Linearly implicit full discretization of surface evolution coupled to diffusion on the surface

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Contents

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1 Introduction

Partial differential equations (PDEs) are used in a wide range of applications, particularly in natural sciences.
A special class of such equations are surface partial differential equations, which operate on a two-dimensional surface and arise in many applications such as fluid dynamics [3] or cell biology [2]. Further applications can be found in [10].
Surface finite elements were first used on the Poisson problem with the Laplace-Beltrami operator on a curved surface, as in [6]. This has later been extended to parabolic equations on stationary surfaces, see [9]. Furthermore, in [8] a finite element method on evolving surfaces (ESFEM) has been developed.
A natural extension of the usual surface PDEs is introduced by coupling the velocity to a PDE on the surface. The space discretization of two different velocity laws were studied in [16]. By using BDF (backward differential formulae) as a time discretization, [17] then analyses full discretizations of both velocity laws. The first velocity law studied was a regularized mean curvature flow, i.e.

$$v - \alpha\Delta_{\Gamma(X)}v + \beta H_{\Gamma(X)}\nu_{\Gamma(X)} = g(x,t)\nu_{\Gamma(X)},$$

with $\alpha > 0, \beta \geq 0$ fixed parameters and $g : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ smooth. If $\alpha = 0$ and $g \equiv 0$ this corresponds with to the mean curvature flow, for which convergence results for an ESFEM discretization have been shown in [15].

Furthermore a dynamic velocity law was studied, in the form of

$$\partial^\bullet v + v\nabla_{\Gamma(X)} \cdot v - \alpha\Delta_{\Gamma(X)}u = f(x,t)\nu_{\Gamma(X)},$$

with $\alpha > 0$ and $f : \mathbb{R}^3 \times [0,T] \to \mathbb{R}$ smooth. In addition, a system that couples both PDEs was then analyzed. In [17] full discretizations of both velocity laws were studied and optimal order convergence results were given.

The purpose of this work is to combine the techniques to of [16] that gives an analysis of the space discretization of the coupled system with the results from [17] to give a complete convergence analysis of the fully discretized coupled system.

The main theorem of this work has already been formulated in [17], Theorem 8.1, however without a concrete proof. This thesis gives a proof and in doing so formulates the techniques used in greater detail. The following work is organized as follows.

In section 2 we describe the Problem and the weak formulation. The notation for ESFEM is introduced and basic properties are recalled. A polynomial finite element space discretization of order $k > 1$ is combined with a linearly implicit BDF time discretization to construct a numerical scheme. In doing so, a matrix-vector form is formulated which is used not only for the implementation, but also plays a key role in the stability analysis later.

Section 3 collects auxiliary results from [16] and [17], which are central tools in the convergence analysis later.

The stability analysis is found in section 4. The errors are estimated in terms of defects by making use of the matrix-vector formulation and the auxiliary results. Furthermore, multiplier techniques from [19] were used, therefore the proof is only applicable to BDF methods of order

$p \leq 5$. Geometric estimates do not enter in this part. Three stability equations are derived and then combined to prepare the convergence result.

Geometric estimates are collected in section 5, which are the necessary tools to later bound the defects.

Section 6 contains the consistency error analysis, which bounds the defects defined in the stability analysis. This is the part of the convergence analysis in which the geometry plays a central role, in the form of the geometric estimates from the section before. The bounds of the defects then proof convergence together with the results from the stability analysis.

The final section describes the implementation and numerical experiments. The reference element technique that was used to implement the matrix assembly is described and a simple class of test functions on an evolving sphere is constructed. Numerical results are presented and illustrate the convergence results.

2 Problem formulation

In the following section, we recall some definitions and introduce the basic notation for the thesis, which is taken mainly from [6],[10] and [16] .

2.1 Definition of a surface

We characterize a two-dimensional surface with the help of a distance function $d : \mathbb{R}^3 \to \mathbb{R}$. We define a surface $\Gamma \subset \mathbb{R}^3$ by

$$\Gamma = \{x \in \mathbb{R}^3 \quad | \quad d(x) = 0\}. \tag{2.1}$$

Generally, we do not need d to be defined on whole $\mathbb{R}^3$, but instead choose a subset $U \subset \mathbb{R}^3$ as domain for d, which then contains the surface Γ.

2.2 Evolution of surfaces

In the following, we consider an evolving two-dimensional closed surface $\Gamma(t) \subset \mathbb{R}^3$ as the image of a sufficiently smooth map $X : \Gamma^0 \times [0,T] \to \mathbb{R}^3$ with $X(p,0) = p$, where Γ^0 is a smooth closed initial surface. We define:

$$\Gamma(t) = \{X(p,t) \quad | \quad p \in \Gamma^0\}.$$

With this definition, $X(\cdot,t)$ is the surface at a certain time t, whereas $X(p,\cdot)$ can be imagined as the trajectory of a single particle $p \in \Gamma^0$ from the initial state. In the following, we mainly use the notation

$$\Gamma(X(\cdot,t)) = \Gamma(t),$$

to indicate the dependence of X on the surface. When the time is clear from the context we will omit t and write briefly $\Gamma(X)$.

A real valued function f that operates on an evolving surface is then a function on the space-time $\mathcal{G}_T$, defined by

$$f : \mathcal{G}_T = \bigcup_{t\in[0,T]} \{t\} \times \Gamma(X(\cdot,t)) \to \mathbb{R}.$$

Remark 1. *The surface at a given time t is then defined by the image of a function X instead of a distance function as in section 2.1 . For theoretical reasons it is sometimes useful to have access to a distance function. For a given surface $\Gamma(X)$ we can define a signed distance function by*

$$d_X(y) = \begin{cases} \inf_{x\in\Gamma(X)} |x-y| & \text{, for } y \text{ outside of } \Gamma(X) \\ -\inf_{x\in\Gamma(X)} |x-y| & \text{, otherwise,} \end{cases}$$

where $|\cdot|$ denotes the euclidian norm, for all $y \in \mathbb{R}^3$. On a computational level, we only need the distance function to generate the initial surface and do not use it at any other time.
Without loss of generality we can always assume a distance function to have the above form.

Since the gradient of a distance function in the form of Remark 1 is normalized in a neighborhood $U \subset \mathbb{R}^3$ around Γ, we can define the outer normal vector at a point $x \in \Gamma$ by

$$\nu_\Gamma(x) = \nabla d(x).$$

Therefore we can define a projection $p : U \to \Gamma$, that is defined by being the unique solution of

$$x = p(x) + d(x)\nu_\Gamma(p(x)), \text{ for } x \in U. \tag{2.2}$$

For a function $f : \Gamma \to \mathbb{R}$ we define the surface gradient to be

$$\nabla_\Gamma f = \nabla f - (\nabla f \cdot \nu_\Gamma)\, \nu_\Gamma,$$

where ∇f denotes the gradient of an arbitrary extension $\widetilde{f} : U \to \mathbb{R}$ of $f : \Gamma \to \mathbb{R}$. The Laplace–Beltrami operator is then given by the surface divergence of the surface gradient, i.e.

$$\Delta_\Gamma f = \nabla_\Gamma \cdot \nabla_\Gamma f \ .$$

2.3 Evolving surfaces

The definition of the surface now allows us to define the velocity $v : \mathcal{G}_T \to \mathbb{R}^3$ simply by taking the derivative of the trajectory of a particle from the initial position, i.e.

$$v(x,t) = \partial_t X(q,t), \tag{2.3}$$

for a point $x = X(q,t) \in \Gamma(X)$. Given a velocity $v : \mathbb{R}^3 \mapsto \mathbb{R}^3$ one can obtain the evolution X of the surface by solving the ordinary differential equation (ODE) above.
For a function $u : \mathcal{G}_T \to \mathbb{R}$ we write the material derivative as

$$\partial^\bullet u(x,t) = \frac{\mathrm{d}}{\mathrm{dt}} u(X(p,t),t) \quad \text{for} \quad x = X(p,t).$$

Applying the chain rule connects the material derivative with the regular time derivation by

$$\partial^\bullet u(x,t) = \partial_t u(x,t) + v \cdot \nabla u(x,t),$$

given these quantities exist. At a point $x \in \Gamma(X)$, to a time $t \in [0,T]$ we also define with $\nu_{\Gamma(X)}(x,t)$ the outer normal, by $\nabla_{\Gamma(X)} u(x,t)$ the surface gradient and with $\Delta_{\Gamma(X)} u(x,t)$, the Laplace-Beltrami operator applied to u. Furthermore we write $\nabla_{\Gamma(X)} \cdot v(x,t)$ as the surface divergence of v.
As in [10] we define the mean curvature by

$$H = \nabla_{\Gamma(X)} \cdot \nu_{\Gamma(X)}.$$

We use the identity

$$\Delta_{\Gamma(X)} x_{\Gamma(X)} = -H \nu_{\Gamma(X)}, \tag{2.4}$$

from [6], where $x_{\Gamma(X)} : \mathcal{G}_T \to \mathcal{G}_T$ denotes the identity map on the surface $\Gamma(X(\cdot,t))$ at all times $t \in [0,T]$.

2.4 Sobolev Spaces

We assume $\Gamma(X)$ to be sufficiently smooth at all times and define, for measurable functions $f{:}\,\Gamma(X) \to \mathbb{R}$ and a given time t the Sobolev norms

$$\|f\|^p_{L^p(\Gamma(X))} = \int_{\Gamma(X)} |f|^p \mathrm{dA} \text{ for } 1 \le p < \infty \text{ and}$$
$$\|f\|_{L^\infty(\Gamma(X))} = \operatorname{ess}\sup_{x \in \Gamma(X)} |f(x)|,$$

where dA denotes the surface measure. In the following it is always clear by which measure we integrate and therefore drop it from the notation to increase readability. A basic property of integrals is Green's formula, on a closed surface $\Gamma(X)$ it is given by

$$-\int_{\Gamma(X)} (\Delta_{\Gamma(X)} u)\varphi = \int_{\Gamma(X)} \nabla_\Gamma u \cdot \nabla_\Gamma \varphi, \tag{2.5}$$

for $u, \varphi : \Gamma(X) \to \mathbb{R}$ sufficiently smooth, for all expression above to exist. The boundary term vanishes, since $\Gamma(X)$ is closed.

