (\nabla u^\varepsilon) - \varepsilon \Delta u_t^\varepsilon = 0 \quad \text{in} \quad \Omega_T := (0, T) \times \Omega,
(1.2) \quad u^\varepsilon = 0 \quad \text{on} \quad \partial \Omega_T := (0, T) \times \partial \Omega,
(1.3) \quad u^\varepsilon (0, \cdot) = u_0, \quad u^\varepsilon (0, \cdot) = v_0 \quad \text{on} \quad \Omega,
\]
for \( \varepsilon \geq 0 \). Here, \( \Omega \subset \mathbb{R}^n \) is a bounded domain, and \( T > 0 \). The stress \( \sigma = D\phi \) is a non-monotone
function, with nonconvex potential \( \phi : \mathbb{R}^{m \times n} \to \mathbb{R} \).

Problem (1.1)–(1.3) has been studied in several analytical contributions [1, 6, 7, 8, 10, 11, 12, 14, 15, 17, 18, 19]. Existence of weak solutions
for \( \varepsilon > 0 \) has been verified in [10], where the implicit Euler method is used to semidiscretize the problem in time: Let \( I_k = \{t_j\}_{j=1}^J \)
an an equidistant net of mesh width \( k = t_j - t_{j-1} \) to discretize \( [0, T] \), and \( j \geq 1 \). We set
\[
(1.4) \quad u_{\varepsilon}^{-1} := u_0 - kv_0, \quad \text{and} \quad u_0^\varepsilon := u_0.
\]
For \( j \geq 1 \), and given \( u_{\varepsilon}^{j-i} : \Omega \to \mathbb{R}^m \), for \( i = 1, 2 \), then \( u_{\varepsilon}^j : \Omega \to \mathbb{R}^m \) solves
\[
(1.5) \quad u_{\varepsilon}^j - 2u_{\varepsilon}^{j-1} + u_{\varepsilon}^{j-2} - k^2 \text{div} (\nabla u_{\varepsilon}^j) - \varepsilon k \Delta [u_{\varepsilon}^j - u_{\varepsilon}^{j-1}] = 0 \quad \text{in} \quad \Omega,
(1.6) \quad u_{\varepsilon}^j = 0 \quad \text{on} \quad \partial \Omega.
\]
Provided that \( \phi \) satisfies certain smoothness and asymptotic growth assumptions, together with
the Andrews-Ball condition (H3) given below, weak solutions of (1.1)–(1.3) may be constructed as
proper limits of existing solutions to the semidiscretized problem. Uniqueness of weak solutions
in the case of globally Lipschitz-continuous \( \sigma \) is also verified in [10]. Qualitative studies, including
long-time dynamics are performed in [11, 12, 1].

Existence of generalized solutions to (1.1)–(1.3) in the case \( \varepsilon = 0 \) is studied in [6, 7], and [14]. In
the first references, a time-discretization by implicit Euler method is again employed to construct
Young-measure valued solutions for every \( t_j \) (\( j \in \mathbb{N} \)), using relaxation theory. This direct approach,
however, requires convexity of the relaxed energy to validate the crucial energy decay property
for corresponding first moments of the Young-measure valued solutions, to afterwards identify limits
for \( k \to 0 \) as generalized solutions of (1.1)–(1.3). A different approach is proposed in [14], where
generalized solutions to (1.1)–(1.3) (\( \varepsilon = 0 \)) are constructed as proper limits of corresponding weak

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solutions for $\varepsilon > 0$; here, no (poly-)convexity assumption of related relaxed energies as in [8] is needed, and Andrews-Ball condition is sufficient to verify solvability.

So far, numerical analysis for (1.1)–(1.3) is only rare. In [3], the authors establish strong convergence with optimal rates for a fully discrete version of (1.5)–(1.6) in the case $\varepsilon = 1$, using low order finite elements. However, the error analysis requires additional regularity assumptions for solutions of (1.1)–(1.3) which are not known to hold in general, and it is pointed out in [3, Remarks (4,6) on p.s 76/77] that convergence towards weak solutions is not known, and that ‘a mild constraint on discretization parameters in space and time might be necessary to guarantee convergence’ in this context [3, Remark (5) on p. 77]. Moreover, usage of discrete Gronwall’s lemma leads to an exponential factor which grows in terms of $\varepsilon^{-1}T$ for general $\varepsilon > 0$, thus leaving unclear how to efficiently balance temporal and spatial discretization parameters with the regularization parameter to converge to solutions of (1.5)–(1.6), for $\varepsilon = 0$.

In [4], a fully discrete scheme is proposed for $\varepsilon = 0$, where a discrete Young-measure valued solution which solves the problem is shown to exist. As is argued in [4, Remark 3.7], by first driving the spatial mesh size $h$ to zero, and afterwards tending $k \to 0$, measure-valued solutions to the limiting problem (1.1)–(1.3) may be recovered. However, this leaves open the practically relevant question on whether solutions of the proposed scheme converge to those of (1.1)–(1.3) when spatial and temporal discretization parameters simultaneously tend to zero, and whether a mesh constraint $F(k, h) \geq 0$ applies to make convergence hold for $(k, h) \to 0$, or not.

The goal of this paper is to approximate weak ($\varepsilon > 0$) and generalized ($\varepsilon = 0$) solutions of (1.1)–(1.3). In a first step, we study a finite-element based discretization (see Scheme A in Section 3) for values $\varepsilon > 0$, verify a discrete energy decay property for corresponding iterates, and establish unconditional convergence in Theorem 3.1 for $(k, h) \to 0$ towards weak solutions of (1.1)–(1.3), provided triangulations $T_h$ of $\Omega \subset \mathbb{R}^n$ meet the requirements of [5] — which are satisfied in the case of quasiuniform meshes, for example. This result answers the questions from above raised in [3]. Then, we prove a discrete energy decay property for iterates of Scheme A in case that a mild constraint $F(\varepsilon, k) \geq 0$ is additionally met, and eventually show convergence towards generalized solutions of (1.1)–(1.3) (for $\varepsilon = 0$) when all, $(\varepsilon, k, h)$ simultaneously tend to zero; see Theorem 3.2. Hence, the latter result justifies convergence of solutions of the proposed Scheme A towards generalized solutions of (1.1)–(1.3) (for $\varepsilon = 0$), as opposed to a more costly direct discretization which uses discrete Young-measure valued solutions.

