APPROXIMATE EULER METHOD FOR PARABOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS DRIVEN BY SPACE-TIME LÉVY NOISE

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ABSTRACT. We consider parabolic stochastic partial differential equations driven by space-time Lévy noise. Different discretization methods to accurately simulate jumps are proposed and analyzed in the context of an implicit time discretization. Computational studies based on a finite element discretization are provided to illustrate combined truncation and time-discretization effects.

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1. INTRODUCTION

In recent years, stochastic partial differential equations driven by a space-time Lévy noise have received increasing interest; see e.g., Applebaum and Wu [AW00], Knoche [Kno04], Mueller [Mue98], Saint Loubert Bié [SLB98]. One motivation for this work is the model of river pollution proposed by Kwakernaak [Kwa75] and studied by Curtain [Cur76]. Let \( \mathcal{O} := (0,1) \times (0,3) \subset \mathbb{R}^2 \), and \( A = D\Delta - \gamma \frac{\partial}{\partial \xi_2} - a \); here, \( D > 0 \) is the dispersion rate, \( \gamma > 0 \) scales convection in the \( \xi_2 \)-direction, and \( a \geq 0 \) is the leakage rate. Then \( u : (0,T) \times \mathcal{O} \to \mathbb{R} \) describes the concentration of a chemical substance which solves the following problem (see Figure 1)

\[
\begin{align*}
\frac{\partial u}{\partial t} &= Au \quad \text{in } \mathcal{O}_T := (0,T) \times \mathcal{O}, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } (0,T) \times \Gamma_N, \quad \text{and} \quad u = 0 \quad \text{on } (0,T) \times \Gamma_D.
\end{align*}
\]

A chemical substance is now deposited in the region \( G \subset \mathcal{O} \). The deposits leak, and substance is dripping out with a certain mean rate \( \lambda > 0 \). This dripping out is modeled by a space-time Lévy noise, and the related concentration process \( \{ u(t, \xi); t \in [0,T], \xi \in \mathcal{O} \} \) solves the stochastic partial differential equation (SPDE)

\[
\begin{align*}
\frac{\partial u(t, \xi)}{\partial t} &= Au(t, \xi) + \dot{\mathbf{L}}(t, \xi) \quad \forall (t, \xi) \in \mathcal{O}_T, \\
u(0, \xi) &= 0 \quad \forall \xi \in \mathcal{O}, \\
\frac{\partial}{\partial \xi_2} u(t, \xi) &= 0 \quad \forall (t, \xi) \in (0,T) \times \Gamma_N, \quad u(t, \xi) = 0 \quad \forall (t, \xi) \in (0,T) \times \Gamma_D.
\end{align*}
\]

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where $\dot{L}$ represents space-time Lévy noise. Problem (1) is an example of the more general setting, which we consider in this work. Figures 1 and 2 show simulations for one path, which are based on the approximate Euler scheme (‘Scheme B’) that is proposed and studied below.

A first numerical analysis of such SPDE’s is [HM06]. It is, however, that increments of the driving Lévy process were exactly simulated in these works. It is only for a small number of Lévy processes that the exact distribution function of the increments is available, while in general it is only the Lévy measure of the driving noise that is given. Therefore, the goal of this paper is to propose different strategies to approximate a given Lévy process, and an analysis of errors inherent to a corresponding time-discretization of (13).
Several approaches exist to simulate SDEs driven by Lévy processes. Rubenthaler [Rub03] studied the approximate Euler scheme, which combines (explicit) time-discretization, and simulatable approximations of a given Lévy process (with unknown distribution) by truncating jumps below a certain threshold $\epsilon > 0$: the numerical strategy then uses ‘interlacing’, i.e., simulates subsequent stopping times where jumps occur, corresponding jump heights, and an intermediate Brownian motion. We remark that this strategy is less efficient in high dimensions, and hence also for SPDE’s (after discretization in space). Convergence in law with rates for the scheme is shown in [Rub03]. For related numerical works to solve SDE’s without approximating the driving Lévy noise, we refer to cited works in [Rub03].

Such a truncation strategy ignores small jumps, which may cause a rough approximation of Lévy measures of infinite variation, and a reduced precision of iterates of the related approximate Euler scheme; Asmussen and Rosiński [AR01] have shown that small jumps may be represented by a Wiener process, leading to an improved approximation of the given Lévy process. In [JKMP05], weak rates of convergence for iterates of the approximate Euler scheme for SDEs that employs this strategy are shown; strong rates of convergence are obtained in [Fou10], where the proof relies on a central limit theorem when using the quadratic Wasserstein distance. In this work, we give a different proof (cf. Proposition 3.4) that is based on Fourier transforms and uses tail estimates to show a corresponding $L^p$ estimate ($1 \leq p \leq 2$), which is the proper setting of (1) resp. (13) where mild solutions are $L^p$-valued.

The main goal of this work is to obtain accurate, simulatable increments with Lévy measure $\nu\{|x| \geq \epsilon\}$, $\kappa \geq 1$ by truncating small (and large) jumps, and approximating the remaining jump heights by a sum of $O(\epsilon^{-1})$ atoms, such that $\epsilon > 0$ controls both, truncation and discretization effects. For this purpose, we consider different meshing strategies to assemble a related compound Poisson process, to simultaneously represent all, small, medium, and large jumps accurately:

- local sizes of an equally weighted mesh $\{D_j\}_{j=1}^{\epsilon^{-1}}$ covering $\mathbb{R} \setminus (-\epsilon, \epsilon)$ (cf. [Sch03]) are determined by the requirement $\nu(D_j) = \epsilon \nu(\mathbb{R} \setminus (-\epsilon, \epsilon))$. Meshes depend on the given Lévy measure $\nu$, will be shown to only roughly approximate small jumps, and only work reliably for certain Lévy measures; cf. Proposition 3.3, Theorem 4.1, and the computational experiments summarized in Figure 8.
- Equally spaced meshes allow for an accurate simulation of bounded jumps of size $[\epsilon, 1]$, while small and large jumps suffer a rough approximation.
- The newly introduced stretched meshes accurately resolve small jumps up to size $O(\epsilon^2)$, and (relatively less frequent) large jumps up to size $O(\epsilon^{-\gamma})$ for $0 \leq \gamma < \frac{1}{2}$; cf. Proposition 3.3, Theorem 4.1 and Figure 8 computational evidence.

A related analysis shows that stretched meshes lead to the smallest approximation errors of the jump term, which is also confirmed by computational studies in Section 5. Thus a proper configuration of (discrete) approximate Lévy measures complements the strategy of an additional Gaussian variable to model small jumps, and may even make it dispensable; see Proposition 3.4 and Section 5.1 for computational evidence.

In Section 4, we derive $L^p$-error estimates for the (implicit) approximate Euler scheme (‘Scheme B’) to numerically solve problem (13). The corresponding analysis accounts for both, time discretization, and approximation of the space-time Lévy white noise. The error estimate in Theorem 4.1 quantifies how truncation and discretization effects (using the different mesh strategies above) of the space-time Lévy white noise affect accuracy of the time-discrete approximates of (13). In Section 5, we use a space-time discretization based on finite elements to study the effects of truncation and different mesh strategies to approximate the Lévy measure, as well as time discretization; computational studies are reported that evidence improved approximation properties of the (implicit) approximate Euler method using stretched meshes.
The paper is organized as follows. Necessary background material on the space-time Lévy noise is provided in the preliminaries. Strong rates of convergence for the approximate Lévy random walk for different meshes, and the gain in accuracy by using a compensatory Wiener process are studied in Section 3. The approximate Euler method to numerically solve problem (13) is analyzed in Section 4. Computational studies are reported in Section 5.

2. Preliminaries

In this section, we recall the notion of space-time Lévy processes.

2.1. Lévy processes and Lévy measures. Let $\mathcal{L}(\mathbb{R})$ denote the set of Lévy measures supported on $\mathbb{R}$, i.e., the set of positive measures $\nu$ on $\mathbb{R}$ such that

$$
\nu(\{0\}) = 0, \quad \nu \text{ is } \sigma\text{-finite and } \int_{\mathbb{R}}(|x|^2 \wedge 1) \nu(dx) < \infty.
$$

The characteristic function of a Lévy process $L = \{L(t) : t \geq 0\}$ is uniquely decomposable by the Lévy-Khinchine formula: for any real-valued Lévy process $L = \{L(t) : t \geq 0\}$ there exists a non-negative number $Q$, a non-negative measure $\nu \in \mathcal{L}(\mathbb{R})$, and an element $m \in \mathbb{R}$ such that

$$
E[e^{ix(L(1),x)}] = \exp\left(i mx - \frac{1}{2}Qx^2 - \int_{\mathbb{R}} \left(1 - e^{ix} + 1_{\{|y|<1\}}(y)ixy\right) \nu(dy)\right) \quad \forall x \in \mathbb{R}.
$$

Here, $\nu$ denotes the Lévy measure of the Lévy process $L$; the triplet $(Q, m, \nu)$ uniquely determines the law of the Lévy process.

Given a real-valued Lévy process $L$ on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, one can construct the integer-valued Poisson random measure $\mu_L : \mathcal{B}(\mathbb{R}^+ \times \mathbb{R}) \times \mathcal{B}(\mathbb{R} \setminus \{0\}) \rightarrow \mathbb{N}_0 \cup \{\infty\}$ by

$$
\mu_L(I \times D) := \#\{s \in I \mid \Delta L(s) \in D\},
$$

where the jump process $\Delta L = \{\Delta L(t) : 0 \leq t < \infty\}$ is given by $\Delta L(t) := L(t) - L(t^-) = L(t) - \lim_{\epsilon \to 0} L(t^- - \epsilon)$ for $t > 0$, and $\Delta L(0) = 0$.

The intensity of the Poisson random measure $\mu_L$ is the Lévy measure $\nu = \mathbb{E}[\mu_L([0,1] \times \cdot \cdot \cdot)]$. Vice versa, if $\mu$ is a given (time-homogeneous) Poisson random measure on $\mathbb{R} \setminus \{0\}$ over a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and with finite Lévy measure $\nu \in \mathcal{L}(\mathbb{R})$, then the stochastic process $L$ defined by

$$
[0,\infty) \ni t \mapsto L(t) = \int_0^t \int_{\mathbb{R}} \zeta \mu(ds,d\zeta)
$$

is a compound Poisson process; if the Lévy measure is only $\sigma$-finite, $\mu$ has to be replaced by the compensated Poisson random measure $\tilde{\mu}(ds,d\zeta) = \mu(ds,d\zeta) - \nu(d\zeta)ds$; see e.g. [A1].