2.5 Leibniz formula

The Leibniz formula is a central tool in the theory of evolving surfaces, details can be found in [10]. Let $u\colon \Gamma(X) \times [0,T] \to \mathbb{R}$ a sufficiently smooth function on an evolving surface $\Gamma(X)$ with velocity $v : \bigcup_{t\in[0,T]} \Gamma(X(\cdot,t)) \times \{t\} \to \mathbb{R}^3$. It is given by

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Gamma(X)} u = \int_{\Gamma(X)} \partial^\bullet u + \int_{\Gamma(X)} u \nabla_{\Gamma(X)} \cdot v. \tag{2.6}$$

2.6 Coupled PDE with diffusion on the surface

The parabolic surface PDE coupled with a regularised velocity law, described in [16] is given by:

$$\partial^\bullet u + u\nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u = f(u, \nabla_{\Gamma(X)} u), \tag{2.7}$$

$$v - \alpha\Delta_{\Gamma(X)} v + \beta H_{\Gamma(X)} \nu_{\Gamma(X)} = g(u, \nabla_{\Gamma(X)} u)\nu_{\Gamma(X)}, \tag{2.8}$$

$$\partial_t X(q,t) = v(X(q,t),t), \tag{2.9}$$

for all $q \in \Gamma^0$. Here $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ are given continuously differentiable functions, $\alpha > 0, \beta \geq 0$ are fixed parameters.

2.7 Weak formulation

Now all the necessary tools are introduced to recall the weak formulation of the above differential equation, which is given in [16], Section 2. We multiply the surface PDE (2.7) with a test function $\varphi(.,t) \in H^1(\Gamma(X(.,t)))$ with $\partial^\bullet\varphi = 0$ and the velocity law (2.8) with $\psi(.,t) \in H^1(\Gamma(X(.,t)))^3$. Integration on both sides then yields:

$$\int_{\Gamma(X)} \partial^\bullet u\varphi + \int_{\Gamma(X)} u\varphi\nabla_{\Gamma(X)} \cdot v - \int_{\Gamma(X)} \Delta_{\Gamma(X)} u\varphi = \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u)\varphi, \tag{2.10}$$

$$\int_{\Gamma(X)} v \cdot \psi - \alpha \int_{\Gamma(X)} \Delta_{\Gamma(X)} v \cdot \psi + \int_{\Gamma(X)} \beta H_{\Gamma(X)} \nu_{\Gamma(X)} \cdot \psi = \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u)\nu_{\Gamma(X)} \cdot \psi, \tag{2.11}$$

Since we assume $\partial^\bullet\varphi = 0$ we have with the product rule $\partial^\bullet(u\varphi) = (\partial^\bullet u)\varphi$. This merges the first two summands of (2.10) with the Leibniz formula. Using Green's formula on the last summand then yields

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Gamma(X)} u\varphi + \int_{\Gamma(X)} \nabla_{\Gamma(X)} u \cdot \nabla_{\Gamma(X)}\varphi = \int_{\Gamma(X)} f(u, \nabla_{\Gamma(X)} u)\varphi, \tag{2.12}$$

for the weak formulation of the surface PDE.
In the integrated velocity law (2.11) we use the definition of X on the surface (2.4) and then Green's formula on both the second and the third term to obtain

$$\int_{\Gamma(X)} v\cdot\psi + \alpha \int_{\Gamma(X)} \nabla_{\Gamma(X)} v \cdot \nabla_{\Gamma(X)}\psi \tag{2.13}$$

$$+\beta \int_{\Gamma(X)} \nabla_{\Gamma(X)} X \cdot \nabla_{\Gamma(X)}\psi = \int_{\Gamma(X)} g(u, \nabla_{\Gamma(X)} u)\nu_{\Gamma(X)} \cdot \psi. \tag{2.14}$$

The complete weak formulation is therefore:

Find $u(\cdot,t) \in H^1(\Gamma(X(\cdot,t))), v(\cdot,t) \in W^{1,\infty}(\Gamma(X(\cdot,t)))^3$ and $X : \Gamma^0 \to \mathbb{R}^3$ such that (2.12),(2.13) and (2.9) hold for all $\varphi \in H^1(\Gamma(X(\cdot,t))), \psi \in H^1(\Gamma(X(\cdot,t)))^3$.

2.8 Finite element spaces

The spatial discretization we use is given by the surface finite element method, as in [7] and [8]. We use piecewise polynomial basis functions of degree k and simplificial elements as in [5] and [13]. We triangulate the initial surface Γ^0 by a suitable family of triangulations $\mathcal{T}_h$ where h is the maximum element diameter defined by

$$h = \max_{T\in\mathcal{T}_h}(\text{diam}(T)).$$

For a fixed h we collect the nodes of the triangulation into a vector $\mathbf{x}^0 = (x_1^0, ..., x_N^0) \in \mathbb{R}^{3N}$. By piecewise polynomial interpolation of degree k this defines an approximate surface Γ_h^0 that interpolates Γ^0 in the points x_j^0. We denote the time dependent nodes by the vector

$$\mathbf{x}(t) = (x_1(t), ..., x_N(t)) \in \mathbb{R}^{3N},$$

where $x_j(t)$ can be understood as the position of the particle $x_j^0 = x_j(0)$ at a time t.
Under the assumption that $x_j(t)$ is close enough to the exact trajectory of the same initial state $x_{*,j}(t) := X(x_j^0, t)$ the nodal vector $\mathbf{x}(t)$ corresponds to an admissible triangulation. Since the dependence on t is clear and omnipresent we omit the argument when no confusion can arise in the context.

Given a nodal vector $\mathbf{x}$ we define $\Gamma_h[\mathbf{x}]$ to be the piecewise polynomial surface of order k that interpolates the nodes collected in $\mathbf{x}$. We define finite element basis functions

$$\phi_j[\mathbf{x}] : \Gamma_h(\mathbf{x}) \to \mathbb{R}, \quad j = 1, ..., N,$$

where the pullback onto the reference triangle is a polynomial of degree k and satisfies

$$\phi_j[\mathbf{x}](x_k) = \delta_{jk} \quad \text{for all} \quad j, k = 1, ..., N.$$

The finite element space on $\Gamma_h(\mathbf{x})$ is then given by the span of these functions by

$$S_h(\mathbf{x}) = \text{span}\{\phi_1[\mathbf{x}], ..., \phi_N[\mathbf{x}]\}.$$

For a finite element function $u_h \in S_h(\mathbf{x})$ the tangential gradient is then defined piecewise.

We now set our approximation of X as

$$X_h(q_h,t) = \sum_{j=1}^{N} x_j(t)\phi[\mathbf{x}(0)](q_h), \quad q_h \in \Gamma_h^0,$$

which has the properties

$$X_h(q_k,t) = \sum_{j=1}^{N} x_j(t)\phi_j[\mathbf{x}(0)](p_k) = \sum_{j=1}^{N} x_j(t)\delta_{jk} = x_k(t) \qquad k = 1,...,N \ ,$$

$$X_h(q_h,0) = \sum_{j=1}^{N} x_j(0)\phi_j[\mathbf{x}(0)](q_h) = q_h \qquad \forall q_h \in \Gamma_h^0.$$

In the first equation we use the property of the basis function in the nodes, the second equation follows through the definition of triangulation of the initial state Γ^0. Furthermore, we define

$$\Gamma(X_h(.,t)) = \Gamma_h(\mathbf{x}(t)),$$

since both evolving surfaces are piecewise polynomial of order k and have the same nodes.
The discrete velocity $v_h(x,t) \in \mathbb{R}^3$ at a point $x = X_h(p_h,t)$ is given by

$$v_h(X_h(q_h,t),t) = \partial_t X_h(q_h,t).$$

A key feature of the basis function is given by the transport property

$$\frac{\mathrm{d}}{\mathrm{dt}}\big(\phi_j[\mathbf{x}(t)](X_h(q_h,t))\big) = 0,$$

for details see [10]. Through integration from 0 to an arbitrary $t \in [0,T]$ this yields

$$\phi_j[\mathbf{x}(t)](X_h(q_h,t)) = \phi_j[\mathbf{x}(0)](X_h(q_h,t)), \quad \text{for all} \quad q_h \in \Gamma_h^0.$$

From this we can simplify the discrete velocity at a point $x = X_h(q_h,t) \in \Gamma(X_h(.,t))$ using the product rule

$$v_h(x,t) = \partial_t X_h(q_h,t) = \frac{\mathrm{d}}{\mathrm{dt}}\sum_{j=1}^{N} x_j(t)\phi[\mathbf{x}(0)](q_h) = \frac{d}{dt}\sum_{j=1}^{N} x_j(t)\phi[\mathbf{x}(t)](X_h(q_h,t)) \tag{2.15}$$

$$= \sum_{j=1}^{N} \dot{x}_j(t)\phi[\mathbf{x}(t)](X_h(q_h,t)) + \sum_{j=1}^{N} x_j(t)\left(\frac{d}{dt}\phi[\mathbf{x}(t)](X_h(q_h,t))\right) \tag{2.16}$$

$$= \sum_{j=1}^{N} v_j(t)\phi[\mathbf{x}(t)](X_h(q_h,t)), \tag{2.17}$$

where we denote $v_j(t) = \dot{x}_j(t)$ and used the integrated transport property for the third inequality and the original transport property in the last inequality.
The discrete material derivative is defined naturally for a finite element function

$$u_h(x,t) = \sum_{j=1}^{N} u_j(t)\phi[\mathbf{x}(t)](x), \quad \text{for} \quad x \in \Gamma_h(\mathbf{x}(t)),$$

by

$$\partial_h^\bullet u_h(x,t) = \frac{\mathrm{d}}{\mathrm{dt}} u_h(X_h(q_h,t)) \qquad \text{for} \quad x = X_h(q_h,t).$$

2.9 ESFEM spatial semi-discretization

The finite element spatial semi-discretization is now through the weak formulation restricted on the finite element spaces. Find the nodal vector $\mathbf{x}(t) \in \mathbb{R}^{3N}$, the finite element function $u_h(.,t) \in S_h[\mathbf{x}(t)]$ and the finite element function $v_h(.,t) \in S_h[\mathbf{x}(t)]^3$ such that, for all $\phi_h \in S_h[\mathbf{x}(t)], \psi \in S_h[\mathbf{x}(t)]^3$