The remainder of the paper is as follows: Section 2 provides necessary background material for problem (1.1)–(1.3), as well as further details on used notation. Section 3 proposes a fully discrete scheme and studies both, stability and convergence for $(k, h) \to 0$, and $(\varepsilon, k, h) \to 0$. Section 4 reports on computational experiments to study formation and propagation of two-dimensional microstructures in the case of positive and vanishing values of $\varepsilon \geq 0$ in (1.1)–(1.3).

2. Preliminaries and Notation

2.1. Weak solutions ($\varepsilon > 0$) resp. generalized solutions ($\varepsilon = 0$) of (1.1)–(1.3). For $\varepsilon > 0$, weak solutions to (1.1)-(1.3) are constructed in [10], provided $\sigma = D\phi$ satisfies the following hypotheses:

\begin{itemize}
  \item[(H1)] $\phi \in C^2(\mathbb{R}^{m \times n})$,
  \item[(H2)] There exist constants $C_i > 0$, and $2 \leq p < \infty$, such that
    \[ C_1 |F|^p - C_2 \leq \phi(F) \leq C_2(|F|^p + 1), \quad |\sigma^{(i)}(F)| \leq C_2(|F|^{p-1-i} + 1) \quad \text{for } i = 0, 1. \]
  \item[(H3)] There exists $K > 0$, such that
    \[ \langle \sigma(F_1) - \sigma(F_2), F_1 - F_2 \rangle \geq -K|F_1 - F_2|^2 \quad \forall F_1, F_2 \in \mathbb{R}^{m \times n}. \]
\end{itemize}
Note that some polynomial asymptotic growth behavior for first and second order derivatives of \( \phi \) is assumed in (H2). Hypothesis (H3) is often referred to as Andrews-Ball condition. In the following, let \( E[u, v] = \int_{\Omega} \left[ \phi(|\nabla u|) + \frac{1}{2}|v|^2 \right] \, dx \) denote the corresponding energy functional.

**Definition 2.1.** Let \( \Omega \subset \mathbb{R}^n \) be open and bounded, and \( \phi \) satisfies assumptions (H1), (H2), (H3). Suppose that \( (u_0, v_0) \in W_0^{1,p}(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m) \). A function \( u^\varepsilon : \Omega_T \to \mathbb{R}^m \) is called a weak solution of (1.1)–(1.3), for \( \varepsilon > 0 \), if for all \( T > 0 \) there hold

(i) \( u^\varepsilon \in L^\infty(0, T; W_0^{1,p}(\Omega, \mathbb{R}^m)) \cap W^{1,\infty}(0, T; L^2(\Omega, \mathbb{R}^m)) \cap W^{1,2}(0, T; W_0^{1,2}(\Omega, \mathbb{R}^m)) \),

(ii) initial data are attained, i.e., for \( t \to 0 \),

\[
 u^\varepsilon(t, \cdot) \to u_0 \quad \text{in} \quad W_0^{1,p}(\Omega, \mathbb{R}^m), \\
 u^\varepsilon_t(t, \cdot) \to v_0 \quad \text{in} \quad L^2(\Omega, \mathbb{R}^m).
\]

(iii) for all \( \xi \in C_0^\infty([0, T) \times \Omega; \mathbb{R}^m) \) there holds

\[
 \int_0^T \int_{\Omega} \left[ -u_t \cdot \xi_t + \sigma(\nabla u^\varepsilon) \cdot \nabla \xi + \varepsilon \nabla u^\varepsilon_t \cdot \nabla \xi \right] \, dx \, dt = \int_{\Omega} v_0 \cdot \xi(0, \cdot) \, dx.
\]

(iv) for almost every \( t \in [0, T] \),

\[
 E[u^\varepsilon(t, \cdot), u^\varepsilon_t(t, \cdot)] - E[u_0, v_0] \leq -\varepsilon \int_0^t \int_{\Omega} |\nabla u^\varepsilon|^2 \, dx \, ds.
\]

Existence of weak solutions to (1.1)–(1.3) \( (\varepsilon > 0) \) has been shown in [10]; moreover, weak solutions are unique provided \( \sigma \) is globally lipschitz. — For \( \varepsilon = 0 \), gradients of solutions of (1.1)–(1.3) are measure-valued, which leads to the following notion of generalized solutions. In the following, we employ Radon measures \( \nu \equiv \{\nu_{t,x}\} \in L_w^\infty(\Omega_T, \mathcal{M}(\mathbb{R}^{m \times n})) := (L^1(\Omega_T; C_0(\mathbb{R}^{m \times n})))' \).

**Definition 2.2.** Let \( \Omega \subset \mathbb{R}^n \) be open and bounded, and \( \phi \) satisfies assumptions (H1), (H2), (H3). Suppose that \( (u_0, v_0) \in W_0^{1,p}(\Omega, \mathbb{R}^m) \times W^{1,2}(\Omega, \mathbb{R}^m) \). A tuple \( (u, \nu) \) is called a generalized solution of (1.1)–(1.3) for \( \varepsilon = 0 \), if for all \( T > 0 \) there hold

(i) \( (u, \nu) \in \left[ L^\infty(0, T; W_0^{1,p}(\Omega, \mathbb{R}^m)) \cap W^{1,\infty}(0, T; L^2(\Omega, \mathbb{R}^m)) \right] \times L_w^\infty(\Omega_T, \mathcal{M}(\mathbb{R}^{m \times n})) \), where

\[
 \nu = \{\nu_{t,x}\}_{t,x} \quad \text{is a probability measure},
\]

(ii) initial data are attained, i.e., for \( t \to 0 \),

\[
 u(t, \cdot) \to u_0 \quad \text{in} \quad W_0^{1,p}(\Omega, \mathbb{R}^m), \\
 u^\varepsilon_t(t, \cdot) \to v_0 \quad \text{in} \quad L^2(\Omega, \mathbb{R}^m).
\]

(iii) for all \( \xi \in C_0^\infty([0, T) \times \Omega; \mathbb{R}^m) \),

\[
 \int_0^T \int_{\Omega} \left[ -u_t(x, t) \cdot \xi_t(x, t) + \nabla \xi(x, t) \cdot \int_{\mathbb{R}^{m \times n}} \sigma(\lambda) \, d\nu_{t,x}(\lambda) \right] \, dx \, dt = \int_{\Omega} v_0 \cdot \xi(0, \cdot) \, dx,
\]

and

\[
 \nabla u(t, x) = \int_{\mathbb{R}^{m \times n}} \lambda \, d\nu_{t,x}(\lambda) \quad \text{a.e. in} \quad \Omega_T.
\]

Existence of generalized solutions to (1.1)–(1.3) has been shown in [6, 7] for cases where the quasiconvexification of \( \phi \) is convex. In [14], generalized solutions are obtained as proper limits of weak solutions of (1.1)–(1.3) for \( \varepsilon \to 0 \), for which assumptions (H1), (H2), (H3) are sufficient.