A finite $\alpha$-th variation resp. a finite $\beta$-th moment of the Lévy measure $\nu \in \mathcal{L}(\mathbb{R})$ characterize small resp. large jumps, and are relevant quantities to describe its approximation.

**Definition 2.1.** We call a Lévy measure $\nu$ of type $(\alpha, \beta)$, for $\alpha \in [0,2]$ and $\beta \in [0,\infty]$, iff

$$
\alpha = \inf\left\{\alpha \geq 0 : \lim_{x \to 0} \nu((x, \infty)) x^{\alpha} < \infty, \text{ and } \lim_{x \to 0} \nu((-\infty, x)) x^{\alpha} < \infty\right\},
$$

$$
\beta = \sup\left\{\beta \geq 0 : \int_{\{|x|>1\}} |x|^\beta \nu(dx) < \infty\right\}.
$$

The index $\alpha$ is known as ‘Blumenthal-Getoor’ index.
2.2. The space-time Lévy noise or the space-time Poisson noise. The space-time Lévy white noise is a generalization of the space-time Gaussian white noise. Let \( M(S) \) be the set of all \( \mathbb{R} \)-valued measures on the measurable space \( (S, \mathcal{S}) \), and \( \mathcal{M}(S) \) be the \( \sigma \)-field on \( M(S) \) generated by functions \( i_\Phi : M(S) \ni \mu \mapsto \mu(B) \in \mathbb{R} \) for \( B \in \mathcal{S} \). In the following, we put \( S = [0, \infty) \times \mathbb{R}^d \), and \( \mathcal{S} = \mathcal{B}([0, \infty)) \times \mathcal{B}(\mathbb{R}^d) \).

**Definition 2.2.** Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete filtered probability space, \( \mathcal{O} \subset \mathbb{R}^d \) be a bounded domain with smooth boundary, and \( \nu \in \mathcal{L}(\mathbb{R}) \). The space-time Lévy white noise on \( \mathcal{O} \) with jump size intensity \( \nu \in \mathcal{L}(\mathbb{R}) \) is a measurable mapping

\[
\mathbf{L} : (\Omega, \mathcal{F}) \to \left( M([0, \infty) \times \mathcal{O}), \mathcal{M}([0, \infty) \times \mathcal{O}) \right)
\]

such that

(i) for all \( I \times B \in \mathcal{B}([0, \infty)) \times \mathcal{B}(\mathcal{O}) \), the random variable \( \mathbf{L}(I \times B) \) is real-valued, infinitely divisible, and has the characteristic exponent

\[
\exp\left( i\theta \mathbf{L}(I \times B) \right) = \exp\left( \frac{\lambda(I) \lambda_d(B)}{\lambda(I) \lambda_d(B) < \infty} \right),
\]

whenever \( \lambda(I) \lambda_d(B) < \infty \);  

(ii) if the sets \( I_1 \times B_1, I_2 \times B_2 \in \mathcal{B}([0, \infty)) \times \mathcal{B}(\mathcal{O}) \) are disjoint, then the random variables \( \mathbf{L}(I_1 \times B_1) \) and \( \mathbf{L}(I_2 \times B_2) \) are independent and \( \mathbf{L}(I_1 \times B_1 \cup I_2 \times B_2) = \mathbf{L}(I_1 \times B_1) + \mathbf{L}(I_2 \times B_2) \);  

(iii) the measure-valued process \( t \mapsto \mathbf{L}([0, t) \times \cdot) \) is \( \mathbb{F} \)-adapted.

**Remark 2.1.** Fix a non-negative function \( \varphi \in C_0(\mathcal{O}) \). The real-valued process \( L(\varphi) = \{L(t, \varphi) : 0 \leq t < \infty\} \) defined by

\[
[0, \infty) \ni t \mapsto L(t, \varphi) := \int_0^t \int_{\mathcal{O}} \varphi(\xi) \mathbf{L}(ds, d\xi)
\]

is an \( \mathbb{R} \)-valued Lévy process with Lévy measure \( \nu_\varphi : C_0(\mathbb{R}) \ni B : \to (\int_{\mathcal{O}} \varphi(\xi) d\xi) \nu(B) \).

For more details we refer to [PZ07] and [BHau10, Chapter 7].

3. Simulation of a Lévy walk

Let \( \mu \) be a time-homogeneous Poisson random measure on \( \mathbb{R} \) with intensity \( \nu \in \mathcal{L}(\mathbb{R}) \). The \( \mathbb{R} \)-valued mapping

\[
L : [0, \infty) \ni t \mapsto L(t) := \int_0^t \int_{\mathbb{R}} \zeta \tilde{\mu}(ds, d\zeta)
\]

is well-defined, and \( L = \{L(t) : 0 \leq t < \infty\} \) defines a time-homogeneous \( \mathbb{R} \)-valued Lévy process. The Lévy process may be approximated in different ways, where one is to generate a Lévy random walk: For a given time-step size \( \tau > 0 \), it is a sequence of random variables

\[
(\Delta_0^m L, \Delta_1^m L, \Delta_2^m L, \ldots, \Delta_m^m L, \ldots),
\]

where

\[
\Delta_m^m L := L(t_{m+1}) - L(t_m) = \int_{t_m}^{t_{m+1}} \int_{\mathbb{R}} \zeta \tilde{\mu}(ds, d\zeta) \quad \forall m \in \mathbb{N}.
\]

In general, the distribution of \( \Delta_m^m L \) is not known, such that \( \Delta_m^m L \) cannot be simulated directly. In the following, we study two strategies to simulate approximations of the Lévy random walk \( \{\Delta_m^m L : m \geq 0\} \): truncation, and discretization of the measure \( \nu \) using (i) stretched meshes to accurately resolve small, medium, and large jumps, and (ii) more common meshes covering (compact subsets) of \( \mathbb{R} \), combined with a proper Gaussian variable to account for small jumps.
The goal of this work is to simulate space-time white Lévy noise. This is done by a (finite) sum of independent one-dimensional Lévy processes, each having the Lévy measure \( \nu \). To conclude, two parameters will be involved in the discretization below: the time-step size \( \tau \) and the truncation parameter \( \epsilon \).

We start with a Lévy process \( L \) having characteristic measure \( \nu \) and ask for a good approximation of the Lévy random walk given in (2) with time-step size \( \tau \).

**Truncation of small jumps:** Let \( \kappa \geq 1 \). Fix a truncation parameter \( 0 < \epsilon < 1 \), and define the approximate Lévy measure
\[
\nu_{\epsilon,\kappa} : B(\mathbb{R}) \ni D \mapsto \nu\left(D \cap \left(\mathbb{R} \setminus (-\epsilon^\kappa, \epsilon^\kappa)\right)\right).
\]

Let \( \mu_{\epsilon,\kappa} \) be the Poisson random measure induced by truncating small jumps, i.e.,
\[
\mu_{\epsilon,\kappa} : B(\mathbb{R}^+) \times B(\mathbb{R}) \ni I \times D \mapsto \int_I \int_{R \setminus (-\epsilon^\kappa, \epsilon^\kappa)} 1_D(\zeta) \mu(ds, d\zeta) \in \mathbb{N} \cup \{\infty\}.
\]

The approximate Lévy process \( L_{\epsilon,\kappa} = \{L_{\epsilon,\kappa}(t) : 0 \leq t < \infty\} \) is defined as
\[
L_{\epsilon,\kappa}(t) := \int_0^t \int_{\mathbb{R}} \zeta \tilde{\mu}_{\epsilon,\kappa}(ds, d\zeta) \quad \forall \, t \geq 0,
\]
where \( \tilde{\mu}_{\epsilon,\kappa} \) denotes the compensated Poisson random measure.

**Proposition 3.1.** Let \( \kappa \geq 1 \). Assume \( 1 \leq p \leq 2 \), let \( \tau > 0 \) be the time-step size, and \( \epsilon > 0 \) the truncation parameter. Suppose that \( \nu \in \mathcal{L}(\mathbb{R}) \) is of type \( (\alpha, \beta) \) with \( 0 < \alpha < p \). There exists a constant \( C > 0 \), such that
\[
E\left[|\Delta^m_{\tau} L - \Delta^m_{\tau} L_{\epsilon,\kappa}|^p\right] \leq C \tau \epsilon^{(p-\alpha)} \quad (m \in \mathbb{N}).
\]

**Proof.** By Burkholder’s inequality, for \( 1 \leq p \leq 2 \),
\[
E\left[|\Delta^m_{\tau} L - \Delta^m_{\tau} L_{\epsilon,\kappa}|^p\right] = E\left[\int_{t_m}^{t_{m+1}} \int_{\mathbb{R}} \zeta \tilde{\mu}(ds, d\zeta) - \int_{t_m}^{t_{m+1}} \int_{\mathbb{R}} \zeta \tilde{\mu}_{\epsilon,\kappa}(ds, d\zeta)|^p\right]
\]
\[
= E\left[\int_{t_m}^{t_{m+1}} \int_{-\epsilon^\kappa}^{\epsilon^\kappa} \zeta \tilde{\mu}(ds, d\zeta)|^p\right]
\]
\[
\leq C\tau \int_{-\epsilon^\kappa}^{\epsilon^\kappa} |\zeta|^p \nu(d\zeta).
\]
Since \( \nu \) is of type \( (\alpha, \beta) \), there exists some constant \( C > 0 \) such that
\[
\nu(d\zeta) \leq C |\zeta|^{-\alpha-1} d\zeta, \quad |\zeta| \leq \epsilon^\kappa.
\]
Thus, we obtain for \( 0 < \alpha < p \)
\[
\int_{-\epsilon^\kappa}^{\epsilon^\kappa} |\zeta|^p \nu(d\zeta) \leq C \int_{-\epsilon^\kappa}^{\epsilon^\kappa} |\zeta|^p |\zeta|^{-\alpha-1} d\zeta \leq C \epsilon^{k(p-\alpha)}.
\]
\( \square \)
Approximating the Lévy measure by a discrete measure. The second step is to approximate the truncated Poisson random measure $\mu^{\epsilon, \kappa}$ by a Poisson random measure $\tilde{\nu}^{\epsilon, \kappa}$ which has the discrete Lévy measure $\nu^{\epsilon, \kappa}$, where

$$\tilde{\nu}^{\epsilon, \kappa} := \sum_{j=1}^{J} \nu(D_j) \delta_{d_j} \in \mathcal{L}(\mathbb{R} \setminus (-\epsilon^\kappa, \epsilon^\kappa)).$$

Here $D_1, \ldots, D_J$ are disjoint sets such that $\bigcup_{j=1}^{J} D_j \subset \mathbb{R} \setminus (-\epsilon^\kappa, \epsilon^\kappa)$, with $d_j \in D_j$, $j = 1, \ldots, J$, and $\delta_{d_j} \in M(\mathbb{R})$ denotes the Dirac measure supported at $d_j \in D_j$. We choose $J = O(\epsilon^{-1})$ to control both, truncation and discretization errors for a given Poisson random measure $\nu$, and the Lévy random walk in (2) is approximated by

$$\left(\tilde{\alpha}^{0}_{\tau} L_{\epsilon, \kappa}, \tilde{\alpha}^{1}_{\tau} L_{\epsilon, \kappa}, \tilde{\alpha}^{2}_{\tau} L_{\epsilon, \kappa}, \ldots, \tilde{\alpha}^{m}_{\tau} L_{\epsilon, \kappa}, \ldots\right).$$

The following result bounds the error $|\Delta_{\tau}^{m} L_{\epsilon, \kappa} - \tilde{\alpha}_{\tau}^{m} L_{\epsilon, \kappa}|$ in terms of the difference $\nu - \tilde{\nu}^{\epsilon, \kappa}$; its simple proof is based on Burkholder’s inequality.