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Gamma_h(\mathbf{x})} u_h \phi_h + \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} u_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \phi_h = \int_{\Gamma_h(\mathbf{x})} f(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h) \phi_h,$$

$$\int_{\Gamma_h(\mathbf{x})} v_h \cdot \psi_h + \alpha \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} v_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \psi_h$$
$$+ \beta \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} X_h \cdot \nabla_{\Gamma_h(\mathbf{x})} \psi_h = \int_{\Gamma_h(\mathbf{x})} g(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h) \nu_{\Gamma_h[\mathbf{x}]} \cdot \psi_h,$$

together with the ODE that relates the surface position to its velocity

$$\partial_t X_h(q_h, t) = v_h(X_h(q_h, t), t), \qquad \forall q_h \in \Gamma_h^0.$$

The initial values for u_h and $\mathbf{x}$ are values of the interpolation of the initial surface, i.e.

$$x_j(0) = x_j^0, \quad u_j(0) = u(x_j^0, 0), \quad \text{for} \quad j = 1, ..., N.$$

2.10 Matrix-vector formulation

In the following we collect the coefficients of the basis functions for the Galerkin-approximations u_h, v_h in respective nodal vectors $\mathbf{u} \in \mathbb{R}^N$ and $\mathbf{v} \in \mathbb{R}^{3N}$.
With the definition of the basis functions we have

$$u_h = \sum_{j=1}^{N} u_j \phi_j[\mathbf{x}], \quad \text{for} \quad u(x_j) = u_j \in \mathbb{R},$$
$$v_h = \sum_{j=1}^{N} v_j \phi_j[\mathbf{x}], \quad \text{for} \quad v(x_j) = v_j \in \mathbb{R}^3.$$

By using the given basis functions we now define the usual mass and stiffness matrices on the surface.
Since the surface is determined by the nodes $\mathbf{x}$ we denote this dependence as an argument. For the basis $(\phi_j[\mathbf{x}] \in S_h[\mathbf{x}])_{j=1}^N$ we then define

$$\mathbf{M}(\mathbf{x})|_{jk} = \int_{\Gamma_h(\mathbf{x})} \phi_j[\mathbf{x}] \phi_k[\mathbf{x}], \qquad \forall j, k = 1, ..., N,$$
$$\mathbf{A}(\mathbf{x})|_{jk} = \int_{\Gamma_h(\mathbf{x})} \nabla_{\Gamma_h(\mathbf{x})} \phi_j[\mathbf{x}] \cdot \nabla_{\Gamma_h(\mathbf{x})} \phi_k[\mathbf{x}], \qquad \forall j, k = 1, ..., N.$$

Since we have to work with three-dimensional finite element functions like v_h and X_h, we need a notation of the finite element matrices that allows matrix-vector operations on the nodal vectors

$\mathbf{x}, \mathbf{v} \in \mathbb{R}^{3N}$. This is achieved in an easy way by the tensor notation for matrices. We let therefore

$$\mathbf{M}^{[3]}(\mathbf{x}) = I_3 \otimes \mathbf{M}(\mathbf{x}) \quad \text{and}$$
$$\mathbf{A}^{[3]}(\mathbf{x}) = I_3 \otimes \mathbf{A}(\mathbf{x}).$$

We furthermore define

$$\mathbf{K}(\mathbf{x}) = \mathbf{M}^{[3]}(\mathbf{x}) + \alpha \mathbf{A}^{[3]}(\mathbf{x}).$$

When it is clear from the context we drop the upper index and write $\mathbf{M}(\mathbf{x})$ for $\mathbf{M}^{[3]}(\mathbf{x})$, $\mathbf{A}(\mathbf{x})$ for $\mathbf{A}^{[3]}(\mathbf{x})$ and $\|\cdot\|_{H^1(\Gamma)}$ for $\|\cdot\|_{H^1(\Gamma)^3}$ and so on. With these matrices we can now formulate a system of differential-algebraic equations (DAEs) which the nodal vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{x}$ satisfy.
The matrix–vector formulation, as described in [16], is now given by choosing the finite element basis as test functions and the linearity of the integrals and reads:

$$\frac{\mathrm{d}}{\mathrm{dt}} (\mathbf{M}(\mathbf{x})\mathbf{u}) + \mathbf{A}(\mathbf{x})\mathbf{u} = \mathbf{f}(\mathbf{x}, \mathbf{u}),$$
$$\mathbf{K}(\mathbf{x})\mathbf{v} + \beta \mathbf{A}(\mathbf{x})\mathbf{x} = \mathbf{g}(\mathbf{x}, \mathbf{u}),$$
$$\dot{\mathbf{x}} = \mathbf{v},$$

where the right-hand side vectors $\mathbf{f}(\mathbf{x}, \mathbf{u}), \mathbf{g}(\mathbf{x}, \mathbf{u})$ are given by

$$\mathbf{f}(\mathbf{x}, \mathbf{u})|_j = \int_{\Gamma_h[\mathbf{x}]} f(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h) \phi_j[\mathbf{x}],$$
$$\mathbf{g}(\mathbf{x}, \mathbf{u})|_{3(j-1)+l} = \int_{\Gamma_h[\mathbf{x}]} g(u_h, \nabla_{\Gamma_h(\mathbf{x})} u_h)(\nu_{\Gamma_h[\mathbf{x}]})_l \phi_j[\mathbf{x}],$$

for $j = 1, ..., N$ and $l = 1, 2, 3$.

2.11 Linearly implicit time discretization

We separate the interval $[0, T]$ in equidistant $t_n = n\tau \le T$ and write $\mathbf{u}^n \in \mathbb{R}^N$, $\mathbf{x}^n, \mathbf{v}^n \in \mathbb{R}^{3N}$ for the approximations of $\mathbf{u}(t_n), \mathbf{x}(t_n)$ and $\mathbf{v}(t_n)$.
For the p-step linearly implicit BDF discretization we use an extrapolation of the old position vectors $\mathbf{x}^{n-j}$ for $j = 1, .., p$. The extrapolated position vector $\widetilde{\mathbf{x}}^n$ is then a linear combination of old position vectors $\mathbf{x}^{n-p}, ..., \mathbf{x}^{n-1}$

$$\widetilde{\mathbf{x}}^n = \sum_{j=0}^{p-1} \gamma_j \mathbf{x}^{n-p+j}, \tag{2.18}$$

with $\gamma_j \in \mathbb{R}$ for $j = 0, ..., p-1$. In the same way we define an extrapolated value vector $\widetilde{\mathbf{u}}^n$ and later extrapolations $\widetilde{\mathbf{x}}^n_*$ and $\widetilde{\mathbf{u}}^n_*$ of the exact nodal vectors $\mathbf{x}^n_*$ and $\mathbf{u}^n_*$.

$$\frac{1}{\tau} \sum_{j=0}^{p} \delta_j \mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) \mathbf{u}^{n-j} + \mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{u}^n = \mathbf{f}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n), \tag{2.19}$$

$$\mathbf{K}(\widetilde{\mathbf{x}}^n)\mathbf{v}^n + \beta \mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{x}^n = \mathbf{g}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n), \tag{2.20}$$

$$\frac{1}{\tau} \sum_{j=0}^{p} \delta_j \mathbf{x}^{n-j} = \mathbf{v}^n. \tag{2.21}$$

The coefficients $\delta_j \in \mathbb{R}$ for $j = 0, ..., p$ and $\gamma_j \in \mathbb{R}$ for $j = 0, ..., p-1$ are then given by the coefficients of the polynomials

$$\delta(\zeta) = \sum_{j=0}^{p} \delta_j \zeta^j = \sum_{j=1}^{p} \frac{1}{j}(1-\zeta)^j, \qquad \gamma(\zeta) = \sum_{j=0}^{p} \gamma_j \zeta^j = \frac{1-(1-\zeta)^p}{\zeta}. \tag{2.22}$$

We draw attention to the fact, that in the discretized scheme of the surface PDE (2.19) appear mass matrices $\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})$, at extrapolated nodal values even at times where we already have numerical solutions. This avoids the need to assemble the mass matrix twice for one time step. While the computational cost is reduced quite a bit by this, since the matrix assembly is the biggest computational challenge, the accuracy of the method is basically the same.

We have included some numerical tests, that compare numerical solutions that reuse the old matrices $\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})$ and some that assembly the mass matrices again, once the numerical solution is obtained (See Figure 7.4).
In the following we recall some notation from [17]. We denote with $\mathbf{x}_*^n \in \mathbb{R}^{3N}$ the exact nodal vector, with corresponding (exact) velocities $\mathbf{v}_*^n \in \mathbb{R}^{3N}$ and evaluations of u by $\mathbf{u}_*^n \in \mathbb{R}^N$, i.e.

$$\mathbf{x}_*(t) = (x_{*,j}(t))_{j=1}^N \in \mathbb{R}^{3N} \quad \text{with} \quad x_{*,j}(t) = X(q_j, t). \quad j = 1, ..., N$$

Naturally, this induces the vector of the exact evaluations of u by

$$\mathbf{u}_*(t) = (u(x_{*,j}(t)))_{j=1}^N \in \mathbb{R}^N.$$

The nodal vector defines a discrete surface $\Gamma_h(\mathbf{x}_*(t))$ that interpolates the exact surface $\Gamma(X(.,t))$ for every $t \in [0,T]$. We also collect the velocity in a nodal vector

$$\mathbf{v}_*(t) = (v_{*,j}(t))_{j=1}^N = \dot{\mathbf{x}}_*(t) \in \mathbb{R}^{3N} \quad \text{with} \quad v_{*,j}(t) = \dot{x}_{*,j}(t), \quad j = 1, ..., N.$$

The interpolated exact velocity is then given by

$$v_{*,h}(t) = \sum_{j=1}^{N} v_{*,j}(t)\phi_j[\mathbf{x}_*(t)].$$

Vectors at the discrete times t_n given through the time discretization are denoted by

$$\mathbf{x}_*^n = \mathbf{x}_*(t_n), \quad \mathbf{u}_*^n = \mathbf{u}_*(t_n) \quad \text{and} \quad \mathbf{v}_*^n = \mathbf{v}_*(t_n).$$

The errors of the numerical solutions at a point in the discrete time scheme are denoted with

$$\mathbf{e}_{\mathbf{x}}^n := \mathbf{x}^n - \mathbf{x}_*^n, \quad \mathbf{e}_{\mathbf{u}}^n := \mathbf{u}^n - \mathbf{u}_*^n \quad \text{and} \quad \mathbf{e}_{\mathbf{v}}^n := \mathbf{v}^n - \mathbf{v}_*^n.$$

In the same way as in (2.18), we define extrapolated errors of $\mathbf{x}$ and $\mathbf{u}$. Furthermmore, we set

$$\widetilde{\mathbf{e}}_{\mathbf{x}}^n := \widetilde{\mathbf{x}}^n - \widetilde{\mathbf{x}}_*^n = \sum_{j=0}^{p-1} \gamma_j(\mathbf{x}^{n-p+j} - \mathbf{x}_*^{n-p+j}) = \sum_{j=0}^{p-1} \gamma_j \mathbf{e}_{\mathbf{x}}^{n-p+j},$$
$$\widetilde{\mathbf{e}}_{\mathbf{u}}^n := \widetilde{\mathbf{u}}^n - \widetilde{\mathbf{u}}_*^n = \sum_{j=0}^{p-1} \gamma_j(\mathbf{u}^{n-p+j} - \mathbf{u}_*^{n-p+j}) = \sum_{j=0}^{p-1} \gamma_j \mathbf{e}_{\mathbf{u}}^{n-p+j}.$$

We introduce a new notation, that denotes a continuous extrapolated surface that corresponds to the nodal vector $\widetilde{\mathbf{x}}_*^n$.