2.2. Finite element spaces, discrete time-derivatives, and interpolation. Throughout this paper we assume that \( T_h \) is a regular triangulation of the polygonal or polyhedral bounded Lipschitz domain \( \Omega \subset \mathbb{R}^n \) into triangles or tetrahedra of maximal diameter \( h > 0 \), for \( n = 2, 3 \), which satisfies the requirements stated in [5]. Let

\[
 V_h := \{v_h \in C(\overline{\Omega}, \mathbb{R}^m) : v|_K \text{ affine function } \forall K \in T_h \} \subset W^{1,p}(\Omega)
\]
denote the lowest order conforming finite element space. Given the set of all nodes (or vertices) $N_h$ in $T_h$ and letting $\{\psi_z : z \in N_h\}$ denote the nodal basis in $V_h$, we define the nodal interpolation operator $I_h : C(\Omega, \mathbb{R}^m) \to V_h$ by $I_h \psi := \sum_{z \in N_h} \psi(z) \psi_z$, for $\psi \in C(\Omega, \mathbb{R}^m)$. Projections $P_X : X \to V_h$, for $X = L^2(\Omega), W_0^{1,2}(\Omega)$ are defined via $(u - P_X u, \Phi)_X = 0$, for all $\Phi \in V_h$; in particular, there holds $P_X u \to u$ in $X$, for $h \to 0$.

Given a time-step size $k > 0$ and a sequence $\{\varphi^j\}_{j \geq 0}$ in some vector space $X$, we set for $j \geq 1$, 
\[ d_t \varphi^{j+1} := k^{-1}(\varphi^{j+1} - \varphi^j) \quad \text{and} \quad d^2_t \varphi^{j+1} := d_t(d_t \varphi^{j+1}). \]

Piecewise constant interpolations of $\varphi^j$ are defined on $0 \leq t \leq Jk$ such that $t \in [jk, (j+1)k)$ for some $j \in \{0, \ldots, J - 1\}$ by 
\[ \varphi^j_k(t, \cdot) := \varphi^j \quad \text{and} \quad \varphi^+_k(t, \cdot) := \varphi^{j+1}, \]

and a piecewise affine interpolation is defined through 
\[ \hat{\varphi}_k(t, \cdot) := \frac{t-jk}{k} \varphi^{j+1} + \frac{(j+1)k-t}{k} \varphi^j. \]

Note that $\|\varphi^+_k - \hat{\varphi}_k\|_X + \|\varphi^-_k - \hat{\varphi}_k\|_X \leq 2k\|\varphi_t\|_X$.

Throughout the paper, let $C > 0$ be a generic constant which depends on given data $\Omega, T, u_0, v_0, \phi,$ and geometric properties of $T_h$.

3. A Finite element based discretization of (1.1)–(1.3), with $\varepsilon \geq 0$

In [10], the authors introduce and study the semidiscretization in time given by (1.4)–(1.6) for the case $\varepsilon = 1$, where iterates $u^j : \Omega \to \mathbb{R}^m$, for $j \geq 1$ minimize $E^j_k : W_0^{1,p}(\Omega, \mathbb{R}^m) \to \mathbb{R}$, with 
\[ E^j_k[w] := \int_\Omega \left[ \phi(\nabla w) + \frac{\varepsilon}{2k} |\nabla (w - u^j)|^2 + \frac{1}{2k^2} |w - 2u^j - u^j - 2u^j|^2 \right] \, dx. \]

Attainment of minimizers follows from convexity of $F \mapsto \tilde{\phi}(F) := \phi(F) + \frac{\varepsilon}{2k} |F|^2$ for $k \leq \frac{\varepsilon}{2}$, since (H3) implies 
\[ \langle D\tilde{\phi}(F_1) - D\tilde{\phi}(F_2), F_1 - F_2 \rangle \geq (-K + \frac{\varepsilon}{k}) |F_1 - F_2|^2 \quad \forall F_1, F_2 \in \mathbb{R}^{m 	imes n}. \]

As a consequence, $E^j_k : W_0^{1,p}(\Omega, \mathbb{R}^m) \to \mathbb{R}$ is weakly lower semicontinuous for every $j \geq 1$.

The second step in their construction is to derive a (semi-)discrete energy inequality to control iterates of (1.4)–(1.6). For every $\delta \in (0, 1)$, and all $k \leq 2\delta k_0$, there holds
\[ \sup_{j \in N_0} E[u^j_k, d_t u^j_k] + k \sum_{j=1}^\infty \int_\Omega \left[ \varepsilon (1 - \delta) |\nabla d_t u^j_k|^2 + \frac{\varepsilon k}{2} |d_t u^j_k|^2 \right] \, dx \leq E[u^0_\varepsilon, v^0_\varepsilon]. \]

Therefore, uniform bounds are available, and convergence of globally continuous, piecewise affine interpolations $\{\hat{u}_k\}$ to a limit $u^\varepsilon : \Omega_T \to \mathbb{R}^m$ for $k \to 0$ may be concluded, as well as 
\[ \left| \int_{\Omega_T} (\hat{u}_k u^\varepsilon - u^\varepsilon) \xi_t \, dx \, dt \right| + \varepsilon \int_{\Omega_T} \nabla (\hat{u}_k) \xi_t \cdot \nabla \xi \, dx \, dt \to 0 \quad \forall \xi \in C_0^\infty(\Omega_T, \mathbb{R}^m). \]

The crucial last step now is to identify limits in the nonlinear term to obtain the second term in (iii) of Definition 2.1.

For this purpose, strong convergence $\nabla \hat{u}_k \to \nabla u^\varepsilon$ in $L^2(\Omega_T, \mathbb{R}^m)$ for $k \to 0$ is shown, which ensures enough compactness of sequences to pass to the limit in the second term in (iii) of Definition 2.1 and then proves $u^\varepsilon : \Omega_T \to \mathbb{R}^m$ to be a weak solution of (1.1)–(1.3).