**Proposition 3.2.** Let $\kappa \geq 1$, $\epsilon > 0$ and the time-step size $\tau > 0$ be given. There holds for all $1 \leq p \leq 2$

$$\mathbb{E}\left[|\Delta_{\tau}^{m} L_{\epsilon, \kappa} - \tilde{\alpha}_{\tau}^{m} L_{\epsilon, \kappa}|^p\right] \leq C \tau \int_{\mathbb{R} \setminus (-\epsilon^\kappa, \epsilon^\kappa)} |\zeta|^p (\nu - \tilde{\nu}^{\epsilon, \kappa})(d\zeta) \quad (m \in \mathbb{N}).$$

As a consequence, an accurate approximation of increments of the Lévy noise $\Delta_{\tau}^{m} L_{\epsilon, \kappa}$ is controlled by accurate approximations $\tilde{\nu}^{\epsilon, \kappa} = \sum_{j=1}^{J} \nu(D_j) \delta_{d_j}$, which heavily rests on the capability of the mesh $\bigcup_{j=1}^{J} D_j \subset \mathbb{R} \setminus (-\epsilon^\kappa, \epsilon^\kappa)$ to resolve small and large jumps related to a Lévy measure of type $(\alpha, \beta)$. For the sake of simplicity, suppose that jumps are positive, thus $\bigcup_{j=1}^{J} D_j \subset \mathbb{R}^+ \setminus [0, \epsilon^\kappa)$. In below, we compare three grid strategies:

1. **equally weighted mesh**: Let $D_1 = [\epsilon, x_2)$, $D_j = [x_i, x_{i+1})$, and $D_i = [x_i, x_{i+1})$ such that $\nu(D_i) = \nu(D_j)$, for $1 \leq i, j \leq J \equiv O(\epsilon^{-1})$; see [Sch03].
2. **equally spaced mesh**: Let $D_1 = [\epsilon, x_2)$, and $D_i = [x_i, x_{i+1})$ be such that $\lambda(D_i) = \lambda(D_j)$, for $1 \leq i, j \leq J \equiv O(\epsilon^{-1})$.
3. **stretched mesh**: Let $D_1 = [\epsilon^2, x_2)$, and choose $\epsilon := \lambda(D_j)$ according to

$$\epsilon_j = \begin{cases} 
\frac{\epsilon^2}{j} \quad & \text{if } x_j < 1, \\
\frac{\epsilon^2}{j^{\gamma}} \quad & \text{if } x_j \geq 1,
\end{cases}$$

for some $0 < \gamma < \frac{1}{2}$; see [Pr97, Chapter 10].
While strategy (ii) is limited to accurately resolve a bounded range of jumps $\Delta^p L_{\epsilon,1}$, strategy (i) covers the whole range of jumps $\mathbb{R} \setminus (-\epsilon, \epsilon)$; this mesh is specific to each $\nu$, leading to fine intervals $D_j$ for small jumps, while the meshing in regions supporting large jumps is coarse. The mesh strategy (iii) starts with $\lambda(D_1) = \epsilon^2$ and geometrically widens to resolve frequent small jumps; after $J = O(\epsilon^{-1})$ intervals $D_i = [x_i, x_{i+1})$, the local width is $O(\epsilon)$ to resolve jump heights of order $O(1)$, since $\epsilon_J \leq C\epsilon^{-1}\epsilon^2$, and

$$x_{j+1} = \sum_{j=1}^{J} \epsilon_j = \epsilon^2 \sum_{j=1}^{J} j = \epsilon^2 \frac{J(J + 1)}{2} = O(1).$$

Large jumps up to a height $O(\epsilon^{-\gamma})$ are then resolved on a mesh that coarsens from local mesh size $O(\epsilon)$ to $O(\epsilon^{1-\gamma})$ after another $J = O(\epsilon^{-1})$ intervals.

The following proposition collects bounds for $\int_{\mathbb{R}\setminus(-\epsilon^\alpha,\epsilon^\alpha)} |\zeta|^p (\nu - \hat{\nu}^{\epsilon,\kappa})(d\zeta)$ and mesh strategies (i) through (iii), for $\nu \in \mathcal{L}(\mathbb{R})$ of type $(\alpha, \beta)$; its proof shows improved approximation properties of mesh strategy (iii) which matches the structural properties of $\nu$, and evidences the dependence of the approximation error on $\epsilon > 0$, and $\alpha, \beta \geq 0$.

**Proposition 3.3.** Let $1 \leq p \leq 2$, $\epsilon > 0$, and $\nu \in \mathcal{L}(\mathbb{R})$ be a Lévy measure of type $(\alpha, \beta)$. For some $D \subset \mathbb{R}$, let

$$\mathcal{E}_{\nu,\kappa}(D) := \int_{\mathbb{R}\setminus(-\epsilon^\alpha,\epsilon^\alpha)} \mathbb{1}_D(\zeta)|\zeta|^p (\nu - \hat{\nu}^{\epsilon,\kappa})(d\zeta) \quad (\kappa = 1, 2)$$

be the error due to mesh strategies (i) through (iii) above. Then for every compact $K \subset \mathbb{R}$,

1. $\mathcal{E}_{\nu,1}(\mathbb{R}) \leq C\epsilon^{(1-\alpha)\frac{2-p}{3}}$ for $0 \leq \alpha < 1$ ('equally weighted mesh'),
2. $\mathcal{E}_{\nu,1}(K) \leq C \max\{\epsilon^{p-\alpha}, \epsilon\}$ ('equally spaced mesh'),
3. $\mathcal{E}_{\nu,2}(\mathbb{R}) \leq C \max\{\epsilon^{2(p-\alpha)}, \epsilon^{1-2\gamma}, \epsilon^{\gamma(3-p)}\}$ ('stretched mesh').

Three sources of errors need to be controlled for stretched meshes (iii): those attributed to small jumps, to large jumps, and those not resolved by the mesh for finite $\epsilon > 0$; in particular, choosing $\gamma < \frac{1}{2}$ small causes larger truncation errors in the presence of large jumps. The convergence estimate for the mesh (i) is limited to cases $0 \leq \alpha < 1$. Equally spaced meshes are only competitive in case the Lévy measure is supported on compact subsets of $\mathbb{R}$.

**Proof.** (iii) We independently study the approximation of small, medium, and large jumps. We first bound the error due to approximating $\nu$ by $\hat{\nu}^{\epsilon,2}$ of form (4) on $\bigcup_{j=1}^{J} D_j$. Since $\nu$ is a Lévy measure of type $(\alpha, \beta)$, there exists some constant $C > 0$ such that

$$\nu(d\zeta) \leq C|\zeta|^{-(1+\alpha)}d\zeta \quad \forall \zeta \in \mathbb{R} : |\zeta| = o(1).$$

Hence,

$$\nu(D_j) \leq C|x_j|^{-(1+\alpha)}\lambda(D_j), \quad \text{and} \quad x_j = \epsilon^2 \sum_{\ell=1}^{j} \ell \leq C_j^2 \epsilon^2 = C_j \epsilon_j.$$
Because of $[\epsilon^2, 1] = \bigcup_{j=1}^J D_j$, fundamental theorem of calculus, the identity $x_{j+1} - x_j = \epsilon_{j+1}$, the bound $|x_{j+1}| \leq 2|x_j|$, and the above estimates,

$$\int_{[\epsilon^2, 1]} |\zeta|^p (\nu - \nu^\circ \zeta)(d\zeta) \leq C \sum_{j=1}^J \left( \int_{D_j} |\zeta|^p \nu(d\zeta) - |x_j|^p \nu(D_j) \right)$$

$$\leq C \sum_{j=1}^J \left( |x_{j+1}|^p - |x_j|^p \right) \nu(D_j) \leq C \sum_{j=1}^J |x_{j+1}|^{p-1} (x_{j+1} - x_j) \nu(D_j)$$

$$\leq C \sum_{j=1}^J |x_j|^{p-1-(1+\alpha)} \epsilon_j^2 \leq C \sum_{j=1}^J \epsilon_j^{p-\alpha} j^{p-\alpha-2} \leq C \epsilon^{2(p-\alpha)} \sum_{j=1}^J j^{p-\alpha-2+p-\alpha}.$$  

We conclude for $J = \mathcal{O}(\epsilon^{-1})$ that

$$\sum_{j=1}^J j^{2(p-\alpha-1)} \leq C(1 + J^{2(p-\alpha)-1}) \leq C \max\left\{ 1, \epsilon^{1-2(p-\alpha)} \right\},$$

such that

$$\int_{[\epsilon^2, 1]} |\zeta|^p (\nu - \nu^\circ \zeta)(d\zeta) \leq C \max\left\{ \epsilon^{2(p-\alpha)}, \epsilon \right\}.$$  

Next, we bound this error on $\bigcup_{j=1}^J \tilde{D}_j$, for $\tilde{D}_j := [\tilde{x}_j, \tilde{x}_{j+1}]$, and $\tilde{x}_j := x_{2j}$. We may assume that there exists a constant $C > 0$ such that

$$\nu(d\zeta) \leq C |\zeta|^{-(1+\beta)} d\zeta \quad \forall \zeta \in \mathbb{R} : |\zeta|^{-1} = o(1).$$  

We then conclude that for $1 \leq j \leq J$,

$$\nu(\tilde{D}_j) \leq C|\tilde{x}_j|^{-(1+\beta)} \lambda(\tilde{D}_j)$$

$$\tilde{x}_j = x_{2j} + \epsilon \sum_{\ell=1}^j \tilde{\gamma}_\ell \leq C(1 + J^{1+\gamma}\epsilon) = C(1 + j\epsilon_j).$$

Similar to before,

$$\int_{[x_{2j}, x_{3j}]} |\zeta|^p (\nu - \nu^\circ \zeta)(d\zeta) \leq C \sum_{j=1}^J |\tilde{x}_j|^{p-2-\beta} \epsilon_j^2.$$  

Noting $|\tilde{x}_j|^{p-2-\beta} \leq C$ for all $1 \leq j \leq J$, and $\sum_{j=1}^J j^2 \gamma^2 \leq C J^{2\gamma+1} \epsilon^2$, we obtain

$$\int_{[x_{2j}, x_{3j}]} |\zeta|^p (\nu - \nu^\circ \zeta)(d\zeta) \leq C \epsilon^{1-2\gamma}.$$  

Finally, since $x_{3j} = \mathcal{O}(\epsilon^{-\gamma})$, truncating larger jumps introduces an error that is bounded by

$$\int_{[x_{3j}, \infty]} |\zeta|^p \nu(d\zeta) \leq C \int_{x_{3j}, \infty} |\zeta|^{p-(1+\beta)} d\zeta \leq C \epsilon^{-\gamma(p-\beta)}.$$  

(ii) This assertion easily follows from the arguments above.