2.11.1 A continuous extrapolated surface

Let $X_h^* : \Gamma_h^0 \times [0,T] \to \mathbb{R}^3$ be the time discretized evolution with exact nodal values. Then we define

$$\widetilde{X}_h^*(\cdot,t) = \sum_{j=0}^{p-1} \gamma_j X_h^*(\cdot, t-(j+1)\tau) \ , \ t \in [\tau p, T].$$

This discrete surface corresponds, at a given time t_n with the nodal vector $\widetilde{\mathbf{x}}_*^n$, since

$$\widetilde{X}_h^*(q_i,t_n) = \sum_{j=0}^{p-1} \gamma_j X_h^*(q_i, t_{n-j-1}) = \sum_{j=0}^{p-1} \gamma_j \left(\mathbf{x}_*^{n-j-1}\right)_i = (\widetilde{\mathbf{x}}_*^n)_i \, .$$

We denote the material derivative on this surface with $\widetilde{\partial}_h^\bullet$. The finite element space and the corresponding basis functions are defined in the same way as before and are denoted by $\widetilde{\phi}_j$ for $j = 1, ..., N$. The corresponding velocity to $\widetilde{X}_h^*$ is denoted by $\widetilde{v}_h^*$ and fulfills by construction

$$\begin{aligned}\widetilde{v}_h^*(\widetilde{X}_h^*(p_h,t_n),t_n) = \frac{\mathrm{d}}{\mathrm{dt}}\widetilde{X}_h^*(p_h,t_n) = \frac{\mathrm{d}}{\mathrm{dt}}\sum_{j=0}^{p-1}\gamma_j X_h^*(p_h,t_{n-j-1}) = \sum_{j=0}^{p-1}\gamma_j \frac{\mathrm{d}}{\mathrm{dt}} X_h^*(p_h,t_{n-j-1}) \\ = \sum_{j=0}^{p-1}\gamma_j v_h(X_h^*(p_h,t_{n-j-1}),t_{n-j-1}).\end{aligned}$$

2.12 Lifts

For a function $\eta_h : \Gamma_h(X_h^*) \to \mathbb{R}$ we denote its lift by $\eta_h^l : \Gamma(X) \to \mathbb{R}$ and define it, by using the projection p from (2.2), in the following way:

$$\eta_h^l(p(x)) = \eta_h(x), \text{ for } x \in \Gamma_h(X_h^*).$$

The projection is well defined for a neighborhood $U \subset \mathbb{R}^3$ small enough, which makes the lift, defined as above, well defined for h small enough.

For a function $\eta_h : \Gamma_h(X_h) \to \mathbb{R}$, corresponding to a nodal vector $\boldsymbol{\eta} \in \mathbb{R}^N$ we write

$$\widehat{\eta}_h = \sum_{j=0}^{N} \eta_j \phi_j[\mathbf{x}_*^n]. \tag{2.23}$$

We use this, to define its lift onto denote its lift onto $\Gamma(X)$ via the lift of the finite element function on $\Gamma_h(X_h^*)$, i.e.

$$\eta_h^L = \widehat{\eta}_h^l.$$

2.13 Statement of the main result

We are now in the position to formulate our main result, which was formulated in [17].

Theorem 2.1. *We consider the linearly implicit full discretization (ESFEM/BDF) (2.19)–(2.21) of the coupled system (2.7)–(2.9). We use polynomial finite elements of order $k \geq 2$ and BDF methods of order $2 \leq p \leq 5$. Furthermore, we assume the triangulation Γ_h^0 of the initial surface Γ^0 to be quasi-uniform and admissible. Suppose that the problem has sufficiently smooth solutions u, X, v, such that the exact evolution $X(\cdot,t) : \Gamma^0 \to \Gamma(t)$ is non-degenerate for $0 \leq t \leq T$ and that its interpolation $X_h^*(\cdot,t) : \Gamma_h^0$ admits an admissible triangulation for $0 \leq t \leq T$. Suppose further, that the starting values are sufficiently accurate and the mild step size restriction $\tau^{p-1} \leq c_0 h$ and the mild mesh size restriction $h^{k-1} \leq c_1 \tau$ hold for $c_0, c_1 > 0$. Then, there exist $h_0, \tau_0 > 0$, such that for $h < h_0$, $\tau < \tau_0$ the following error bound holds, on the exact surface $\Gamma(X(\cdot,t_n))$, $0 \leq t_n \leq T$:*

$$\left\|(u_h^n)^L - u(\cdot,t_n)\right\|_{L^2(\Gamma(t_n)^3)} + \left(\tau \sum_{j=p}^{n} \left\|\left(u_h^j\right)^L - u(\cdot,t_j)\right\|^2_{H^1(\Gamma(t_j))^3}\right)^{1/2} \leq C(h^k + \tau^p), \tag{2.24}$$

$$\left\|(v_h^n)^L - v(\cdot,t_n)\right\|_{H^1(\Gamma(t_n)^3)} \leq C(h^k + \tau^p), \tag{2.25}$$

$$\left\|(x_h^n)^L - x_{\Gamma(X)}\right\|_{H^1(\Gamma(t_n)^3)} \leq C(h^k + \tau^p). \tag{2.26}$$

The constant C does not depend on h, τ or n, for $n\tau \leq T$. It depends on the Lipschitz constants of the coupling terms f and g, bounds of higher derivatives of the exact solution (u, v, X) and the length of the time interval T.

Remark 2. *The mild restrictions on the step and mesh sizes, as stated in Theorem 2.1, only enter in the estimation of (4.23). For the rest of the proof it is sufficient to demand $C\tau^p \leq h$.*

In the following, C will denote a constant that exists, while γ will denote an arbitrary small constant. For example:

$$ab \leq \gamma a^2 + Cb^2 \tag{2.27}$$

reads as: for $\gamma > 0$ arbitrary small, there exists a $C > 0$, such that (2.27) holds for all $a, b \in \mathbb{R}$.

3 Auxiliary results

The finite element matrices induce in a natural way discrete versions of the Sobolev norms. Let $\mathbf{w} = (w_j) \in \mathbb{R}^N$ be an arbitrary real vector and $w_h = \sum_{j=1}^N w_j \phi_j[\mathbf{x}] \in S_h(\mathbf{x})$. We then have

$$\|\mathbf{w}\|^2_{\mathbf{M}(\mathbf{x})} := \mathbf{w}^t \mathbf{M}(\mathbf{x}) \mathbf{w} = \|w_h\|^2_{L^2(\Gamma_h[\mathbf{x}])},$$
$$\|\mathbf{w}\|^2_{\mathbf{A}(\mathbf{x})} := \mathbf{w}^t \mathbf{A}(\mathbf{x}) \mathbf{w} = \left\|\nabla_{\Gamma_h(\mathbf{x})} w_h\right\|^2_{L^2(\Gamma_h[\mathbf{x}])}.$$

One of the main challenges in the stability analysis of ESFEM is the relation of different surfaces. To overcome this we use an approach where one transforms two surfaces linearly into each other.

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{3N}$ be two nodal vectors, that each define a discrete surface $\Gamma_h[\mathbf{x}]$ and $\Gamma_h[\mathbf{y}]$. We write $\mathbf{e} = \mathbf{x} - \mathbf{y} \in \mathbb{R}^{3N}$. We introduce a parameter $\theta \in [0,1]$ and consider the intermediate surface $\Gamma_h^\theta := \Gamma_h(\mathbf{y} + \theta\mathbf{e})$. On this surface we have a finite element space $S_h(\mathbf{y} + \theta\mathbf{e})$ and therefore, the finite element function, corresponding to $\mathbf{e}$, is given by

$$e_h^\theta = \sum_{j=1}^N e_j \phi_j[\mathbf{y} + \theta\mathbf{e}]. \tag{3.1}$$

For an arbitrary vector $\mathbf{w} \in \mathbb{R}^N$ we then write

$$w_h^\theta = \sum_{j=1}^N w_j \phi_j[\mathbf{y} + \theta\mathbf{e}].$$

In the following we collect some technical Lemma from [16], done as such in [17].

Lemma 3.1. *Let $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$ be arbitrary vectors, with corresponding finite element functions $w_h, z_h \in S_h(\Gamma_h^\theta)$. Then, the following statements hold.*

(i) We have

$$\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\boldsymbol{y}))\mathbf{z} = \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta (\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) z_h^\theta \, d\theta,$$
$$\mathbf{w}^T(\mathbf{A}(\mathbf{x}) - \mathbf{A}(\boldsymbol{y}))\mathbf{z} = \int_0^1 \int_{\Gamma_h^\theta} (\nabla_{\Gamma_h^\theta} w_h^\theta)(D_{\Gamma_h^\theta} e_h^\theta)(\nabla_{\Gamma_h^\theta} z_h^\theta) \, d\theta,$$

where we write $D_{\Gamma_h^\theta} e_h^\theta = trace(E) I_3 - (E + E^T)$ with $E = \nabla_{\Gamma_h^\theta} e_h^\theta \in \mathbb{R}^{3\times3}$.