The goal in this paper is to study convergence properties of a fully discrete version of (1.4)–(1.6), where lowest order conforming finite elements are used. In below, we drop the index $\varepsilon$, such that
$U^j \equiv U^j \in V_h$ for every $j \geq -1$, and $V^j := k^{-1}(U^j - U^{j-1})$, for every $j \geq 0$. The practical scheme reads as follows, for a given positive triple $(\varepsilon, k, h) > 0$:

**Scheme A:** Let $U^0 = P_{I_h^0}u_0 \in V_h$, and $V^0 = P_{I_h^2}v_0 \in V_h$. For $j \geq 1$, and given $U^{j-1}, U^{j-2} \in V_h$, find $U^j \in V_h$, such that for all $\Phi \in V_h$ there holds

$$\sup_{j \in \mathbb{N}_0} E_{k,h}^j[W] := \int_{\Omega} \left[ \phi(\nabla W) + \frac{\varepsilon}{2k} |\nabla (W - U^{j-1})|^2 + \frac{1}{2k^2} W^2 - 2U^{j-1} + U^{j-2} \right] dx. $$

The scheme has been proposed in [3]. Existence of solutions $\{U^j\}_{j \geq 1} \subset V_h$ follows from continuity and coercivity properties of $E_{k,h}^j$. — The following lemma states a discrete energy inequality for iterates $\{U^j\}_{j \geq 1} \subset V_h$, in the case $k = \mathcal{O}(\varepsilon)$.

**Lemma 3.1.** Let $\{U^j\}_{j \geq 1} \subset V_h$ solve Scheme A. Given $\delta \in (0,1)$, for all $k \leq \frac{\delta}{\mathcal{O}(\varepsilon)}$, there holds

$$\sup_{j \in \mathbb{N}_0} E[U^j, V^j] + k \sum_{j=1}^{\infty} \int_{\Omega} \left[ \varepsilon (1 - \delta) |\nabla V^j|^2 + \frac{k}{2} |d_t V^j|^2 \right] dx \leq E[U^0, V^0]. $$

**Proof.** The proof is same as in the context of semi-discretization [10, Lemma 2.2], which combines convexity properties of the above $\phi$, and uses (3.2), with $\Phi = d_t U^j$. \hfill \Box

We remark that the dissipative character of Scheme A is the reason for the last term on the left-hand side of (3.3). As will be evident below, this control is necessary to verify convergence for the scheme.

We use $V^j = d_t U^j \in V_h$, to restate Scheme A for $j \geq 0$ as follows,

$$(d_t V^j, \Phi) + (\sigma(\nabla U^j), \Phi) + \varepsilon (\nabla d_t U^j, \nabla \Phi) = 0, \quad (V^j, \Phi) - (d_t U^j, \Phi) = 0 \quad \forall \Phi \in V_h. $$

We now use $(U^+_k, \check{U}_k)$ resp. $(V^+_k, \check{V}_k)$, which are constant and linear interpolations in time of the sequences $\{U^j\}$ and $\{V^j\}$. For notational brevity, most time we omit sub-indices $k$ and use $(U^+, \check{U})$ and $(V^+, \check{V})$ to stand for $(U^+_k, \check{U}_k)$ and $(V^+_k, \check{V}_k)$.

In the next lemma, we restate Scheme A for $\check{U}, \check{V} \in C([0,T]; V_h)$.

**Lemma 3.2.** Suppose that the assumptions of Lemma 3.1 are valid. For all $T > 0$, and $\Psi \in W^{1,1}(0, T; V_h) \cap L^2(0, T; V_h)$, there holds

$$\left| \int_0^T [-(\check{U}_t, \Psi_t) + (\sigma(\nabla \check{U}), \nabla \Psi) + \varepsilon (\nabla \check{U}_t, \nabla \Psi)] dt + (\check{V}(T, \cdot), \Psi(T, \cdot)) - (V^0, \Psi(0, \cdot)) \right| \leq \int_0^T (\check{V} - V^+, \Psi_t) dt + \int_0^T (\sigma(\nabla \check{U}) - \sigma(\nabla U^+), \nabla \Psi) dt.$$

**Proof.** We rewrite (3.4) as follows: for all $\Psi, \Phi \in W^{1,1}(0, T; V_h) \cap L^2(0, T; V_h)$,

$$\int_0^T [\check{V}_t, \Psi_t) + (\sigma(\nabla U^+), \nabla \Psi) + \varepsilon (\nabla \check{U}_t, \nabla \Psi)] ds = 0, $$

$$\int_0^T [(V^+, \Phi) - (\check{U}_t, \Phi)] ds = 0. $$
Integration by parts in time in the first term of (3.6) leads to
\[
\int_0^T \left[ -\langle \dot{V}, \Psi_t \rangle + (\sigma(\nabla U^+), \nabla \psi) + \varepsilon (\nabla \dot{U}_t, \nabla \psi) \right] \, ds + \langle \dot{V}(T, \cdot), \Psi(T, \cdot) \rangle - \langle \dot{V}(0, \cdot), \Psi(0, \cdot) \rangle = 0.
\]
Because of (3.7), we obtain
\[
\int_0^T \left[ -\langle \dot{U}_t, \Psi_t \rangle + (\sigma(\nabla U), \nabla \psi) + \varepsilon (\nabla \dot{U}_t, \nabla \psi) \right] \, ds + \langle \dot{V}(0, \cdot), \Psi(0, \cdot) \rangle
\]
\[
= \int_0^T \left[ (V^+ - \dot{V}, \Psi_t) + (\sigma(\nabla U) - \sigma(\nabla U^+), \nabla \psi) \right] \, ds.
\]
\[\square\]

The following result gives uniform bounds for \(\{\dot{V}\}\).

**Lemma 3.3.** Suppose that the assumptions of Lemma 3.1 are valid. Then
\[
\|\dot{V}\|_{L^2(0,T;W^{-1,p'}(\Omega,\mathbb{R}^m))} + \|\dot{V}\|_{L^2(0,T;W^{1,2}^r(\Omega,\mathbb{R}^m))} \leq C.
\]

**Proof.** Use (3.6), (H2), and [5, Thm. 3 & 4] to conclude
\[
\|\dot{V}(s,\cdot)\|_{W^{-1,p'}} := \sup_{\varphi \in W^{1,p}_0} \frac{\langle \dot{V}(s,\cdot), P_{L^p} \varphi \rangle}{\|\varphi\|_{W^{1,p}}} \leq C\|\nabla U^+(s,\cdot)\|_{L^p} + C\varepsilon\|\nabla \dot{U}(s,\cdot)\|_{L^2}.
\]

Estimate (3.3) then implies the first part of the assertion. The second part follows from (3.3), and
\[
\|\dot{V}\|_{L^2(0,T;W^{1,2})} \leq \|V^-\|_{L^2(0,T;W^{1,2})} + \|V^+\|_{L^2(0,T;W^{1,2})}.
\]
\[\square\]