(i) Note that $\nu([\epsilon, \infty)) \leq C(\epsilon^{-\alpha} + 1)$, such that $\nu(D_j) = \frac{1}{\epsilon} \nu([\epsilon, \infty)) \leq C \epsilon^{-\alpha}$, where $J = \mathcal{O}(\epsilon^{-1})$. Because of (8), which applies for $0 \leq \alpha < 1$, there holds $|x_j| = \mathcal{O}(\epsilon^{-\frac{1-\alpha}{p}})$. Hence, since $p \geq 1$,

$$\int_{[\epsilon, \infty]} |\zeta|^p (\nu - \nu^\circ \zeta)(d\zeta) \leq C \sum_{j=1}^J |x_j|^{p-1} \lambda(D_j) \nu(D_j) \leq C \epsilon^{1-\alpha} |x_j|^{p-1} \sum_{j=1}^J \lambda(D_j) = C \epsilon^{1-\alpha} |x_j|^p,$$
which implies the assertion. \hfill \Box

**Remark 3.1.** 1. The given strategy based on the stretched grid (iii) may easily be generalized: Let $D_1 = [\epsilon, x_2]$ and choose $\epsilon_j := \lambda(D_j)$ according to

$$
\epsilon_j = \begin{cases} 
  j^b\epsilon^a & \text{if } x_j < 1, \\
  j^d\epsilon^c & \text{if } x_j \geq 1,
\end{cases}
$$

for some $0 < a, c$ and $0 \leq b, d$. Since we only allow $J = O(\epsilon^{-1})$ intervals, this implies $a = b + 1$.

The stretched mesh (iii) corresponds to $(a, b, c, d) = (2, 1, 1, \gamma)$, while the equally spaced mesh results from $(a, b, c, d) = (1, 0, 1, 0)$.

The corresponding approximation error may be shown accordingly,

$$
\mathcal{E}_{\tau, \alpha}(\mathbb{R}) \leq C \max\{\epsilon^{a(p-\alpha)}, \epsilon^{2(c-d)-1}, \epsilon^{d+1-c}(-p-\beta)\}.
$$

Then, choices $a \geq 2$, and $\frac{d}{2} < c < d + 1$ may lead to furtherly reduced approximation errors of $\hat{D}^\epsilon$.

2. Equally weighted meshes depend on the given Lévy measure $\nu$. For $J = O(\epsilon^{-1})$ many atoms, the measure approximation error converges to 0 if $\alpha < 1$; in situations where $\alpha \geq 1$, $J = o(\epsilon^{-\alpha})$ is sufficient to guarantee convergence.

**Approximating the small jumps by a Wiener process:** Let $\kappa = 1$. So far, small jumps are truncated, but may be approximated by a Wiener process: at each time-step $m \in \mathbb{N}$, we generate a Gaussian random variable $\Delta^m W_\epsilon$, where

$$
\Delta^m W_\epsilon \sim \mathcal{N}\left(0, \sigma^2(\epsilon)\tau\right), \quad \text{and} \quad \sigma(\epsilon) := \left(\int_{-\epsilon}^{\epsilon} \zeta^2 \nu(d\zeta)\right)^{\frac{1}{2}}.
$$

Then, the Lévy random walk in (2) is approximated by

$$
\left(\hat{\Delta}^0_{\tau} L_{\epsilon, 1} + \Delta^0_{\tau} W_\epsilon, \hat{\Delta}^1_{\tau} L_{\epsilon, 1} + \Delta^1_{\tau} W_\epsilon, \hat{\Delta}^2_{\tau} L_{\epsilon, 1} + \Delta^2_{\tau} W_\epsilon, \ldots, \hat{\Delta}^m_{\tau} L_{\epsilon, 1} + \Delta^m_{\tau} W_\epsilon, \ldots\right).
$$

A proper balancing of truncation and time discretization yields improved convergence rates if the small jumps are additionally accounted for in this way.

**Proposition 3.4.** Let $1 \leq p \leq 2$, and $\nu \in L(\mathbb{R})$ be of type $(\alpha, \beta)$, $\alpha \in (0, 2]$. Let $L_\epsilon^\tau$ be a Lévy process with characteristic measure $\nu|(-\epsilon, \epsilon)$, and $W_\epsilon$ be a Wiener process of variance $\sigma^2(\epsilon)\rho$. Suppose $\epsilon \leq C\tau^{1/\alpha}$. Then

$$
\mathbb{E}\left[|\Delta^m_{\tau} L_\epsilon^\tau - \Delta^m W_\epsilon|^{p}\right] \leq C\tau^{p} \epsilon^{p(1-\frac{2}{p})} \quad (m \in \mathbb{N}),
$$

where $\Delta^m_{\tau} L_\epsilon^\tau := \Delta^m_{\tau} L - \Delta^m_{\tau} L_{\epsilon, 1}$.

**Proof.** We calculate the difference by its Fourier transform using tail estimates. First, note that for some $\delta > 0$

$$
\mathbb{E}\left[|\Delta^m_{\tau} L_\epsilon^\tau - \Delta^m_{\tau} W_\epsilon|^{p}\right] = C \int_{0}^{\infty} \mathbb{P}\left[|\Delta^m_{\tau} L_\epsilon^\tau - \Delta^m_{\tau} W_\epsilon| \geq x\right] x^{p-1} \, dx \\
\leq C \left(\delta^p + \int_{\delta}^{\infty} \mathbb{P}\left[|\Delta^m_{\tau} L_\epsilon^\tau - \Delta^m_{\tau} W_\epsilon| \geq x\right] x^{p-1} \, dx\right).
$$

In order to estimate the second integral, we use [Kal02, Lemma 5.1, p. 85]: For any random variable $X$ with Fourier transform $\hat{X}$, we have

$$
\mathbb{P}\left[|X| \geq x\right] \leq \frac{x}{2} \int_{-\frac{2}{10}}^{\frac{2}{10}} \left(1 - \Im(\hat{X}(\lambda))\right) \, d\lambda.
$$

(10)
By the independence of the Wiener process and the Poisson random measure, we infer that
\[
\mathbb{E}[e^{i\lambda(L^\nu_t - L^\nu_0)}] = \mathbb{E}[e^{i\lambda L^\nu_t}] \mathbb{E}[e^{-i\lambda L^\nu_0}]
\]
where \(\nu := \nu|_{(-\epsilon, \epsilon)}\). We substitute (11) into (10) and use the identity \(e^{iz} = \cos(z) + i\sin(z)\),

\[
\frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left(1 - \text{Re} \left( \mathbb{E} \left[ e^{i(\nu L^\nu - \nu W)} \right] \right) \right) \, d\lambda
\]

\[
= \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left\{ 1 - \text{Re} \exp \left( \tau \int_{\mathbb{R}} (e^{iy\lambda} - 1 - iy\lambda) \nu^\top(dy) + \frac{\tau}{2} \sigma^2(\epsilon) \lambda^2 \right) \right\} \, d\lambda
\]

\[
= \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left\{ 1 - \text{Re} \exp \left( \tau \int_{\mathbb{R}} (\cos(y\lambda) - 1) \nu^\top(dy) + \frac{\tau}{2} \sigma^2(\epsilon) \lambda^2 + i\tau \int_{\mathbb{R}} (\sin(y\lambda) - y\lambda) \nu^\top(dy) \right) \right\} \, d\lambda
\]

\[
= \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left\{ 1 - \exp \left( \tau \int_{\mathbb{R}} (\cos(y\lambda) - 1) \nu^\top(dy) + \frac{\tau}{2} \sigma^2(\epsilon) \lambda^2 \right) \times \cos \left( \tau \int_{\mathbb{R}} (\sin(y\lambda) - y\lambda) \nu^\top(dy) \right) \right\} \, d\lambda
\]

\[
= \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left\{ 1 - e^{z_1} \cos(z_2) \right\} \, d\lambda,
\]

where \(z_1\) and \(z_2\) are defined as

\[
z_1 := \tau \int_{\mathbb{R}} (\cos(y\lambda) - 1) \nu^\top(dy) + \frac{\tau}{2} \sigma^2(\epsilon) \lambda^2,
\]

\[
z_2 := \tau \int_{\mathbb{R}} (\sin(y\lambda) - y\lambda) \nu^\top(dy).
\]

Next, we apply the Taylor expansion around zero to \((z_1, z_2) \mapsto 1 - e^{z_1} \cos(z_2)\): there exists a constant \(C > 0\) such that

\[
|e^{z_1} \cos(z_2) - 1 - z_1| \leq C|z_1|^2 + C|z_2|^2 + C|e^{\xi_1} \cos(\xi_2)| \cdot |z_1| \left( |z_2|^2 + |z_1|^2 \right)
\]

\[
+ C|e^{\xi_1} \sin(\xi_2)| \cdot |z_2| \left( |z_1|^2 + |z_2|^2 \right)
\]

for some \(|\xi_1| \leq |z_1|\) and \(|\xi_2| \leq |z_2|\). Hence, we obtain

\[
\frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left(1 - \text{Re} \left( \mathbb{E} \left[ e^{i(\nu L^\nu - \nu W)} \right] \right) \right) \, d\lambda \leq \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left| 1 + z_1 - e^{z_1} \cos(z_2) \right| \, d\lambda
\]

\[
\leq \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left\{ C|z_1|^2 + C|z_2|^2 + C|e^{\xi_1} \cos(\xi_2)| \cdot |z_1| \left( |z_2|^2 + |z_1|^2 \right)
\]

\[
+ C|e^{\xi_1} \sin(\xi_2)| \cdot |z_2| \left( |z_1|^2 + |z_2|^2 \right) + |z_1| \right\} \, d\lambda.
\]