(ii) If $\left\|\nabla_{\Gamma_h^\theta} \cdot e_h^\theta\right\|_{L^\infty(\Gamma_h^\theta)} \le \mu$ and $\left\|D_{\Gamma_h^\theta} e_h^\theta\right\|_{L^\infty(\Gamma_h^\theta)} \le \rho$ for $0 \le \theta \le 1$, then

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{y}+\theta\mathbf{e})} \le e^{\mu/2} \|\mathbf{w}\|_{\mathbf{M}(\mathbf{y})} \quad \text{and}$$
$$\|\mathbf{w}\|_{\mathbf{A}(\mathbf{y}+\theta\mathbf{e})} \le e^{\rho/2} \|\mathbf{w}\|_{\mathbf{A}(\mathbf{y})} \quad \forall\ 0 \le \theta \le 1.$$

(iii) If $\|\nabla_{\Gamma_h[\boldsymbol{y}]} e_h^0\|_{L^\infty(\Gamma_h^\theta)} \le \frac{1}{2}$ then, for $0 \le \theta \le 1$

$$\left\|\nabla_{\Gamma_h^\theta} w_h^\theta\right\|_{L^p(\Gamma_h^\theta)} \le C_p \left\|\nabla_{\Gamma_h^0} w_h^0\right\|_{L^p(\Gamma_h^0)} \quad \text{for} \quad 1 \le p \le \infty,$$

where C_p only depends on p.

(iv) Let $y_h^\theta \in \Gamma_h^\theta$ be defined as $y_h^\theta = \sum_{j=1}^N (y_j + \theta e_j)\phi_j[\mathbf{y}](q_h)$, for $q_h \in \Gamma_h[\mathbf{y}]$.
If $\|\nabla_{\Gamma_h[\boldsymbol{y}]} e_h^0\|_{L^\infty(\Gamma_h^\theta)} \le \frac{1}{2}$, then for the corresponding unit normal vectors it follows that

$$|\nu_{\Gamma_h^\theta}(y_h^\theta) - \nu_{\Gamma_h^0}(y_h^0)| \le C\theta |\nabla_{\Gamma_h^0} e_h^0(y_h^0)|,$$

with C independent of h and $q_h \in \Gamma_h[\mathbf{y}]$. Furthermore, we have

$$|\partial_\theta^\bullet \nu_{\Gamma_h^\theta}| \le C|\nabla_{\Gamma_h^0} e_h^0(y_h^0)|. \tag{3.2}$$

Proof. The proofs can be found in [16]. However, to give an understanding of the structure of the techniques used, we give a proof of (i).
With the notation above and the fundamental theorem of calculus we see

$$\mathbf{w}^T(\mathbf{M}(\mathbf{x}) - \mathbf{M}(\mathbf{y}))\mathbf{z} = \int_{\Gamma_h[\mathbf{x}]} w_h^1 z_h^1 - \int_{\Gamma_h[\mathbf{y}]} w_h^0 z_h^0 = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta \mathrm{d}\theta.$$

Now we can use the Leibniz formula from [10], Theorem 5.1, which yields

$$\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta \mathrm{d}\theta = \int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet (w_h^\theta z_h^\theta) + w_h^\theta (\nabla_{\Gamma_h^\theta} \cdot e_h^\theta) z_h^\theta \mathrm{d}\theta$$

However, the first part of the sum in the integral vanishes, since we obtain with the product rule and the transport property

$$\begin{aligned}\partial_\theta^\bullet(w_h^\theta z_h^\theta) &= w_h^\theta(\partial_\theta^\bullet z_h^\theta) + z_h^\theta(\partial_\theta^\bullet w_h^\theta)\\ &= w_h^\theta\left(\sum_{j=1}^N z_j \partial_\theta^\bullet \phi_j[\mathbf{y} + \theta\mathbf{e}]\right) + z_h^\theta\left(\sum_{j=1}^N w_j \partial_\theta^\bullet \phi_j[\mathbf{y} + \theta\mathbf{e}]\right) = 0.\end{aligned}$$

The second identity is shown in the same way, using formulas from [8] and [12, Lemma 3.1]. □

In the following we always let $\Gamma(X(.,t)), t \in [0,T]$ be a smoothly evolving family of smooth closed surfaces. The next result has been shown in [12, Lemma 4.1].

Lemma 3.2. *For $0 \le s,t \le T$ and arbitrary $\mathbf{w},\mathbf{z}$ we have*

$$\mathbf{w}^T(\mathbf{M}(\mathbf{x}_*(t)) - \mathbf{M}(\mathbf{x}_*(s)))\mathbf{z} \le C(t-s)\,\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*(t))}\,\|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}_*(t))},$$
$$\mathbf{w}^T(\mathbf{A}(\mathbf{x}_*(t)) - \mathbf{A}(\mathbf{x}_*(s)))\mathbf{z} \le C(t-s)\,\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*(t))}\,\|\mathbf{z}\|_{\mathbf{A}(\mathbf{x}_*(t))}.$$

The norms for different times are uniformly equivalent for $0 \le s,t \le T$, i.e.

$$\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*(t))} \le C\,\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*(s))}, \qquad \|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*(t))} \le C\,\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*(s))},$$

where the constant only depends on a bound of the $W^{1,\infty}$-norm of the surface velocity.

The next Lemma compares the finite element surfaces that are defined by the exact and extrapolated nodes.

Lemma 3.3. *We denote the nodal vector with the exact nodes to the time t_n by $\mathbf{x}_*^n = \mathbf{x}_*(t_n)$. The extrapolated values are written by $\widetilde{\mathbf{x}}_*^n = \sum_{j=0}^{p-1} \gamma_j \mathbf{x}_*^{n-1-j}$. Then we have the following estimates for all $\mathbf{w},\mathbf{z} \in \mathbb{R}^N$*

$$\mathbf{w}^T(\mathbf{M}(\widetilde{\mathbf{x}}_*^n) - \mathbf{M}(\mathbf{x}_*^n))\mathbf{z} \le C\tau^p\,\|\mathbf{w}\|_{\mathbf{M}(\mathbf{x}_*^n)}\,\|\mathbf{z}\|_{\mathbf{M}(\mathbf{x}_*^n)}$$
$$\mathbf{w}^T(\mathbf{A}(\widetilde{\mathbf{x}}_*^n) - \mathbf{A}(\mathbf{x}_*^n))\mathbf{z} \le C\tau^p\,\|\mathbf{w}\|_{\mathbf{A}(\mathbf{x}_*^n)}\,\|\mathbf{z}\|_{\mathbf{A}(\mathbf{x}_*^n)}.$$

The Lemma above implies a norm equivalence for sufficiently small step sizes $\tau > 0$, between the extrapolated $\mathbf{K}(\widetilde{\mathbf{x}}_*^n)$ and the exact $\mathbf{K}(\mathbf{x}_*^n)$ surface norms, i.e.

$$\frac{1}{2}\,\|\mathbf{w}\|^2_{\mathbf{K}(\mathbf{x}_*^n)} \le \|\mathbf{w}\|^2_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)} \le \frac{3}{2}\,\|\mathbf{w}\|^2_{\mathbf{K}(\mathbf{x}_*^n)}\,. \tag{3.3}$$

We introduce some further notation regarding difference quotients, by defining

$$\mathbf{V}^n = \frac{\mathbf{x}^n - \mathbf{x}^{n-1}}{\tau}, \quad \mathbf{V}_*^n = \frac{\mathbf{x}_*^n - \mathbf{x}_*^{n-1}}{\tau} \in \mathbb{R}^{3N}.$$

This notation is extended to the extrapolated nodal vectors, i.e.

$$\widetilde{\mathbf{V}}^n = \frac{\widetilde{\mathbf{x}}^n - \widetilde{\mathbf{x}}^{n-1}}{\tau}, \quad \widetilde{\mathbf{V}}_*^n = \frac{\widetilde{\mathbf{x}}_*^n - \widetilde{\mathbf{x}}_*^{n-1}}{\tau} \in \mathbb{R}^{3N}.$$

The difference quotients of the position error vectors are denoted with

$$\mathbf{E}_{\mathbf{V}}^n = \mathbf{V}^n - \mathbf{V}_*^n, \qquad \widetilde{\mathbf{E}}_{\mathbf{V}}^n = \widetilde{\mathbf{V}}^n - \widetilde{\mathbf{V}}_*^n.$$

The nodal vector corresponds to a finite element function on discrete surfaces in the usual way. From [1] we take the identity

$$\widetilde{E}_V^n = \frac{\widetilde{e}_x^n - \widetilde{e}_x^{n-1}}{\tau} = C\sum_{j=0}^{n-p} \chi_j\left(\widetilde{e}_v^{n-j} + \widetilde{d}_x^{n-j}\right),$$

with $|\chi_j| \le \vartheta^j$ for a constant $0 < \vartheta < 1$. Here, $\widetilde{d}_x^{n-j}$ denotes the defect corresponding to the extrapolated nodal vector of the past defects.

In the following, we present a new result, that estimates a discrete differential quotient, which plays an important role in the stability analysis of the dynamic surface PDE.

Lemma 3.4. *Let $\mathbf{w}, \mathbf{z} \in \mathbb{R}^N$ be arbitrary, $\mathbf{x}^n$ the numerical approximation of $\mathbf{x}(t_n)$ and $\widetilde{\mathbf{x}}^n$ the extrapolation of the p past values. Then we have*

$$\begin{aligned}&\frac{1}{\tau}\mathbf{w}^T(\mathbf{M}(\widetilde{\mathbf{x}}^n) - \mathbf{M}(\widetilde{\mathbf{x}}^{n-1}) - \big(\mathbf{M}(\widetilde{\mathbf{x}}_*^n) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-1})\big))\mathbf{z}\\ &\le C\,\|w_h\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\Big(\|z_h\|^2_{L^2(\Gamma_h[\mathbf{x}_*^n])} + \gamma\,\|\widetilde{E}_V^n\|^2_{H^1(\Gamma_h[\mathbf{x}_*^n])} + \|\widetilde{e}_x^n\|^2_{H^1(\Gamma_h[\mathbf{x}_*^n])} + \|\widetilde{e}_x^{n-1}\|^2_{H^1(\Gamma_h[\mathbf{x}_*^n])}\Big),\end{aligned}$$

for a $\gamma > 0$ arbitrary small but constant.