It follows from Lemmas 3.1, 3.3, and (H2), that under the given mesh constraint, there exist \(u^\varepsilon \in L^\infty(0,T;W_0^{1,p}(\Omega,\mathbb{R}^m)) \cap W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^m)) \cap W^{1,2}(0,T;W_0^{1,2}(\Omega))\), and a convergent subsequence of \(\{\dot{U}\}_{k,h}\) such that for \((k,h) \to 0\),
\[
\dot{U} \xrightarrow{\ast} u^\varepsilon \quad \text{in} \quad L^\infty(0,T;W_0^{1,p}(\Omega,\mathbb{R}^m)) \cap W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^m)),
\]
\[
\to u^\varepsilon \quad \text{in} \quad W^{1,2}(0,T;W_0^{1,2}(\Omega,\mathbb{R}^m)),
\]
\[
\sigma(\nabla \dot{U}) \xrightarrow{\ast} \sigma \quad \text{in} \quad L^\infty(0,T;L^p(\Omega,\mathbb{R}^m)),
\]
\[
\dot{V} \to u_t \quad \text{in} \quad W^{1,2}(0,T;W_0^{-1,p}(\Omega,\mathbb{R}^m)) \cap L^2(0,T;W_0^{1,2}(\Omega,\mathbb{R}^m)) ,
\]
\[
\dot{U} \to u^\varepsilon \quad \text{in} \quad W^{1,2}(0,T;L^2(\Omega,\mathbb{R}^m)),
\]
thanks to (3.7), and (3.8) for a consequence of the control
\[
\|V^+ - \dot{V}\|_{L^2(\Omega_T)} \leq \sqrt{k}[\sqrt{k}\|\dot{V}\|_{L^2(\Omega_T)}] \leq \sqrt{\frac{k}{\varepsilon} E[U_0,V_0]},
\]
and (3.8) follows from Aubin-Lions compactness result. As can be concluded from e.g. [16, Prop. 1.2, p. 106], \(u^\varepsilon \in C([0,T];L^2(\Omega,\mathbb{R}^m))\). — We are ready to prove the first main convergence theorem of this paper.

**Theorem 3.1.** Let \(T > 0, \varepsilon > 0\), and \((u_0,v_0) \in W_0^{1,p}(\Omega,\mathbb{R}^m) \times L^2(\Omega,\mathbb{R}^m)\). Let assumptions (H1), (H2), (H3) be satisfied, and \(k \leq \frac{\varepsilon}{2k}\). Then there exist \(u^\varepsilon : \Omega_T \to \mathbb{R}^m\) and a subsequence of solutions \(\{\dot{U}\}_{k,h}\) (denoted by same notation) of Scheme A such that for \((k,h) \to 0\)
\[
\dot{U} \xrightarrow{\ast} u^\varepsilon \quad \text{in} \quad L^\infty(0,T;W_0^{1,p}(\Omega,\mathbb{R}^m)) \cap W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^m)).
\]
Moreover, \(u^\varepsilon\) is a weak solution of (1.1)–(1.3).
This result complements the one in [3], because it verifies unconditional convergence of solutions of Scheme A to weak solutions of (1.1)-(1.3), and hence answers questions raised in [3, Remarks (5), (6) on p. 76/77]. Note that the result in the theorem is for a subsequence \( \{\hat{U}\}_{k,h} \); weak convergence of the whole sequence holds in the case of unique weak solutions, which is e.g. valid for globally lipshitz-continuous \( \sigma \). Finally, thanks to [5] there holds

\[
(U^0, V^0) \to (u_0, v_0) \quad \text{in } W^{1,p}(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m) \quad (h \to 0).
\]

**Proof. Step 1:** We may use (3.8)\_2, and (3.8)\_4, to verify for \((k, h) \to 0\)

\[
\left| \int_0^T (\hat{U}_t - u^+_t, \Psi_t) \, ds \right| + \varepsilon \left| \int_0^T (\nabla [\hat{U}_t - u^+_t], \nabla \Psi) \, ds \right|
\]

\[
+ \left| (V^0 - v_0, \Psi(0, \cdot)) \right| + \left| (\hat{V}(T, \cdot) - u^+_T(\cdot, \cdot), \Psi(T, \cdot)) \right| \to 0,
\]

for all \( \Psi = \mathcal{I}_h \xi \), and \( \xi \in W^{1,1}(0; T; C_0^\infty(\Omega, \mathbb{R}^m)) \cap L^2(0, T; C_0^\infty(\Omega, \mathbb{R}^m)) \). Since \( \mathcal{I}_h \xi \to \xi \) in any \( W^{1,p}(\Omega) \ (1 \leq p \leq \infty) \) for \( h \to 0 \), on using again (3.8), we may conclude that (3.10) holds for any \( \Psi \in W^{1,1}(0; T; C_0^\infty(\Omega, \mathbb{R}^m)) \cap L^2(0, T; C_0^\infty(\Omega, \mathbb{R}^m)) \).

Terms on the right-hand side of (3.5) vanish, which is a consequence of Lemma 3.1: we compute

\[
\left| \int_0^T (V^+ - \hat{V}, \Psi_t) \, ds \right| \leq C \sqrt{h} \left( k \int_0^T \|\hat{V}_t\|^2_{L^2} \, ds \right)^{1/2} \left( \int_0^T \|\Psi_t\|^2_{L^2} \, ds \right)^{1/2}
\]

\[
\leq C \sqrt{h} \left( \int_0^T \|\Psi_t\|^2_{L^2} \, ds \right)^{1/2},
\]

by Lemma 3.1. For the second term on the right-hand side of (3.5), we use (H2), and Lemma 3.1; without restriction, assume \( p > 2 \) for the following calculation,

\[
\left| \int_0^T (\sigma(\hat{U}) - \sigma(\nabla U^+), \nabla \Psi) \, ds \right| \leq C \int_{\Omega_T} \left[ 1 + |\nabla \hat{U}|^{p-2} + |\nabla U^+|^{p-2} \right] |\nabla [\hat{U} - U^+]| \|\nabla \Psi\| \, dx \, ds
\]

\[
\leq C \left[ \int_0^T \left( \int_{\Omega} \left[ 1 + |\nabla \hat{U}|^p + |\nabla U^+|^p \right] \, dx \right)^{p/(p-2)} \left( \int_{\Omega} |\nabla [\hat{U} - U^+]|^2 \, dx \right)^{2/p} \, ds \right] \|\Psi\|_{L^\infty(0,T;W^{1,\infty})} \|\nabla [\hat{U} - U^+]\|_{L^{p/2}} \, ds.
\]