Since \(\nu\) is of type \((\alpha, \beta)\), there exists some \(C > 0\) such that

\[
\nu(dy) \leq C |y|^{-(1+\alpha)} \, dy, \quad |y| \leq \epsilon.
\]
Thus, we can simplify each of the three summands in the integral. Taking into account a Taylor expansion around 0 of the function $x \mapsto \cos(x)$, we obtain by using the definition of $\sigma^2(\epsilon)$

$$|z_1| = \tau \left| \int_{\mathbb{R}} (\cos(y\lambda) - 1) \nu'(dy) + \frac{1}{2} \sigma^2(\epsilon) \lambda^2 \right| = \tau \left| \int_{\mathbb{R}} -\frac{y^2}{2} \lambda^2 + \frac{1}{4!} y^4 \lambda^4 \nu'(dy) + \frac{1}{2} \sigma^2(\epsilon) \lambda^2 \right| \
\leq C \tau \lambda^4 \int_{-\epsilon}^{\epsilon} |y|^{3-\alpha} \ dy \leq C \tau \lambda^4 |\epsilon|^{4-\alpha}.$$

Hence, $|z_1|^2 \leq C \tau^2 \lambda^8 |\epsilon|^{8-2\alpha}$. By Taylor expansion around 0 of the function $x \mapsto \sin(x)$,

$$|z_2| = \tau \left| \int_{\mathbb{R}} (\sin(y\lambda) - y\lambda) \nu'(dy) \right| \leq \tau \left| \int_{\mathbb{R}} |y\lambda|^3 + \frac{1}{5!} |y\lambda|^5 \nu'(dy) \right| \leq C \tau \lambda^3 \int_{-\epsilon}^{\epsilon} |y|^{2-\alpha} \ dy + C \tau \lambda^5 \int_{-\epsilon}^{\epsilon} |y|^{6-\alpha} \ dy \leq C \tau \lambda^3 |\epsilon|^{3-\alpha} + C \tau \lambda^5 |\epsilon|^{5-\alpha},$$

and hence, $|z_2|^2 \leq C \tau^2 (\lambda^6 |\epsilon|^{6-2\alpha} + \lambda^{10} |\epsilon|^{10-2\alpha})$. Using $\epsilon \leq C \tau^{1/\alpha}$ and setting $\delta := \tau^{\frac{1}{2}} \epsilon^{1-\frac{3}{4} \alpha}$, we obtain $|z_1| \leq C$ and $|z_2| \leq C$.

We now insert these estimates in (12) and get

$$\frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left( 1 - \text{Re} \left[ e^{i(\Delta^m \bar{L}_\epsilon \Delta^m W_\epsilon)} \right] \right) \ d\lambda \leq \frac{x}{2} \int_{-\frac{x}{2}}^{\frac{x}{2}} \left\{ C \tau^2 \lambda^8 |\epsilon|^{8-2\alpha} + C \tau^2 \lambda^6 |\epsilon|^{6-2\alpha} + C \tau^2 \lambda^{10} |\epsilon|^{10-2\alpha} + C \tau p \lambda^4 |\epsilon|^{4-\alpha} + C \tau^3 \lambda^{11} |\epsilon|^{11-3\alpha} + C \tau^3 \lambda^9 |\epsilon|^{9-3\alpha} + C \tau^3 \lambda^{12} |\epsilon|^{12-3\alpha} + C \tau^3 \lambda^{10} |\epsilon|^{10-3\alpha} + C \tau^3 \lambda^{14} |\epsilon|^{14-3\alpha} + C \tau^3 \lambda^{13} |\epsilon|^{13-3\alpha} + C \tau^3 \lambda^{15} |\epsilon|^{15-3\alpha} \right\} \ d\lambda \leq C \tau^2 x^{-8} |\epsilon|^{8-2\alpha} + C \tau^2 x^{-6} |\epsilon|^{6-2\alpha} + C \tau^2 x^{-10} |\epsilon|^{10-2\alpha} + C \tau x^{-4} |\epsilon|^{4-\alpha} + C \tau^3 x^{-12} |\epsilon|^{12-3\alpha} + C \tau^3 x^{-10} |\epsilon|^{10-3\alpha} + C \tau^3 x^{-14} |\epsilon|^{14-3\alpha},$$

and hence,

$$\mathbb{E} \left[ |\Delta^m \bar{L}_\epsilon - \Delta^m W_\epsilon|^p \right] \leq C \delta^p + C \tau^2 \delta^p |\epsilon|^{8-2\alpha} + C \tau^2 \delta^p |\epsilon|^{6-2\alpha} + C \tau^2 \delta^p |\epsilon|^{10-2\alpha} + C \tau^3 \delta^p |\epsilon|^{12-3\alpha} + C \tau^3 \delta^p |\epsilon|^{10-3\alpha} + C \tau^3 \delta^p |\epsilon|^{14-3\alpha}.$$

By setting $\delta := \tau^{\frac{1}{3}} \epsilon^{1-\frac{3}{4} \alpha}$ we obtain under the restriction $\epsilon \leq C \tau^{1/\alpha}$ that the terms $\tau^r \delta^p |\epsilon|^{r-\alpha}$ converge as fast as $\delta^p$, and thus

$$\mathbb{E} \left[ |\Delta^m \bar{L}_\epsilon - \Delta^m W_\epsilon|^p \right] \leq C \tau^5 \epsilon^{p-\frac{5}{2} \alpha} \leq C \tau^5 \epsilon^{p-\frac{5}{2} \alpha}.$$

□

We may now combine Propositions 3.1 through 3.3 to control both, truncation and discretization errors to approximate $\nu \in \mathcal{L}(\mathbb{R})$; an additional compensation of small jumps by a Gaussian random variable (9) for choices $\epsilon \leq C \tau^{1/\alpha}$ (for meshes (i) and (ii)), and $\epsilon^2 \leq C \tau^{1/\alpha}$ (for mesh (iii)) may reduce the truncation error from $\mathcal{O}(\tau \epsilon^{p-\alpha})$ (for meshes (i) and (ii)), resp. $\mathcal{O}(\tau \epsilon^{2(p-\alpha)})$ (for mesh (iii)) to $\mathcal{O}(\tau^{\frac{5}{2}} \epsilon^{p-\frac{5}{2} \alpha})$ resp. to $\mathcal{O}(\tau^{\frac{5}{2}} \epsilon^{2(p-\frac{5}{2} \alpha)})$; cf. Proposition 3.4. To conclude, using a stretched mesh (iii) seems a promising alternative to using an equally spaced mesh (ii) together with additional compensation by the Gaussian variable (9).
4. The Implicit Euler scheme with truncated noise

Fix $1 < p \leq 2$. Let $f : \mathbb{R} \times \mathcal{O} \to \mathbb{R}$ be Lipschitz continuous in the first variable, and of linear growth. For $T > 0$, we consider

$$
\frac{\partial u(t, \xi)}{\partial t} = \Delta u(t, \xi) + f(u(t, \xi), \xi) + \hat{L}(t, \xi) \quad \forall (t, \xi) \in \mathcal{O}_T,
$$

(13) 

$$
u(t, \xi) = 0 \quad \forall (t, \xi) \in (0, T) \times \partial \mathcal{O},
$$

$$u(0, \xi) = u_0(\xi) \quad \forall \xi \in \mathcal{O},$$

where $u_0 \in L^p(\mathcal{O})$, and $\hat{L}(t, \xi)$ denotes the formal time derivative of the space-time Lévy noise.

**Definition 4.1.** A weak solution of equation (13) is a process $u = \{u(t, \cdot) : t \geq 0\}$, such that

(i) $u$ satisfies the following integral equation for any $\phi \in C_0^\infty(\mathcal{O})$, any $0 \leq t \leq T$ and $\mathbb{P}$-almost surely

$$\int_{\mathcal{O}} u(t, \xi) \phi(\xi) d\xi - \int_{\mathcal{O}} u(0, \xi) \phi(\xi) d\xi = \int_0^t \int_{\mathcal{O}} u(s, \xi) \Delta \phi(\xi) ds d\xi$$

$$+ \int_0^t \int_{\mathcal{O}} f(u(s, \xi), \xi) \phi(\xi) ds d\xi + \int_0^t \int_{\mathcal{O}} \phi(\xi) L(ds, d\xi).$$

(ii) for all $\phi \in C_0^\infty(\mathcal{O})$, the real-valued process $\{\langle u(t), \phi \rangle : t \in [0, T]\}$ is adapted to $\{\mathcal{F}_t\}_{t \geq 0}$ and càdlàg.

It is shown in Saint Lubert Bié [SLB98] or Hausenblas [Hau05] that if $p < \frac{2}{m} + 1$, then there exists a unique weak solution of problem (13) such that $u \in L^p(\Omega_T; L^p(\mathcal{O}))$. In fact, the semigroup approach is used in [Hau05] to construct a mild solution.

Next, we use the approximate Euler method to discretize (13) in time. For this purpose, consider the space-time random walk

$$\left(\Delta^0_\tau L, \Delta^1_\tau L, \Delta^2_\tau L, \ldots , \Delta^m_\tau L, \ldots \right).$$

(14)

For every $m \in \mathbb{N}$, the $M(\mathcal{O})$-valued random variable $\Delta^m_\tau L := L([t_m, t_{m+1}], \cdot)$ acts on functions $\phi \in C_0(\mathcal{O})$ via

$$\langle \Delta^m_\tau L, \phi \rangle_{M \times C_0} := \int_{t_m}^{t_{m+1}} \int_\mathcal{O} \int_\mathbb{R} \phi(\xi) \zeta \tilde{\nu}(ds, d\xi, d\zeta).$$

(15)

Here, $\mu$ denotes the space-time Poisson random measure on $\mathcal{O}_T \times \mathbb{R} \setminus \{0\}$ with jump size intensity $\nu$, and $\tilde{\nu}(ds, d\xi, d\zeta) = \mu(ds, d\xi, d\zeta) - \nu(d\zeta)ds d\xi$ the compensated Poisson random measure; see e.g. [A1, PZ07].