Proof. In the following we introduce an important idea taken from [16] and in this form [17], that will play an important role here and throughout the stability analysis.
In order to use the auxiliary results from Lemma 3.1 we need to control the $W^{1,\infty}$ norm of the position error $\widetilde{e}_x^n$, which is the finite element function corresponding to the nodal vector

$\widetilde{\mathbf{e}}_{\mathbf{x}}^n = \widetilde{\mathbf{x}}^n - \widetilde{\mathbf{x}}_*^n$. We assume that (2.26) holds true for $p, ..., n-1$. Then we use an inverse inequality and the norm equivalence from (3.3) to obtain

$$\begin{aligned} \left\|\nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{e}_x^n\right\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} &\leq ch^{-1}\left\|\nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{e}_x^n\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\ &\leq ch^{-1}\|\widetilde{\mathbf{e}}_{\mathbf{x}}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)} \leq ch^{-1}\|\widetilde{\mathbf{e}}_{\mathbf{x}}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} \\ &\leq ch^{-1}\sum_{j=1}^{p}\left\|\mathbf{e}_{\mathbf{x}}^{n-j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)} \\ &\leq ch^{-1}\cdot c\gamma h \leq c\gamma, \end{aligned} \tag{3.4}$$

for $\gamma > 0$ arbitrary small, but constant. In view of this result we can use 3.1 (iii) and (iv) for sufficiently small ϑ.
Let $\Theta = (\theta, \xi)$, $\Gamma_h^\theta = \Gamma_h[(1-\theta)\widetilde{\mathbf{x}}^n + \theta\widetilde{\mathbf{x}}^{n-1}]$ and $\Gamma_{h,*}^\theta = \Gamma_h[(1-\theta)\widetilde{\mathbf{x}}_*^n + \theta\widetilde{\mathbf{x}}_*^{n-1}]$.
Starting from Lemma 3.1 (i) we obtain

$$\begin{aligned} &\frac{1}{\tau}\mathbf{w}^T(\mathbf{M}(\widetilde{\mathbf{x}}^n) - \mathbf{M}(\widetilde{\mathbf{x}}^{n-1}) - \big(\mathbf{M}(\widetilde{\mathbf{x}}_*^n) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-1})\big))\mathbf{z} \\ &= \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta \left(\nabla_{\Gamma_h^\theta}\cdot\frac{\widetilde{X}_h^n - \widetilde{X}_h^{n-1}}{\tau}\right) - \int_{\Gamma_{*,h}^\theta} w_h^\theta z_h^\theta\left(\nabla_{\Gamma_{*,h}^\theta}\cdot\frac{\widetilde{X}_{*,h}^n - \widetilde{X}_{*,h}^{n-1}}{\tau}\right)\mathrm{d}\theta \\ &= \int_0^1 \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta\left(\nabla_{\Gamma_h^\theta}\cdot\widetilde{V}_h^{n,\theta}\right) - \int_{\Gamma_{*,h}^\theta} w_h^\theta z_h^\theta\left(\nabla_{\Gamma_{*,h}^\theta}\cdot\widetilde{V}_{*,h}^{n,\theta}\right)\mathrm{d}\theta, \end{aligned}$$

where $\widetilde{V}_h^{n,\theta}$ and $\widetilde{V}_{*,h}^{n,\theta}$ simply denote the difference quotient, corresponding to the nodal vectors $\widetilde{\mathbf{V}}^n$ and $\widetilde{\mathbf{V}}_*^n$.

We now use the fundamental theorem of calculus, similarly to the proof of Lemma 3.1. To simplify the notation we introduce two new nodal vectors. We choose $\widetilde{\mathbf{y}}^{n,\theta} = (1-\theta)\widetilde{\mathbf{x}}^n + \theta\widetilde{\mathbf{x}}^{n-1}$ and $\widetilde{\mathbf{y}}_*^{n,\theta} = (1-\theta)\widetilde{\mathbf{x}}_*^n + \theta\widetilde{\mathbf{x}}_*^{n-1}$ with an intermediate surface $\Gamma_h^\Theta = \Gamma_h[\xi\widetilde{\mathbf{y}}^{n,\theta} + (1-\xi)\widetilde{\mathbf{y}}_*^{n,\theta}]$. A finite element function $w_h^\Theta \in S_h^\Theta[\xi\widetilde{\mathbf{y}}^{n,\theta} + (1-\xi)\widetilde{\mathbf{y}}_*^{n,\theta}]$, characterized by a nodal vector $\mathbf{w} \in \mathbb{R}^N$ is then given by

$$w_h^\Theta = \sum_{j=0}^N w_j\phi_j[\xi\widetilde{\mathbf{y}}^{n,\theta} + (1-\xi)\widetilde{\mathbf{y}}_*^{n,\theta}].$$

Furthermore, we denote

$$\widetilde{V}_h^{\Theta,n} = \sum_{j=1}^N\left((\widetilde{\mathbf{V}}_*^n)_j + \xi\left(\widetilde{\mathbf{E}}_{\mathbf{V}}^n\right)_j\right)\phi_j[\xi\widetilde{\mathbf{y}}^{n,\theta} + (1-\xi)\mathbf{y}_*^{n,\theta}] = \widetilde{V}_{*,h}^{\Theta,n} + \xi\widetilde{E}_V^{\Theta,n}.$$

Now, using the fundamental theorem of calculus with the notation above yields

$$\int_0^1 \int_{\Gamma_h^\theta} w_h^\theta z_h^\theta \left(\nabla_{\Gamma_h^\theta} \cdot \widetilde{V}_h^{n,\theta}\right) - \int_{\Gamma_{*,h}^\theta} w_h^\theta z_h^\theta \left(\nabla_{\Gamma_{*,h}^\theta} \cdot \widetilde{V}_{*,h}^{n,\theta}\right) \mathrm{d}\theta \tag{3.5}$$

$$= \int_0^1 \int_0^1 \int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta \partial_\xi^\bullet \left(\nabla_{\Gamma_h^\Theta} \cdot \widetilde{V}_h^{\Theta,n}\right)$$

$$+ \int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta \left(\nabla_{\Gamma_h^\Theta} \cdot \widetilde{V}_h^{\Theta,n}\right)\left(\nabla_{\Gamma_{*,h}^\Theta} \cdot \widetilde{e}_y^{\Theta,n}\right) \mathrm{d}\theta \mathrm{d}\xi$$

$$= \int_0^1 \int_0^1 \int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta \partial_\xi^\bullet \left(\nabla_{\Gamma_h^\Theta} \cdot \left(\widetilde{V}_{*,h}^{\Theta,n} + \xi \widetilde{E}_V^{\Theta,n}\right)\right) \tag{3.6}$$

$$+ \int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta \left(\nabla_{\Gamma_h^\Theta} \cdot \widetilde{V}_h^{\Theta,n}\right)\left(\nabla_{\Gamma_h^\Theta} \cdot \widetilde{e}_y^{\Theta,n}\right) \mathrm{d}\theta \mathrm{d}\xi \tag{3.7}$$

where the in the last line the usual auxiliary results (Lemma 3.1) and norm equivalences were used. We use [11, Lemma 2.6], which describes how the surface gradient and the material derivative commute to obtain

$$\begin{aligned} \partial_\xi^\bullet \nabla_{\Gamma_h^\Theta} \cdot \widetilde{V}_h^{\Theta,n} &= \nabla_{\Gamma_h^\Theta} \cdot \partial_\xi^\bullet \widetilde{V}_h^{\Theta,n} - \left(\nabla_{\Gamma_h^\Theta} \cdot \widetilde{e}_y^{\Theta,n} - \nu_{\Gamma_h^\Theta} (\nu_{\Gamma_h^\Theta})^T (\nabla_{\Gamma_h^\Theta} \widetilde{e}_y^{\Theta,n})^T\right) \nabla_{\Gamma_h^\Theta} \cdot \widetilde{V}_h^{\Theta,n} \\ &= \nabla_{\Gamma_h^\Theta} \cdot \widetilde{E}_V^{\Theta,n} - \left(\nabla_{\Gamma_h^\Theta} \cdot \widetilde{e}_y^{\Theta,n} - \nu_{\Gamma_h^\Theta} (\nu_{\Gamma_h^\Theta})^T (\nabla_{\Gamma_h^\Theta} \cdot \widetilde{e}_y^{\Theta,n})^T\right) \nabla_{\Gamma_h^\Theta} \cdot \left(\widetilde{V}_{*,h}^{\Theta,n} + \xi \widetilde{E}_V^{\Theta,n}\right). \end{aligned}$$

We continue by estimating (3.6) with the Cauchy–Schwartz inequality to obtain

$$\begin{aligned} &\left| \int_0^1 \int_0^1 \int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta \partial_\xi^\bullet \left(\nabla_{\Gamma_{*,h}^\Theta} \cdot \left(\widetilde{V}_{*,h}^{\Theta,n} + \xi \widetilde{E}_V^{\Theta,n}\right)\right) \mathrm{d}\theta \mathrm{d}\xi \right| \\ &\leq \int_0^1 \int_0^1 \int_{\Gamma_h^\Theta} \left| w_h^\Theta z_h^\Theta \nabla_{\Gamma_{*,h}^\Theta} \cdot \widetilde{E}_V^{\Theta,n} \right| \\ &+ \left| w_h^\Theta z_h^\Theta \left(\nabla_{\Gamma_h^\Theta} \widetilde{e}_y^{\Theta,n} - \nu_{\Gamma_h^\Theta} (\nu_{\Gamma_h^\Theta})^T (\nabla_{\Gamma_h^\Theta} \widetilde{e}_y^{\Theta,n})^T\right) \nabla_{\Gamma_h^\Theta} \widetilde{V}_h^{\Theta,n} \right| \mathrm{d}\theta \mathrm{d}\xi \\ &\leq \int_0^1 \int_0^1 C \left\| w_h^\Theta \right\|_{L^\infty(\Gamma_h^\Theta)} \left\| z_h^\Theta \right\|_{L^2(\Gamma_h^\Theta)} \left\| \widetilde{E}_V^{\Theta,n} \right\|_{H^1(\Gamma_h^\Theta)} \\ &+ C \left\| w_h^\Theta \right\|_{L^\infty(\Gamma_h^\Theta)} \left\| z_h^\Theta \right\|_{L^2(\Gamma_h^\Theta)} \left\| \widetilde{e}_x^{n,\Theta} + \widetilde{e}_x^{n-1,\Theta} \right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \left\| v_*^{n,\Theta} \right\|_{L^\infty(\Gamma_h^\Theta)} \\ &+ C \left\| w_h^\Theta \right\|_{L^\infty(\Gamma_h^\Theta)} \left\| z_h^\Theta \right\|_{L^2(\Gamma_h^\Theta)} \left\| \nu_{\Gamma_h^\Theta} \right\|_{L^\infty(\Gamma_h^\Theta)}^2 \left\| \widetilde{e}_x^{n,\Theta} + \widetilde{e}_x^{n-1,\Theta} \right\|_{W^{1,\infty}(\Gamma_h^\Theta)} \left\| \widetilde{E}_V^{n,\Theta} \right\|_{L^2(\Gamma_h^\Theta)} \mathrm{d}\theta \mathrm{d}\xi \end{aligned}$$