The following interpolation result is only needed for \( p > 4 \),

\[
\|\nabla [U^+ - \hat{U}]\|_{L^{p/2}} \leq C \|\nabla [U^+ - \hat{U}]\|_{L^p}^{\frac{p-4}{2}} \|\nabla [U^+ - \hat{U}]\|_{L^2}^{\min\{1, \frac{2}{p-2}\}},
\]

where \([.]^+ := \max\{., 0\}\). Hence, the last integral term in (3.12) may be controlled as follows, using again Lemma 3.1,

\[
\int_0^T \|\nabla [U^+ - \hat{U}]\|_{L^{p/2}} \, ds \leq C \|\nabla U^+\|_{L^\infty(0,T;L^p)}^{\frac{p-4}{2}} \|\nabla [U^+ - \hat{U}]\|_{L^2}^{\min\{1, \frac{2}{p-2}\}} \int_0^T \|\hat{U}_t\|_{L^2}^{\min\{1, \frac{2}{p-2}\}} \, ds
\]

\[
\leq C k^{\min\{1, \frac{2}{p-2}\}} \|\nabla U^+\|_{L^\infty(0,T;L^p)}^{\frac{p-4}{2}} \|\nabla \hat{U}_t\|_{L^2(\Omega_T)}^{\min\{1, \frac{2}{p-2}\}}
\]

\[
\leq C k^{\min\{1, \frac{2}{p-2}\}} e^{\max\{-\frac{1}{p-2}, -\frac{1}{p-2}\}}.
\]

Consequently, terms on right-hand sides of (3.11) and (3.12) tend to zero for \( k \to 0 \), and hence the right-hand side of (3.5) tends to zero. By (3.10), we may replace \( \hat{U} \) and \( \Psi \) in terms one, three, and
four on the left-hand side of (3.5) by corresponding ones, with \( u^\varepsilon \) and \( \xi \). Therefore it remains to verify that
\[
\lim_{k, h \to 0} \int_0^T (\sigma(\nabla \hat{U}) - \sigma(\nabla u^\varepsilon), \nabla \xi) \, ds = 0 \quad \forall \xi \in W^{1,1}(0, T; C_0^\infty(\Omega, \mathbb{R}^m)) \cap L^2(0, T; C_0^\infty(\Omega, \mathbb{R}^m)),
\]
which is done in the next step.

**Step 2:** In order to identify \( \sigma = \sigma(\nabla u^\varepsilon) \) almost everywhere in \( \Omega_T \), we verify strong convergence \( \hat{U} \to u^\varepsilon \) in \( L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^m)) \), for \( (k, h) \to 0 \). So far, we have that
\[
\int_0^T \left[ -(u^\varepsilon_t, \xi_t) + (\bar{\sigma}, \nabla \xi) + \varepsilon (\nabla u^\varepsilon_t, \nabla \xi) \right] \, dt + (u^\varepsilon(T, \cdot), \xi(T, \cdot)) - (v_0, \xi(0, \cdot))
\]
(3.14)
\[
= 0 \quad \forall \xi \in W^{1,1}(0, T; C_0^\infty(\Omega, \mathbb{R}^m)) \cap L^2(0, T; C_0^\infty(\Omega, \mathbb{R}^m)).
\]
Note that by density, and (3.8), identity (3.14) holds for all
\[
\xi \in X := W^{1,1}(0, T; W_0^{1,p}(\Omega, \mathbb{R}^m)) \cap L^2(0, T; W_0^{1,2}(\Omega, \mathbb{R}^m)).
\]
We subtract (3.14) from the left-hand side of (3.5), and choose \( \xi = \Psi \in X \cap W^{1,1}(0, T; V_h) \cap L^2(0, T; V_h) \), where for all \( t \in [0, T] \), we take \( \Psi(t) = P_{L^2}(u^\varepsilon - \hat{U}), \) with \( \varepsilon := u^\varepsilon - \hat{U} \).
\[
\left| \int_0^T \left[ -\|P_{L^2}e_t\|^2 + \left( \bar{\sigma} - \sigma(\nabla \hat{U}), \nabla P_{L^2}e \right) + \frac{\varepsilon}{2} \|\nabla e\|^2 \right] \, dt 
+ (e(T, \cdot), P_{L^2}e(T, \cdot)) - (v_0 - V, P_{L^2}e(0, \cdot)) \right| 
\]
(3.16)
\[
\leq \left| \text{RHS}(P_{L^2}e) \right| + \varepsilon \left| \int_0^T (\nabla e_t, \nabla [u^\varepsilon - P_{L^2}u^\varepsilon]) \, ds \right|,
\]
where
\[
\text{RHS}(\Psi) := \left| \int_0^T (V - V^+, \Psi_t) \, dt \right| + \left| \int_0^T (\sigma(\nabla \hat{U}) - \sigma(\nabla \hat{U}^+), \nabla \Psi) \right|.
\]
The left-hand side of (3.16) consists of five terms \( I, II, ..., V \), which we now study independently, for \( (k, h) \to 0 \). The first tends to zero, owing to \( L^2 \)-stability of \( P_{L^2} \), and (3.8)5.
\[
II = \int_0^T \left( \bar{\sigma} - \sigma(\nabla \hat{U}), \nabla [P_{L^2}u^\varepsilon - \nabla u^\varepsilon] \right) \, dt
\]
(3.17)
\[
+ \int_0^T \left[ (\sigma(\nabla u^\varepsilon) - \sigma(\nabla \hat{U})) + [\bar{\sigma} - \sigma(\nabla u^\varepsilon), \nabla [u^\varepsilon - \hat{U}] \right) \, dt
\]
\[
=: II_a + II_b.
\]
We employ \( \|P_{L^2} - \text{Id}\| \varphi \to 0 \) in \( W^{1,p}(\Omega) \) \( (h \to 0) \) for \( \varphi \in W^{1,p}(\Omega) \), which follows from \( W^{1,p}(\Omega) \)-stability of the \( L^2(\Omega) \)-projection from [5, Thms 3 & 4], and
\[
\| \varphi - P_{L^2} \varphi \|_{W^{1,p}} \leq \inf_{W \in V_h} \left[ \| \varphi - W \|_{W^{1,p}} + \| W - P_{L^2} \varphi \|_{W^{1,p}} \right]
\]
\[
\leq C \inf_{W \in V_h} \| \varphi - W \|_{W^{1,p}}.
\]
Together with (H2) and Lemma 3.1 to bound the first argument in \( II_a \), we may conclude that
\[
\lim_{h \to 0} II_a = 0.
\]
For the first bracket \( [..] \) in \( II_b \), we apply (H3) to conclude
\[
II_b \geq -K \int_0^T \| \nabla [u^\varepsilon - \hat{U}] \|_{L^2}^2 \, ds.
\]
Weak convergence of gradients \( \nabla \hat{U} \) from (3.8)1 is now sufficient to verify convergence to zero for \( k, h \to 0 \) for the term in \( II_b \) related to the second bracket \( [..] \).
Finally, we conclude from \(|(e_t(T, \cdot), P_L e(T, \cdot))| \leq \|e_t(T, \cdot)\|_{L^2} \|e(T, \cdot)\|_{L^2}\), and (3.8)\(_5\) that for \((k, h) \to 0\),
\[
\|e(T, \cdot)\|_{L^2} \leq \sqrt{T} \|e_t\|_{L^2(\Omega_T)} + \|e(0, \cdot)\|_{L^2} \to 0.
\]