Since we will not specify approximation in space, we introduce a space-time random walk where we only approximate the jump size intensity. In particular, let $\tilde{\nu}^\epsilon \in \mathcal{L}(\mathbb{R})$ be the measure defined in (4) and let $\tilde{\nu}^\epsilon \kappa$ be a space-time Poisson random measure on $\Omega_T \times \mathbb{R} \setminus \{0\}$ with jump size intensity $\tilde{\nu}^\epsilon \kappa$. Let

$$\left(\hat{\Delta}^0_{\tau \epsilon, \kappa} L_{\epsilon, \kappa}, \hat{\Delta}^1_{\tau \epsilon, \kappa} L_{\epsilon, \kappa}, \hat{\Delta}^2_{\tau \epsilon, \kappa} L_{\epsilon, \kappa}, \ldots , \hat{\Delta}^m_{\tau \epsilon, \kappa} L_{\epsilon, \kappa}, \ldots \right).$$

(16)

be the approximation of the random walk, where

$$\langle \hat{\Delta}^m_{\tau \epsilon, \kappa} L_{\epsilon, \kappa}, \phi \rangle_{M \times C_0} := \int_{t_m}^{t_{m+1}} \int_\mathcal{O} \int_\mathbb{R} \phi(\xi) \zeta \tilde{\nu}^\epsilon \kappa(ds, d\xi, d\zeta) \quad \forall \phi \in C_0(\mathcal{O}).$$

(17)

For every $m \geq 1$, let $v^m_v : \mathcal{O} \times \Omega \to \mathbb{R}$ be a solution of the following implicit Euler scheme at time $t_m$ to approximate $u(t_m, \cdot)$. 

13
Let \( v_\tau^0 = u_0 \). For every \( m \in \mathbb{N} \), the random variable \( v_\tau^{m+1} \) solves

\[
(18) \quad v_\tau^{m+1} = (1 - \tau \Delta)^{-1} \left( v_\tau^m + \tau f(v_\tau^m) + \Delta_\tau^m \mathbf{L} \right),
\]

where \( \{ \Delta_\tau^m : m \in \mathbb{N} \} \) is the family of random variables given in (14).

Note that \( \Delta_\tau^m \mathbf{L} \) is a \( M(O) \)-valued random variable; by [AQ1] for \( 1 \leq q < \infty \), the random variable \( w^m := (1 - \tau \Delta)^{-1} \Delta_\tau^m \mathbf{L} \) is a \( L^p \)-valued distributional solution of

\[
\int_O w^m(\xi)(1 - \tau \Delta)\phi(\xi) \, d\xi = \int_O \phi(\xi) \Delta_\tau^m \mathbf{L}(d\xi) \quad \forall \phi \in W^{1.2}_0(O) \cap W^{2.2}_0(O).
\]

Approximation of the space-time Lévy noise leads to the following approximate Euler scheme.

**Scheme B.** Let \( \hat{v}_\tau^0 = u_0 \). For every \( m \in \mathbb{N} \), let \( \hat{v}_\tau^{m+1} \) be a \( L^p(O) \)-valued random variable that solves

\[
(19) \quad \hat{v}_\tau^{m+1} = (1 - \tau \Delta)^{-1} \left( \hat{v}_\tau^m + \tau f(\hat{v}_\tau^m) + \hat{\Delta}_\tau^m \mathbf{L}_{\epsilon, \kappa} \right),
\]

where \( \{ \hat{\Delta}_\tau^m \mathbf{L}_{\epsilon, \kappa} : m \in \mathbb{N} \} \) is the family of random variables given in (16).

We are ready to formulate our main result.

**Theorem 4.1.** Fix \( T > 0 \), and a polyhedral domain \( O \subset \mathbb{R}^d \), for \( d = 2, 3 \). Let \( u_0 \in L^p(O) \), for some \( 1 < p < \frac{2}{d} + 1 \) be given. Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a complete filtered probability space, and \( t \mapsto \mathbf{L}([0, t] \times \cdot) \) be space-time Lévy noise with jump-size intensity \( \nu \in \mathcal{L}(\mathbb{R}) \) of type \( (\alpha, \beta) \), where \( \alpha \in (0, p) \), and \( \beta \in (p, \infty) \). Then, iterates \( \{ v_\tau^m : m \in \mathbb{N} \} \) of Scheme A, and \( \{ \hat{v}_\tau^m : m \in \mathbb{N} \} \) of Scheme B satisfy

\[
\sup_{0 \leq m \leq M} \mathbb{E} \left[ \| v_\tau^m - \hat{v}_\tau^m \|^p_{L^p(O)} \right] \leq C \left( \epsilon^{(p-\alpha)} + \mathcal{E}_{\tau \mathbb{R}^d}(\mathbb{R}) \right), \quad (\kappa = 1, 2),
\]

depending on the mesh strategies (i) through (iii); see Proposition 3.3.

The following Proposition 4.1 controls errors due to the approximation of the space-time Lévy noise \( \mathbf{L} \).

**Proposition 4.1.** Consider \( \{ \Delta_\tau^m \mathbf{L} : m \in \mathbb{N} \} \) and \( \{ \hat{\Delta}_\tau^m \mathbf{L}_{\epsilon, \kappa} : m \in \mathbb{N} \} \) which is defined by (15) and (17). Then we have for \( m \in \mathbb{N} \), and values \( \frac{d}{q} < \gamma < \frac{2}{p} \)

<table>
<thead>
<tr>
<th>Case</th>
<th>Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>( \mathbb{E} \left[ | \Delta_\tau^m \mathbf{L} - \hat{\Delta}<em>\tau^m \mathbf{L}</em>{\epsilon, \kappa} |^p_{\mathcal{W}^{1.2}<em>0(O)} \right] \leq C \tau \left( \epsilon^{(p-\alpha)} + \mathcal{E}</em>{\tau \mathbb{R}^d}(\mathbb{R}) \right) )</td>
</tr>
<tr>
<td>(ii)</td>
<td>( \mathbb{E} \left[ | \Delta_\tau^m \mathbf{L} |^p_{\mathcal{W}^{1.2}<em>0(O)} \right] + \mathbb{E} \left[ | \hat{\Delta}</em>\tau^m \mathbf{L}<em>{\epsilon, \kappa} |^p</em>{\mathcal{W}^{1.2}_0(O)} \right] \leq C \tau )</td>
</tr>
</tbody>
</table>

**Proof of 4.1.** Let \( p^{-1} + (p')^{-1} = 1 \). Since \( \gamma > \frac{d}{p} = \frac{d}{p} \), we use the continuous embedding \( W^{1,p}_0(O) \hookrightarrow C_0(O) \) to conclude by duality, and (15), (17),

\[
\mathbb{E} \left[ \| \Delta_\tau^m \mathbf{L} - \hat{\Delta}_\tau^m \mathbf{L}_{\epsilon, \kappa} \|^p_{\mathcal{W}^{1.2}_0(O)} \right] \leq C \mathbb{E} \left[ \sup_{\phi \in C_0(O), \| \phi \|_{C(O)} = 1} \left( \Delta_\tau^m \mathbf{L} - \hat{\Delta}_\tau^m \mathbf{L}_{\epsilon, \kappa}, \phi \right)_{M \times C_0} \right]^p.
\]

By Burkholder’s inequality for \( 1 \leq p \leq 2 \), we resume

\[
\leq C \mathbb{E} \left[ \left( \int_{t_m}^{t_{m+1}} \xi(\mu - \hat{\mu}^\epsilon_{\kappa})(ds, d\xi, d\zeta) \right)^p \right] \leq C \tau \mathbb{E} \left[ \int_{\mathbb{R}} |\xi|^p (\nu - \hat{\nu}^\epsilon_{\kappa})(d\zeta) \right],
\]

and Proposition 3.3 then shows assertion (i). — Assertion (ii) follows accordingly. \( \square \)
The following calculation bounds the error between iterates from Scheme A and Scheme B, which implies the assertion of Theorem 4.1.

**Proof.** For $m \geq 1$, iterates $v^m_\tau$ and $\bar{v}^m_\tau$ satisfy

$$v^m_\tau = (1 - \tau \Delta)^{-m} u_0 + \sum_{k=0}^{m-1} (1 - \tau \Delta)^{-(m-k)} \left( \tau f(v^k_\tau) + \Delta^k L \right),$$

and

$$\bar{v}^m_\tau = (1 - \tau \Delta)^{-m} u_0 + \sum_{k=0}^{m-1} (1 - \tau \Delta)^{-(m-k)} \left( \tau f(\bar{v}^k_\tau) + \Delta^k \bar{L}_{c,K} \right).$$

**Step 1. Stability of $\{v^m_\tau : m \in \mathbb{N}\}$ and $\{\bar{v}^m_\tau : m \in \mathbb{N}\}$.** We show stability of iterates in $L^p(\Omega; L^p(\mathcal{O}))$, which is needed in Step 2 below.

$$\mathbb{E}[\|v^m_\tau\|_{L^p}^p] \leq C \|(1 - \tau \Delta)^{-m} u_0\|_{L^p}^p + C \mathbb{E} \left[ \left| \sum_{k=0}^{m-1} (1 - \tau \Delta)^{-(m-k)} \left( \tau f(v^k_\tau) + \Delta^k L \right) \right|_{L^p}^p \right]$$

$$\leq C \|(1 - \tau \Delta)^{-m} u_0\|_{L^p}^p + C \left( \sum_{k=0}^{m-1} \mathbb{E} [\|f(v^k_\tau)\|_{L^p}] \right)^p + C \sum_{k=0}^{m-1} \mathbb{E} \left[ \|(1 - \tau \Delta)^{-(m-k)} \Delta^k L \|_{L^p}^p \right]$$

$$\leq C \|u_0\|_{L^p}^p + C \left( \sum_{k=0}^{m-1} \mathbb{E} [\|f(v^k_\tau)\|_{L^p}] \right)^p + C \sum_{k=0}^{m-1} \frac{\tau}{(m-k)^{2p}}$$

where $d - \frac{d}{p} < \gamma < \frac{2}{p}$. Here, the lower bound of $\gamma$ is needed for the Sobolev inbedding in Proposition 4.1 and the upper bound guarantees the stability of the sum, see below. In the above calculations, we applied first the martingale type $p$ property of the space $L^p(\mathcal{O})$ (for more details see [BHau09, Appendix A]). The product of resolvents $(1 - \tau \Delta)^{-(m-k)}$ inherits the smoothing property of the semigroup, and satisfies (see [HM06, Prop. 4.2] with $X = W^{-\gamma,p}(\mathcal{O})$, $\delta = \frac{\gamma}{2}$ and $\lambda = \frac{1}{p}$) for all $\gamma \geq 0$,

$$\|(1 - \tau \Delta)^{-m} \phi\|_{L^p} \leq \frac{\Gamma(m - \frac{\gamma}{2})}{\Gamma(m)} \tau^{-\frac{\gamma}{2}} \|\phi\|_{W^{-\gamma,p}}$$

$$\leq \frac{\bar{C}}{(\tau m)^{\gamma/2}} \|\phi\|_{W^{-\gamma,p}} \quad \forall m \in \mathbb{N}, \quad \forall \phi \in W^{-\gamma,p}(\mathcal{O}).$$

This bound is used for the third estimate above; the last estimate uses linear growth of $f$, and Proposition 4.1, (ii).