Now we intend to use Lemma 3.1 (ii) and (iii) to estimate the norms on the surface Γ_h^Θ with norms solely on $\Gamma_h[\mathbf{x}_*^n]$. To achieve this we prove that the condition of Lemma 3.1 (iii) is fulfilled. With Taylor's theorem and Peano kernels we have

$$\left\| \nabla_{\Gamma_h(\widetilde{X}_h^*(t_n))} \left(\widetilde{x}_h^*(t_n) - \widetilde{x}_h^*(t_{n-1})\right) \right\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \leq C\tau,$$

where $\widetilde{x}_h^*(t_n)$ denotes the finite element function on $\Gamma_h(\widetilde{X}_h^*(t_n))$ that corresponds to the nodal vector $\widetilde{\mathbf{x}}_*^n$. We therefore can use Lemma 3.1(ii) on $\Gamma_{h,*}^\theta$ to obtain

$$\begin{aligned}\left\|\nabla_{\Gamma_h^\theta}\left(\widetilde{y}^{n,\theta}-\widetilde{y}_*^{n,\theta}\right)\right\|_{L^\infty(\Gamma_h^\theta)} &\leq C\left\|\nabla_{\Gamma_h^\theta}\left((1-\theta)\widetilde{e}_x^{n,\theta}+\theta\widetilde{e}_x^{n-1,\theta}\right)\right\|_{L^\infty(\Gamma_h^\theta)}\\ &\leq C\left\|\nabla_{\Gamma_h^\theta}\left((1-\theta)\widetilde{e}_x^{n,\theta}+\theta\widetilde{e}_x^{n-1}\right)\right\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\\ &\leq C\gamma,\end{aligned}$$

for $\gamma > 0$ arbitrary small but constant. The last inequality is the same as (3.4). Now using Lemma 3.1 (ii) twice, together with the norm equivalence from Lemma 3.3 yields for (3.6)

$$\begin{aligned}&\left|\int_0^1\int_0^1\int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta \partial_\xi^\bullet\left(\nabla_{\Gamma_{*,h}^\Theta}\cdot\left(\widetilde{V}_{*,h}^{\Theta,n}+\xi\widetilde{E}_V^{\Theta,n}\right)\right)\mathrm{d}\theta\mathrm{d}\xi\right|\\ &\leq C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{E}_V^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}\\ &+C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{e}_x^n+\widetilde{e}_x^{n-1}\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}\left\|v_*^n\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\\ &+C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{e}_x^n+\widetilde{e}_x^{n-1}\right\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{E}_V^n\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\\ &\leq C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left(\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}^2+\gamma\left\|\widetilde{E}_V^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2+\left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2+\left\|\widetilde{e}_x^{n-1}\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2\right).\end{aligned}$$

We furthermore estimate (3.7) by

$$\begin{aligned}&\left|\int_0^1\int_0^1\int_{\Gamma_h^\Theta} w_h^\Theta z_h^\Theta\left(\nabla_{\Gamma_h^\Theta}\cdot V_h^{\Theta,n}\right)\left(\nabla_{\Gamma_h^\Theta}\cdot\widetilde{e}_y^{\Theta,n}\right)\mathrm{d}\theta\mathrm{d}\xi\right|\\ &\leq C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\left\|V_{*,h}^n\right\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{e}_x^n+\widetilde{e}_x^{n-1}\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}\\ &+C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{E}_V^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}\left\|\widetilde{e}_x^n+\widetilde{e}_x^{n-1}\right\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*^n])}\\ &\leq C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left(\gamma\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}^2+\gamma\left\|\widetilde{E}_V^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2+\left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2+\left\|\widetilde{e}_x^{n-1}\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2\right).\end{aligned}$$

Combining these results leaves us with

$$\begin{aligned}&\frac{1}{\tau}\mathbf{w}^T(\mathbf{M}(\widetilde{\mathbf{x}}^n)-\mathbf{M}(\widetilde{\mathbf{x}}^{n-1})-\left(\mathbf{M}(\widetilde{\mathbf{x}}_*^n)-\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-1})\right))\mathbf{z}\\ &\leq C\left\|w_h\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\left(\left\|z_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}^2+\gamma\left\|\widetilde{E}_V^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2+\left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2+\left\|\widetilde{e}_x^{n-1}\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}^2\right)\end{aligned}$$

□

4 Stability Analysis

For the stability analysis we introduce defects by plugging in the interpolation of the exact solution. Here we do not have access to a Ritz projection in a natural way, since the surface evolution is part of the solution of the system itself.
With the notation above there exist $\mathbf{d}_\mathbf{u}^n \in \mathbb{R}^N$ and $\mathbf{d}_\mathbf{x}^n, \mathbf{d}_\mathbf{v}^n \in \mathbb{R}^{3N}$ such that

$$
\begin{aligned}
\frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j})\mathbf{u}_*^{n-j}+\mathbf{A}(\widetilde{\mathbf{x}}_*^{n})\mathbf{u}_*^{n}&=\mathbf{f}(\widetilde{\mathbf{x}}_*^{n},\widetilde{\mathbf{u}}_*^{n})+\mathbf{M}(\mathbf{x}_*^{n})\mathbf{d}_{\mathbf{u}}^{n},\\
\mathbf{K}(\widetilde{\mathbf{x}}_*^{n})\mathbf{v}_*^{n}+\beta\mathbf{A}(\widetilde{\mathbf{x}}_*^{n})\mathbf{x}_*^{n}&=\mathbf{g}(\widetilde{\mathbf{x}}_*^{n},\widetilde{\mathbf{u}}_*^{n})+\mathbf{M}(\mathbf{x}_*^{n})\mathbf{d}_{\mathbf{v}}^{n},\\
\frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{x}_*^{n-j}&=\mathbf{v}_*^{n}+\mathbf{d}_{\mathbf{x}}^{n}.
\end{aligned}
\tag{4.1}
$$

By subtracting (4.1) from (2.19) and adding multiple zeroes we obtain the first error equation

$$
\begin{aligned}
\mathbf{M}(\mathbf{x}_*^{n})\frac{1}{\tau}\sum_{j=0}^{p}\delta_j\mathbf{e}_{\mathbf{u}}^{n-j}+\mathbf{A}(\mathbf{x}_*^{n})\mathbf{e}_{\mathbf{u}}^{n}=&\frac{1}{\tau}\sum_{j=0}^{p}\delta_j(\mathbf{M}(\mathbf{x}_*^{n-j})-\mathbf{M}(\mathbf{x}_*^{n}))\mathbf{e}_{\mathbf{u}}^{n-j}\\
&-\frac{1}{\tau}\sum_{j=0}^{p}\delta_j(\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j})-\mathbf{M}(\mathbf{x}_*^{n-j}))\mathbf{e}_{\mathbf{u}}^{n-j}\\
&-\frac{1}{\tau}\sum_{j=0}^{p}\delta_j(\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})-\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}))(\mathbf{u}_*^{n-j}+\mathbf{e}_{\mathbf{u}}^{n-j})\\
&-(\mathbf{A}(\widetilde{\mathbf{x}}_*^{n})-\mathbf{A}(\mathbf{x}_*^{n}))\mathbf{e}_{\mathbf{u}}^{n}-(\mathbf{A}(\widetilde{\mathbf{x}}^{n})-\mathbf{A}(\widetilde{\mathbf{x}}_*^{n}))(\mathbf{u}_*^{n}+\mathbf{e}_{\mathbf{u}}^{n})\\
&+\mathbf{f}(\widetilde{\mathbf{x}}^{n},\widetilde{\mathbf{u}}^{n})-\mathbf{f}(\widetilde{\mathbf{x}}_*^{n},\widetilde{\mathbf{u}}_*^{n})-\mathbf{M}(\mathbf{x}_*^{n})\mathbf{d}_{\mathbf{u}}^{n}.
\end{aligned}
\tag{4.2}
$$

The defects subtracted from the velocity law yields

$$
\begin{aligned}
\mathbf{K}(\widetilde{\mathbf{x}}_*^{n})\mathbf{e}_{\mathbf{v}}^{n}+\beta\mathbf{A}(\widetilde{\mathbf{x}}_*^{n})\mathbf{e}_{\mathbf{x}}^{n}=&-(\mathbf{K}(\widetilde{\mathbf{x}}^{n})-\mathbf{K}(\widetilde{\mathbf{x}}_*^{n}))\mathbf{e}_{\mathbf{v}}^{n}-(\mathbf{K}(\widetilde{\mathbf{x}}^{n})-\mathbf{K}(\widetilde{\mathbf{x}}_*^{n}))\mathbf{v}_*^{n}\\
&-\beta(\mathbf{A}(\widetilde{\mathbf{x}}^{n})-\mathbf{A}(\widetilde{\mathbf{x}}_*^{n}))\mathbf{e}_{\mathbf{x}}^{n}-\beta(\mathbf{A}(\widetilde{\mathbf{x}}^{n})-\mathbf{A}(\widetilde{\mathbf{x}}_*^{n}))\mathbf{x}_*^{n}\\
&+\mathbf{g}(\widetilde{\mathbf{x}}^{n},\widetilde{\mathbf{u}}^{n})-\mathbf{g}(\widetilde{\mathbf{x}}_*^{n},\widetilde{\mathbf{u}}_*^{n})-\mathbf{M}(\mathbf{x}_*^{n})\mathbf{d}_{\mathbf{v}}^{n}.
\end{aligned}
\tag{4.3}
$$

Finally, the defects subtracted from the ODE that connects the velocity to the position of the surface gives

$$
\frac{1}{\tau}\sum_{j=0}^{n}\delta_j\mathbf{e}_{\mathbf{x}}^{n-j}=\mathbf{e}_{\mathbf{v}}^{n}-\mathbf{d}_{\mathbf{x}}^{n}.
\tag{4.4}
$$