Hence, estimate (3.16) leads to
\[
\frac{\varepsilon}{2} \|\nabla[u^\varepsilon(t, \cdot) - \hat{U}(t, \cdot)]\|_{L^2}^2 \leq \frac{\varepsilon}{2} \|\nabla[u^\varepsilon(0, \cdot) - \hat{U}(0, \cdot)]\|_{L^2}^2 + \eta(k, h) + K \int_0^t \|\nabla[u^\varepsilon(s, \cdot) - \hat{U}(s, \cdot)]\|_{L^2}^2 \, ds.
\]
for almost every \(t \in [0, T]\). For brevity, we introduce \(0 \leq \eta(k, h) \leq 1\), such that \(\lim_{k, h \to 0} \eta(k, h) = 0\).

By Gronwall’s inequality, this verifies \(\hat{U} \to u^\varepsilon\) in \(L^\infty(0, T; W^{1,2}(\Omega, \mathbb{R}^m))\) for \((k, h) \to 0\).

But then we may identify \(\tilde{\sigma} = \sigma(\nabla u^\varepsilon)\) a.e. in \(\Omega_T\), which then establishes assertion (iii) of Definition 2.1.

**Step 3:** The energy estimate stated in Lemma 3.2 may be restated as follows,
\[
\sup_{t \geq 0} E[U^+(t, \cdot), \hat{U}_t(t, \cdot)] + \varepsilon \int_0^t (1 - \delta) \|\nabla \hat{U}_t\|^2 \, dx \leq E[U^0, V^0],
\]
for all \(t \geq 0\), and for all \(k \leq \frac{\delta}{2}\), given \(\delta \in (0, 1)\). A problem arises when passing to the limit \((k, h) \to 0\), due to lack of weakly lower semicontinuity property on \(W^{1,p}\) of \(u \mapsto \int_\Omega \phi(\nabla u) \, dx\); the following argument is taken from [10, p. 377]: because of (H3), we exploit the convexity of \(F \mapsto \hat{\phi}(F) := \phi(F) + \frac{K}{2}|F|^2\), to conclude for \(t \geq 0\),
\[
\int_\Omega \hat{\phi}(\nabla u^\varepsilon(t, \cdot)) \, dx = \int_\Omega \hat{\phi}(\nabla u^\varepsilon(t, \cdot)) - \frac{K}{2} |\nabla u^\varepsilon(t, \cdot)|^2 \, dx \leq \liminf_{k, h \to 0} \int_\Omega \hat{\phi}(\nabla \hat{U}(t, \cdot)) - \frac{K}{2} |\nabla u^\varepsilon(t, \cdot)|^2 \, dx
\]
\[
\leq \liminf_{k, h \to 0} \int_\Omega \hat{\phi}(\nabla \hat{U}(t, \cdot)) \, dx + \liminf_{k, h \to 0} \frac{K}{2} \int_\Omega |\nabla \hat{U}(t, \cdot)|^2 - |\nabla u^\varepsilon(t, \cdot)|^2 \, dx
\]
\[
\leq \liminf_{k, h \to 0} \int_\Omega \hat{\phi}(\nabla \hat{U}(t, \cdot)) \, dx,
\]
thanks to \(\hat{U} \to \nabla u^\varepsilon\) in \(L^\infty(0, T; W^{1,p}(\Omega, \mathbb{R}^m))\) for \((k, h) \to 0\), and a.e. \(t \in [0, T]\). Inserting this into (3.19), and using (3.8) else yields to assertion (iv) of Definition 2.1.

**Step 4:** Convergence \(u_t^\varepsilon(t, \cdot) \to v_0\) in \(L^2(\Omega, \mathbb{R}^m)\) \((t \to 0)\) follows from (3.8)\(_1\) which gives boundedness in time of \(t \mapsto u_t^\varepsilon(t, \cdot) \subseteq L^2(\Omega, \mathbb{R}^m)\) a.e. in time, and from (3.8)\(_4\) which validates absolute continuity of the mapping \(t \mapsto u_t^\varepsilon(t, \cdot) \subseteq W^{-1,p'}(\Omega, \mathbb{R}^m)\).

Similarly, \(u^\varepsilon(t, \cdot) \to v^\varepsilon\) in \(W^{1,p}(\Omega, \mathbb{R}^m)\) \((t \to 0)\) follows from (3.8)\(_1\) which gives boundedness in time of \(t \mapsto u^\varepsilon(t, \cdot) \subseteq W^{1,p}(\Omega, \mathbb{R}^m)\) a.e. in time, and from (3.8)\(_4\) which validates absolute continuity of the mapping \(t \mapsto u^\varepsilon(t, \cdot) \subseteq W^{1,2}(\Omega, \mathbb{R}^m)\).

Estimate (3.18) motivates choices \((k, h) \propto \exp(\varepsilon^{-1})\) to converge to weak solutions of (1.1)--(1.3).

In contrast, no strong convergence of gradients \(\{\nabla \hat{U}_t\}_{k,h}\) is needed if generalized solutions to (1.1)--(1.3), for \(\varepsilon = 0\) are approximated, and the strictest mesh constraint comes from (3.9). Then, the following result shows that for simultaneously tending \((\varepsilon, k, h) \to 0\), solutions \(\hat{U} : \Omega_T \to \mathbb{R}^m\) to Scheme A approximate generalized solutions to (1.1)--(1.3), for \(\varepsilon = 0\).