By assumption $\gamma < \frac{2}{p}$, hence $\frac{\gamma}{2} < 1$, such that we can bound $\sum_{k=0}^{m-1} \frac{\tau}{(m-k)^{2p}} \leq \int_0^T x^{-\frac{2p}{2}} dx \leq \bar{C}$.

Note that the constants $\bar{C}$ depend on $p$ and $d$. We obtain

$$\mathbb{E}[\|v^m_\tau\|_{L^p}^p] \leq C \|u_0\|_{L^p}^p + C\tau \sum_{k=0}^{m-1} \mathbb{E} [\|v^k_\tau\|_{L^p}^p].$$

Stability then follows by the discrete version of Gronwall’s lemma. Stability of $\{\bar{v}^m_\tau : m \in \mathbb{N}\}$ follows along the same lines.
Step 2. Bound for $E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right]$. 

$$E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right] \leq E\left[\sum_{k=0}^{m-1} \left(1 - \tau \Delta\right)^{-(m-k)} \left(\tau f(v^k_t) - \tau f(\tilde{v}^k_t) + \Delta^k_{\gamma}L - \tilde{\Delta}^k_{\gamma}L_{\epsilon,\kappa}\right)\right]^{p}_{L^p}.$$ 

By Minkowsky’s inequality, and the martingale type $p$ property give 

$$E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right] \leq E\left[\sum_{k=0}^{m-1} \left(1 - \tau \Delta\right)^{-(m-k)} \left(\tau f(v^k_t) - \tau f(\tilde{v}^k_t)\right)^p\right] + E\left[\sum_{k=0}^{m-1} \left(1 - \tau \Delta\right)^{-(m-k)} \left(\Delta^k_{\gamma}L - \tilde{\Delta}^k_{\gamma}L_{\epsilon,\kappa}\right)\right]^{p}_{L^p}.$$ 

We use again inequality (20) to conclude for $d - \frac{d}{p} < \gamma < \frac{2}{p}$ 

$$E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right] \leq E\left[\left(\sum_{k=0}^{m-1} \tau f(v^k_t) - \tau f(\tilde{v}^k_t)\right)^p\right] + \frac{1}{\tau(m - k)} E\left[\left|\Delta^k_{\gamma}L - \tilde{\Delta}^k_{\gamma}L_{\epsilon,\kappa}\right|_{W^{-\gamma,p}}\right].$$ 

By Lipschitz continuity of $f$, and Proposition 4.1, (ii) we get 

$$E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right] \leq C E\left[\left(\sum_{k=0}^{m-1} \tau v^k_t - \tilde{v}^k_t\right)^p\right] + C \sum_{k=0}^{m-1} \frac{\tau}{(\tau(m - k))^{\frac{p}{2}}} \left(e^{k(p-\alpha)} + E\tau\epsilon,\kappa(\mathbb{R})\right).$$ 

(21) 

Again, since $\frac{p\tau}{2} < 1$ we obtain 

$$E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right] \leq C \tau \sum_{k=0}^{m-1} E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right] + C \left(e^{k(p-\alpha)} + E\tau\epsilon,\kappa(\mathbb{R})\right).$$ 

The discrete version of Gronwall’s lemma applied to the term $E\left[\|v_t^m - \tilde{v}_t^m\|_{L^p}\right]$ gives the assertion. 

\[\Box\]

5. Computational Experiments

5.1. A compensatory Gaussian variable to account for small jumps. In this section, we study the numerical approximation at final time $T = 1$ of the following SDE

$$X(t) = X_0 + \int_0^t \sigma(X(s-)) \, dL(s) \quad (t \geq 0), \quad X_0 = 0,$$

where $\sigma(y) := \sqrt{1 + y^2}$, and $L \equiv \{L(t); t \geq 0\}$ is a one-dimensional Lévy process with Lévy measure

$$\mathcal{B}(\mathbb{R}) \ni I \mapsto \nu_\alpha(I) := \int_I 1_{\{x>0\}}(x) x^{-1-\alpha} \exp(-x^2) \, dx,$$

for $\alpha = 1.5$. We compare the approximate Euler scheme with (Scheme C) and without (Scheme D) a complementary Gaussian variable; cf. [Rub03, Fou10, AR01].

Scheme C. Let $X^0_{\tau,\epsilon} = X_0$, and $\epsilon > 0$. For every $m = 0, \ldots, M$, let $X^m_{\tau,\epsilon}$ be a random variable that solves

$$X^{m+1}_{\tau,\epsilon} = X^m_{\tau,\epsilon} + \sigma(X^m_{\tau,\epsilon}) \tilde{\Delta}^m_t L_{\epsilon,\kappa},$$

where $\sigma(y) := \sqrt{1 + y^2}$, and $L \equiv \{L(t); t \geq 0\}$ is a one-dimensional Lévy process with Lévy measure

$$\mathcal{B}(\mathbb{R}) \ni I \mapsto \nu_\alpha(I) := \int_I 1_{\{x>0\}}(x) x^{-1-\alpha} \exp(-x^2) \, dx,$$
where $\hat{\Delta}^m_{\tau,\epsilon,\kappa}$ denotes the Lévy increment, which is given in Section 3.

**Scheme D.** Let $X^{0,G}_{\tau,\epsilon} = X_0$, and $\epsilon > 0$. For every $m = 0, \ldots, M$, let $X^{m,G}_{\tau,\epsilon}$ be a random variable that solves

$$X^{m+1,G}_{\tau,\epsilon} = X^{m,G}_{\tau,\epsilon} + \sigma(X^{m,G}_{\tau,\epsilon}) \left( \hat{\Delta}^m_{\tau,\epsilon,\kappa} + \Delta^m_{\tau,\epsilon,\kappa} \right),$$

where $(\hat{\Delta}^m_{\tau,\epsilon,\kappa} + \Delta^m_{\tau,\epsilon,\kappa})$ denotes the Lévy increment with compensatory Gaussian increment $\Delta^m_{\tau,\epsilon,\kappa} \sim \mathcal{N}(0, \sigma^2(\epsilon^\kappa)\tau)$; see (9).

We use 100,000 realizations and a time-step size $\tau = 0.01$. Since we cannot simulate the Lévy increments exactly, we approximate the exact solution using a stretched mesh with $\epsilon_{ex} = 0.001$ and $\gamma = 0.1$. In order to compare the stretched mesh with the equally spaced mesh, we fix the amount of intervals to $J = \epsilon_{es}^{-1} = 100$. As a consequence, we have to use a different truncation parameter $\epsilon_s$ for the stretched mesh.

Figures 3 and 4 display Q-Q plots of the solution at final point $T = 1$ in the restricted range $[-20, 20]$, evidencing improved accuracy of the solution in the presence of Gaussian increments.

By using a stretched mesh strategy, small jumps are truncated at $\epsilon_s^\kappa < \epsilon_{es}$ for $\kappa > 1$: as a consequence, an additional approximation by a Gaussian variable is less significant if compared to the equally spaced mesh, since the additional Gaussian variable scales with $\sigma(\epsilon) \leq C\epsilon^{\frac{2-\alpha}{2}}$. Hence, the influence of the additional Gaussian variable grows for growing values $\alpha$ of the Lévy measure and the truncation parameter $\epsilon$: performed computational studies evidence a bad behavior of equally weighted meshes for values $\alpha > 1$, where also assertion (i) of Proposition 3.3 does not apply, and confirm an improved performance of stretched meshes (see also Remark 3.1), where increased stretching reduces the influence of the Gaussian variable.

5.2. **SPDE Simulation Setup.** We report on computational studies for the different discretization strategies from Section 4 in the case of both, finite and infinite driving Lévy measures of type

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**Figure 3.** Equally Spaced Mesh: Quantile-Quantile plots of the solution $X^M_{\tau,\epsilon}$ (left) resp. $X^{M,G}_{\tau,\epsilon}$ (right).
(α, β). Let \( O := (0, 5) \), \( T = 1 \), and consider
\[
\begin{align*}
\frac{\partial u(t, \xi)}{\partial t} &= \frac{1}{10} \cdot \Delta u(t, \xi) + \dot{L}(t, \xi) \quad \text{in } O_T, \\
u(0, \xi) &= \sin \left( \frac{\pi}{5} \xi \right) \quad \text{in } O, \\
u(t, \xi) &= 0 \quad \text{on } (0, 1) \times \partial O.
\end{align*}
\]
We use different driving Lévy processes: the first Lévy measure is \( \nu_\alpha \) defined in (22) for different \( \alpha \in (0, 2) \), which cannot be simulated exactly. The second Lévy measure corresponds to an \( IG(0.5, 1) \) process, where iterates may be simulated exactly; cf. [Sch03, p. 111],
\[
B(\mathbb{R}) \ni I \mapsto \nu_{IG}(I) := \int_I \frac{1}{\sqrt{2\pi}} x^{-3/2} \exp \left( -\frac{1}{2} x \right) dx.
\]
The third Lévy measure describes a Poisson process and can be simulated exactly,
\[
B(\mathbb{R}) \ni I \mapsto \nu_P(I) := \int_I \frac{1}{2} \left( \delta_{0.5}(x) + \delta_1(x) \right) dx.
\]
Note that only positive jumps occur for all, \( \nu_\alpha \), \( \nu_{IG} \), and \( \nu_P \). The following section concerns discretization in space of Scheme B from Section 4.

5.3. Discretization in space using finite elements. Let \( \{ \mathcal{T}_h : 0 < h \leq 1 \} \) be a family of subdivisions of regular triangulations covering \( O \subset \mathbb{R}^d \) of maximum mesh size \( 0 < h \leq 1 \), i.e., \( \overline{O} = \bigcup_{K \in \mathcal{T}_h} \overline{K} \subset \mathbb{R}^d \). We define the finite element space
\[
\mathcal{V}_h := \left\{ v_h \in C_0(\overline{O}) : v_h |_K \in \mathcal{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\},
\]
where \( \mathcal{P}_1 \) denotes the space of polynomials of degree one. Let \( \mathcal{N}_h = \{ N_i : i = 1, \ldots, l \} \) denote the set of nodal points, and \( \{ \phi_i : i = 1, \ldots, l \} \subset \mathcal{V}_h \) related basis functions such that \( \phi_i(N_j) = \delta_{ij} \) \((i, j = 1, \ldots, l)\). The \( L^2(O) \)-orthogonal projection \( P_h : L^2(O) \to \mathcal{V}_h \) is defined via
\[
\left( P_h v - v, \chi \right) = 0 \quad \forall \chi \in \mathcal{V}_h.
\]
In the following, we use approximate Lévy increments \( \Delta^m \tau e^k \) at nodal points \( N_i \) that are defined in (5).
Scheme $\tilde{B}$. Let $\tilde{v}_h^0 = P_h u_0$. For every $m = 0, \ldots, M$, let $\tilde{v}_h^m$ be a $V_h$-valued random variable that solves

\begin{equation}
\left( \tilde{v}_h^m - \tilde{v}_h^{m-1}, \phi \right) + \tau \left( \nabla \tilde{v}_h^m, \nabla \phi \right) = \sum_{i=1}^{l} \left( \tilde{M}_i^m L_i^e \phi_i, \phi \right) \quad \forall \phi \in V_h.
\end{equation}

The results shown in Figure 1 are based on Scheme $\tilde{B}$.