**Theorem 3.2.** Let \(T > 0\), and \((u_0, v_0) \in W^{1,p}(\Omega, \mathbb{R}^m) \times L^2(\Omega, \mathbb{R}^m)\). Let assumptions (H1), (H2), (H3) be satisfied, and \(k = o(\varepsilon)\) in Scheme A. Then there exist \(u : \Omega_T \to \mathbb{R}^m\) and a subsequence of
solutions $\{\hat{U}\}_{\varepsilon,k,h}$ (denoted by same notation) of Scheme A such that as $(\varepsilon,k,h) \to 0$

\[
\hat{U} \xrightarrow{\ast} u \quad \text{in } L^\infty(0,T;W_0^{1,p}(\Omega,\mathbb{R}^m)) \cap W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^m)),
\]

\[
\delta_{\nabla\hat{U}} \xrightarrow{\ast} \nu \quad \text{in } L_w^\infty(\Omega_T;\mathcal{M}(\mathbb{R}^{m \times n})),
\]

Moreover, the tuple $(u,\nu)$ is a generalized solution to (1.1)–(1.3), for $\varepsilon = 0$.

Proof. Step 1: It follows from Lemmas 3.1, 3.3, and (H2) that under the given mesh constraint, there exist

\[
(u,\nu) \in \left[ L^\infty(0,T;W_0^{1,p}(\Omega,\mathbb{R}^m)) \cap W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^m)) \right] \times L_w^\infty(\Omega_T;\mathcal{M}(\mathbb{R}^{m \times n})),
\]

and a convergent subsequence of $\{\hat{U}\}_{\varepsilon,k,h}$ such that for $(\varepsilon,k,h) \to 0$

\[
\hat{U} \xrightarrow{\ast} u \quad \text{in } L^\infty(0,T;W_0^{1,p}(\Omega,\mathbb{R}^m)) \cap W^{1,\infty}(0,T;L^2(\Omega,\mathbb{R}^m)),
\]

\[
\sigma(\nabla\hat{U}) \xrightarrow{\ast} \sigma \quad \text{in } L^\infty(0,T;L^p(\Omega,\mathbb{R}^m)),
\]

\[
V \rightharpoonup u_t \quad \text{in } W^{1,2}(0,T;W^{-1,p}(\Omega,\mathbb{R}^m)),
\]

\[
\hat{U} \rightharpoonup u \quad \text{in } W^{1,2}(0,T;L^p(\Omega,\mathbb{R}^m)),
\]

\[
\delta_{\nabla\hat{U}} \xrightarrow{\ast} \nu \quad \text{in } L_w^\infty(\Omega_T;\mathcal{M}(\mathbb{R}^{m \times n})).
\]

Step 2: We compute the limit $\varepsilon \to 0$ of the third term on the left-hand side of (3.5),

\[
|\varepsilon| \int_0^T \langle \nabla \hat{U}t, \nabla \Psi \rangle \, dt \leq \sqrt{\varepsilon} \left[ \varepsilon \| \nabla \hat{U}t \|_{L^2(\Omega_T)} \right] \| \nabla \Psi \|_{L^2(\Omega_T)} \to 0 \quad \forall \Psi \in L^2(0,T;V_h \cap W^{1,2}(\Omega,\mathbb{R}^m)).
\]

Thanks to [13, Thm. 6.2], there holds

\[
\sigma(t,x) = \int_{\mathbb{R}^{m \times n}} \sigma(\lambda) \, d\nu_{t,x}(\lambda) \quad \text{a.e. in } \Omega_T.
\]

To identify limits in the remaining terms of (3.5) for $(\varepsilon,k,h) \to 0$ is as in the first step of the proof of Theorem 3.1; the arguments given there in step four may be adapted as well, which settles the proof.

4. Computational Experiments

In this section, we report on computational studies for fixed and decreasing values of $\varepsilon > 0$ in (1.1)–(1.3) performed with Scheme A. The nonlinear systems of equations in each time step were approximately solved by a simple fixed point method. The scheme is implemented in MATLAB, with a direct solver for linear systems of equations.

The first example is taken from the literature to allow for direct comparison of our computational results with those obtained with a finite difference method in [18, Section 6.2]; we remark that no results of convergence are given in [18] to theoretically justify their scheme.

Example 4.1. For $\Omega = (1,2)^2$, choose $u_0(x,y) = x^2(x-1)y(y-1)$, $v_0(x,y) = 0$, and $\phi(F) = \frac{1}{4}(|F|^2 - 1)^2$. The evolution (1.1)–(1.3) is approximated by Scheme A, for values $\varepsilon = 0.25$, and $(k,h) = (5 \times 10^{-4}, 2^{-6})$.

Snapshots are displayed in Figure 2, together with the evolution of kinetic, elastic, and total energies, where

\[
E_{\text{kin}}[d_tU^j] = \frac{1}{2} \int_\Omega |d_tU^j|^2 \, dx, \quad E_{\text{ela}}[\nabla U^j] = \int_\Omega \phi(\nabla U^j) \, dx.
\]

The results are in good agreement with those reported in [18, Section 6.2].
In the second example, we study dynamics in the same setup of data for values of $\varepsilon$ tending to zero.

**Example 4.2.** Choose data as in Example 4.1, for values $\varepsilon = 0.1 \cdot 2^{-\ell}$, $0 \leq \ell \leq 4$.

As is known from [6, p. 149], generalized solutions $u : \Omega_T \to \mathbb{R}$ of (1.1)–(1.3) ($\varepsilon = 0$) solve $u_{tt} = \text{div} \, p(\nabla u)$, where $p := D\phi^{**}$, for the existing convexification

$$
\phi^{**} = \sup \{ f : f \leq \phi, \ f \text{ convex} \}.
$$

For our choice of $\phi$, we easily compute $\phi^{**}(F) = \frac{1}{4}([|F|^2 - 1]_+)^2$, where $[\cdot]_+ = \max\{0, \cdot\}$. For the chosen initial data, we then find $u \equiv u_0$ in $\Omega_T$. — The snapshots in Figure 2 for decreasing values of $\varepsilon$ show 2D-microstructures which develop and propagate in time. The qualitative evolution of different energy contributions is comparable to the one depicted in Figure 1.

**REFERENCES**

Figure 2. Example 4.2: Snapshots of $t \mapsto U(t, \cdot)$, at times $t = 1, 2, 4$ (left to right) for values $\varepsilon = 0.1, 0.05, 0.025, 0.0125$ (top to bottom).


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