5.4. Simulation of a Lévy walk. Let $\kappa \geq 1$. Approximate Lévy increments $\{\tilde{\Delta}_k^i L_i^e, \kappa \}_{1 \leq i \leq l}$ will be generated for every $k \leq 0$ by the following algorithm.

**Algorithm.** Let $0 \leq k \leq M$.

a) For each interval $D_j$, simulate an independent Poisson distributed random variable $q_{i,k}^j$ with intensity $\tau \nu(D_j)$.

b) Calculate the increment through

$$\tilde{\Delta}_k^i L_i^e := \sum_{j=1}^{J} d_j \left( q_{i,k}^j - \tau \nu(D_j) \right).$$

c) Optional: Approximation of small jumps.

Calculate increments $\Delta_k^i W_i^e \sim \mathcal{N}(0, \sigma^2(\epsilon^\kappa) \tau)$ and add them to $\tilde{\Delta}_k^i L_i^e$.

See Figure 5 for possible outcomes for (23) with Lévy measure $\nu_\alpha$, for $\alpha = 0.25$ and $\alpha = 1.75$, simulated with a stretched grid with $\gamma = 0.1$ and $\epsilon = 0.001$.

In the following sections, we study the behavior of the error $E\left[ \|u(\tau m) - \tilde{v}_h^m \|_{L^2} \right]$ concerning the time discretization parameter $\tau$, the truncation parameter $\epsilon$ and the measure discretization error $E_{rr,e}(\epsilon)$. After different simulations varying between 100 and 6000 paths we have chosen $I \approx 1000$ paths to conduct reliable statistics.

**Figure 5.** One possible path of the solution of equation (23) with $\alpha = 0.25$ (left) and $\alpha = 1.75$ (right).

5.5. Time-step size error $\tau$. Let $(h, \epsilon) = \left( \frac{1}{200}, \frac{1}{100} \right)$, and $\gamma = 0.1$. We study the error of solutions from Scheme $\tilde{B}$ for different time-steps $\tau_i = 2^{-i}$, with $i = 4, \ldots, 8$; the exact solution $\{\tilde{v}_h^m,ex\}_{m=0}^{M}$ is simulated for the smallest time-step size $\tau_{ex} = 2^{-10}$. In the case of the IG or Poisson process, approximations of the exact solution are simulated using exact increments of the driving noise to study error effects. We consider the $L^2$-norm of the difference between $\{\tilde{v}_h^m,ex\}_{m=0}^{M}$ and $\{\tilde{v}_h^m\}_{m=0}^{M}$ from Scheme $\tilde{B}$ at the final time $T = 1$,

$$\epsilon(\tau_i, \omega_j) := \|\tilde{v}_h^M(\omega_j) - \tilde{v}_h^M,ex(\omega_j)\|_{L^2(\Omega)}.$$
and compute the mean over all $I$ paths

$$S(\tau_i) := \left( \frac{1}{I} \sum_{j=1}^{I} \epsilon(\tau_i, \omega_j) \right)^{\frac{1}{2}} \approx \left( \mathbb{E} \left[ \|\bar{v}_{h,i}^{M} - \bar{v}_{h,ex}^{M}\|_{L^2(O)}^2 \right] \right)^{1/2}.$$ 

To get the approximate rate of convergence $r_s$ we compute

$$\frac{S(\tau_i)}{S(\tau_{i+1})} \approx \left( \frac{\tau_i}{\tau_{i+1}} \right)^{r_s,i} = 2^{r_s,i},$$

and compute $r_s$ as the mean over all $r_s,i$.

For $d = 1$ and $p = 2$, [HM06, Theorem 3.1], the theoretical convergence rate for errors is 0.5. Computed rates of convergence with respect to the time discretization parameter $\tau$ are shown in Figure 6, which slightly change for the different Lévy measures $\nu_{\alpha}$ and are contained in the range $[0.35, 0.45]$, evidencing the theoretical result. For the Lévy measure $\nu_{IG}$ and $\nu_P$, computed rates of convergence are $r_s = 0.55$ and $r_s = 0.41$, respectively.

![Figure 6. Rate of convergence of $\tau$ for the SPDE given in equation (23) driven by the Lévy measures $\nu_{\alpha}$ (left) and the Lévy measures $\nu_{IG}$ and $\nu_P$ (right).](image)

### 5.6. Truncation error $\epsilon$.

We study the effect of truncation of a given Lévy measure $\nu \in \mathcal{L}(\mathbb{R})$ on solutions of the implicit approximate Euler scheme (Scheme $\mathcal{B}$). Here, we just focus on the effect of the truncation. The effect of the measure discretization is treated in Section 5.7 below. For this purpose, a stretched mesh with $\epsilon = 0.001 = \epsilon_{ex}$ and $\gamma = 0.1$ is used, such that $\tilde{\varrho}^{ex,2} = \sum_{j=1}^{J} \nu(D_j) \delta d_j$, and $(h, \tau) = (\frac{1}{260}, 2^{-8})$ are fixed. The exact solution $\{\bar{v}_{h,ex}^{M}\}_{m=0}^{M}$ is simulated using the discrete Lévy measure $\tilde{\varrho}^{ex,2}$. For different truncation parameters $\epsilon_i \in \{0.001, 0.0025, 0.005, 0.0075, 0.01\}$, solutions $\{\bar{v}_{h,i}^{m}\}_{m=0}^{M}$ are then obtained by using the discrete Lévy measure $\tilde{\varrho}^{i,2} := \tilde{\varrho}^{ex,2}[\epsilon_i, \infty)$. The $L^2$-error of the difference between $\{\bar{v}_{h,ex}^{M}\}_{m=0}^{M}$ and $\{\bar{v}_{h,i}^{m}\}_{m=0}^{M}$ at final time $T = 1$ is studied in Figure 7.

Figure 7 (left) shows how computed convergence rates change for different values of $\alpha$. This evidences that the convergence rate changes with respect to the truncation parameter $\epsilon$ for different Lévy measures as stated in Theorem 4.1.

### 5.7. Measure discretization error $\mathcal{E} \mathcal{E} r_{\epsilon,ex}(\mathbb{R})$.

We further study the effect of different mesh strategies proposed in Section 3 on solutions of the implicit approximate Euler scheme (Scheme $\mathcal{B}$). Therefore we fix $(h, \tau) = (\frac{1}{260}, 2^{-8})$. The exact solution $\{\bar{v}_{h,ex}^{M}\}_{m=0}^{M}$ is simulated using the discrete Lévy measure $\tilde{\varrho}^{ex,2} = \sum_{j=1}^{J} \nu(D_{j,ex}) \delta d_{j,ex}$ from Section 5.6. For different truncation parameters $\epsilon_i \in \{0.01, 0.02, 0.03, 0.04, 0.05\}$ solutions $\{\bar{v}_{h,i}^{m,ew}\}_{m=0}^{M}$, resp. $\{\bar{v}_{h,i}^{m,es}\}_{m=0}^{M}$ are obtained using an
Figure 7. Comparison of the theoretical rate (Theorem 4.1) with the observed rates (left). Truncation error: Rate of convergence w.r.t. $\epsilon$ for Lévy measures $\nu_\alpha$ (right)

equally weighted mesh $(d_{j,ew}, D_{j,ew})_{j=1}^{J_i}$, and an equally spaced mesh $(d_{j,es}, D_{j,es})_{j=1}^{J_i}$, to construct the discrete Lévy measures $\tilde{\nu}_{\epsilon,1}^{ew} = \sum_{j=1}^{J_i} \nu(D_{j,ew}) \delta_{d_{j,ew}}$ and $\tilde{\nu}_{\epsilon,1}^{es} = \sum_{j=1}^{J_i} \nu(D_{j,es}) \delta_{d_{j,es}}$, respectively. In order to compare the different resulting convergence rates, we fix the amount of atoms to $J_i = \epsilon_i^{-1}$. Since a stretched mesh with truncation parameter $\epsilon_i$ has a higher amount of intervals, we have to use different truncation parameters $\epsilon_i^s$, to ensure a fair comparison of discretizations $\tilde{\nu}_{\epsilon,1}^{s,2} = \sum_{j=1}^{J_i} \nu(D_{j,s}) \delta_{d_{j,s}}$ of the Lévy measure in terms of effort. The $L^2$-error rate of the difference between $\{\tilde{\nu}_{h,ew}^{m}\}_{m=0}^{M}$ and for each mesh strategy $\{\tilde{\nu}_{h,i}^{m}\}_{m=0}^{M}$ at final time $T = 1$ is depicted in Figure 8.

The $L^2$-rates of convergence given in Figure 8 evidence improved accuracy of the stretched mesh if compared to both, the equally weighted and the equally spaced mesh. The convergence rate obtained for the equally spaced mesh coincides with our theoretical results for a large activity of small jumps ($\alpha \geq 1$): the rate decreases linearly similar to $2 - \alpha$. In contrast, for small values of $\alpha$, we observe huge truncation errors due to large jumps, since jumps of size larger $O(1)$ are ignored for the simulation.

The convergence rates for an equally weighted mesh decrease linearly for a low activity of small jumps ($\alpha < 1$), and stays on a relative low level for bigger activity of the small jumps ($\alpha > 1$); this observation supports our theoretical results in Proposition 3.3, (i). For the stretched mesh with different parameters $\gamma$, the observed convergence rates also fit the theoretical results in Proposition 3.3 (iii), in the case of a big activity of small jumps ($\alpha \geq 1$). There, the rate decreases linearly.

Figure 8. $L^2$-error of solutions from Scheme $\tilde{\mathcal{B}}$, using different mesh strategies.
similar to $2(2 - \alpha)$ for each $\gamma$, which double the rate of convergence of the equally spaced mesh. For smaller values of $\alpha$, the convergence rate stays almost constant in the case of $\gamma = 0.1$ and $\gamma = 0.125$. The parameter $\gamma$ reflects the resolution for large jumps up to height $O(\epsilon^{-\gamma})$, which improves from $\gamma = 0.05$ to $\gamma = 0.125$, resulting in improved rates of convergence in the latter case.

References


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