We need norms to estimate the defects. For the defect $\mathbf{d}\in\mathbb{R}^{3N}$ with the corresponding finite element function $d_h\in S_h(\mathbf{x}_*^n)^3$ we use the dual norm as in [[18]]

$$
\begin{aligned}
\|d_h\|_{H_h^{-1}(\Gamma_h[\mathbf{x}_*^n])}&:=\sup_{0\neq\psi_h\in S_h(\mathbf{x}_*^n)^3}\frac{\int_{\Gamma_h(\mathbf{x}_*^n)}d_h\cdot\psi_h}{\|\psi_h\|_{H^1(\Gamma_h(\mathbf{x}_*^n))^3}}\\
&=\sup_{0\neq\mathbf{z}\in\mathbb{R}^{3N}}\frac{\mathbf{d}^T\mathbf{M}(\mathbf{x}_*^n)\mathbf{z}}{(\mathbf{z}^T\mathbf{K}(\mathbf{x}_*^n)\mathbf{z})^{1/2}}=\sup_{0\neq\mathbf{w}\in\mathbb{R}^{3N}}\frac{\mathbf{d}^T\mathbf{M}(\mathbf{x}_*^n)\mathbf{K}(\mathbf{x}_*^n)^{-1/2}\mathbf{w}}{(\mathbf{w}^T\mathbf{w})^{1/2}}\\
&=\left\|\mathbf{K}(\mathbf{x}_*^n)^{-1/2}\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}\right\|_2=(\mathbf{d}^T\mathbf{M}(\mathbf{x}_*^n)\mathbf{K}(\mathbf{x}_*^n)^{-1}\mathbf{M}(\mathbf{x}_*^n)\mathbf{d})^{1/2}.
\end{aligned}
$$

We then define the norm for the defect vector by

$$\|\mathbf{d}\|^2_{\star,\mathbf{x}^n_*} := \|d_h\|^2_{H_h^{-1}(\Gamma_h[\mathbf{x}^n_*])} = \mathbf{d}^T\mathbf{M}(\mathbf{x}^n_*)\mathbf{K}(\mathbf{x}^n_*)^{-1}\mathbf{M}(\mathbf{x}^n_*)\mathbf{d}.$$

Now we conclude energy estimates from the error equations and collect them in the following proposition.

Proposition 1. *Suppose that the defects of the p-step linearly implicit BDF method are bounded as follows, with a sufficiently small $\vartheta > 0$ independent of h and τ: for $n \geq p$, with $n\tau \leq T$*

$$\|\mathbf{d}^n_{\mathbf{v}}\|_{\star,\mathbf{x}^k_*} \leq \vartheta h. \tag{4.5}$$

We also assume that the initial values fulfill the following error bounds for all $k < p$

$$\|\mathbf{e}^k_{\mathbf{x}}\|_{\mathbf{K}(\mathbf{x}^k_*)} \leq \vartheta h; \qquad \|\mathbf{e}^k_{\mathbf{v}}\|_{\mathbf{K}(\mathbf{x}^k_*)} \leq \vartheta h; \qquad \|\mathbf{e}^k_{\mathbf{u}}\|_{\mathbf{K}(\mathbf{x}^k_*)} \leq \vartheta h. \tag{4.6}$$

Then the following error bound holds for $n \geq p$ such that $n\tau \leq T$

$$\begin{aligned}
&\|\mathbf{e}^n_{\mathbf{x}}\|^2_{\mathbf{K}(\mathbf{x}^n_*)} + \|\mathbf{e}^n_{\mathbf{v}}\|^2_{\mathbf{K}(\mathbf{x}^n_*)} + \sum_{j=1}^{p} \|\mathbf{e}^{n-p+j}_{\mathbf{u}}\|^2_{\mathbf{M}(\mathbf{x}^n_*)} + \tau \sum_{j=1}^{n} \|\mathbf{e}^j_{\mathbf{u}}\|^2_{\mathbf{A}(\mathbf{x}^j_*)} \\
&\leq C\tau \sum_{j=p}^{n} \Big(\|\mathbf{d}^j_{\mathbf{x}}\|^2_{\mathbf{K}(\mathbf{x}^j_*)} + \|\mathbf{d}^j_{\mathbf{v}}\|^2_{*,\mathbf{x}^j_*} + \|\mathbf{d}^j_{\mathbf{u}}\|^2_{*,\mathbf{x}^j_*} \Big) + C \sum_{j=0}^{p-1} \Big(\|\mathbf{e}^j_{\mathbf{x}}\|^2_{\mathbf{K}(\mathbf{x}^j_*)} + \|\mathbf{e}^j_{\mathbf{u}}\|^2_{\mathbf{M}(\mathbf{x}^j_*)} \Big),
\end{aligned} \tag{4.7}$$

Proof. The stability result is the main challenge of the convergence analysis. The developed error equations are tested, rearranged and then the right-hand side is bounded using the auxiliary results.
The proof is divided into four subsections. In the first three subsections each of the error equations is analyzed separately. The final subsection then combines the estimates to obtain the stated result.

4.1 Estimates for the velocity law

In the following we denote with c an arbitrary constant that is not depending on h or τ.
We test (4.3) with $\mathbf{e}^n_{\mathbf{v}}$ and obtain, after rearranging

$$\begin{aligned}
(\mathbf{e}^n_{\mathbf{v}})^T\mathbf{K}(\widetilde{\mathbf{x}}^n_*)\mathbf{e}^n_{\mathbf{v}} = &-(\mathbf{e}^n_{\mathbf{v}})^T(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}^n_*))\mathbf{v}^n_* - (\mathbf{e}^n_{\mathbf{v}})^T(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}^n_*))\mathbf{e}^n_{\mathbf{v}} \\
&- \beta(\mathbf{e}^n_{\mathbf{v}})^T(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}^n_*))\mathbf{x}^n_* - \beta(\mathbf{e}^n_{\mathbf{v}})^T(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}^n_*))\mathbf{e}^n_{\mathbf{x}} \\
&- \beta(\mathbf{e}^n_{\mathbf{v}})^T\mathbf{A}(\widetilde{\mathbf{x}}^n_*)\mathbf{e}^n_{\mathbf{x}} + (\mathbf{e}^n_{\mathbf{v}})^T(\mathbf{g}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n) - \mathbf{g}(\widetilde{\mathbf{x}}^n_*, \widetilde{\mathbf{u}}^n_*)) - (\mathbf{e}^n_{\mathbf{v}})^T\mathbf{M}(\mathbf{x}^n_*)\mathbf{d}^n_{\mathbf{v}}.
\end{aligned} \tag{4.8}$$

We estimate the left-hand side with Lemma 3.1 from below by

$$(\mathbf{e}^n_{\mathbf{v}})^T\mathbf{K}(\widetilde{\mathbf{x}}^n_*)\mathbf{e}^n_{\mathbf{v}} = \|\mathbf{e}^n_{\mathbf{v}}\|_{\mathbf{K}(\widetilde{\mathbf{x}}^n_*)} \geq \frac{1}{2}\|\mathbf{e}^n_{\mathbf{v}}\|_{\mathbf{K}(\mathbf{x}^n_*)}.$$

For $0 \leq \theta \leq 1$, we denote $\Gamma^\theta_h{=}\Gamma_h[\widetilde{\mathbf{x}}^n_* + \theta\widetilde{\mathbf{e}}^n_{\mathbf{x}}]$ where $\widetilde{\mathbf{e}}^n_{\mathbf{x}} = \widetilde{\mathbf{x}}^n - \widetilde{\mathbf{x}}^n_* = \sum_{j=0}^{p-1}\gamma_j\mathbf{e}^{n-p+j}_{\mathbf{x}}$, which is the position error in nodal form.

All finite element functions on Γ_h^θ corresponding to vectors are denoted with a θ as an additional upper index, as defined in (3.1).
In the following we estimate the addends of the right hand side of 4.8 separately and in order.

(i)We start by using Lemma 3.1, which yields

$$\begin{aligned}&(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{v}_*^n\\ &=\int_0^1\int_{\Gamma_h^\theta} e_v^{n,\theta}(\nabla_{\Gamma_h^\theta}\cdot\widetilde{e}_x^{n,\theta})v_*^{n,\theta}\mathrm{d}\theta+\alpha\int_0^1\int_{\Gamma_h^\theta}\nabla_{\Gamma_h^\theta}e_v^{n,\theta}(D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta})\nabla_{\Gamma_h^\theta}v_*^{n,\theta}\mathrm{d}\theta.\end{aligned}$$

Furthermore, we see by the Cauchy–Schwarz inequality

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{v}_*^n\le&\int_0^1\left\|e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|\nabla_{\Gamma_h^\theta}\cdot\widetilde{e}_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|v_*^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)}\mathrm{d}\theta\\ &+\alpha\int_0^1\left\|\nabla_{\Gamma_h^\theta}e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|\nabla_{\Gamma_h^\theta}v_*^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)}\mathrm{d}\theta\\ \le& c\int_0^1\left\|e_v^{n,\theta}\right\|_{H^1(\Gamma_h^\theta)}\left\|\widetilde{e}_x^{n,\theta}\right\|_{H^1(\Gamma_h^\theta)}\left\|v_*^{n,\theta}\right\|_{W^{1,\infty}(\Gamma_h^\theta)}\mathrm{d}\theta.\end{aligned}\tag{4.9}$$

The outer integral can now be discarded by bounding the integrand with an auxiliary result. We use Lemma 3.1(iii) to estimate

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{v}_*^n&\le c\left\|e_v^n\right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\left\|v_*^n\right\|_{W^{1,\infty}(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\\ &\le c\left\|\mathbf{e}_\mathbf{v}^n\right\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}\left\|\widetilde{\mathbf{e}}_\mathbf{x}^n\right\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}(1+c\tau)\left\|v_*^n\right\|_{W^{1,\infty}(\Gamma_h[\mathbf{x}_*^n])},\end{aligned}\tag{4.10}$$

where Lemma 3.3 was used to estimate the last term by the exact norm at the time t_n.
Note that $\mathbf{v}_*^n$ and $\nabla_{\Gamma_h(\mathbf{x})}v_*^n$ are bounded independently from h and τ, since $\mathbf{v}_*^n$ is the exact velocity vector. By using this fact and the norm equivalence from Lemma 3.3 we arrive at

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{v}_*^n&\le c\left\|\mathbf{e}_\mathbf{v}^n\right\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}\left\|\widetilde{\mathbf{e}}_\mathbf{x}^n\right\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}\\ &\le c\left\|\mathbf{e}_\mathbf{v}^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)}\left\|\widetilde{\mathbf{e}}_\mathbf{x}^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)}\\ &\le\sum_{j=1}^p c\left\|\mathbf{e}_\mathbf{v}^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)}.\end{aligned}$$

Using Young's inequality on each summand now yields

$$(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{v}_*^n\le\frac{1}{24}\left\|\mathbf{e}_\mathbf{v}^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2+c\sum_{j=1}^p\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2.$$

(ii) Similarly to (4.9), with an $L^2-L^\infty-L^2$ estimate, we obtain

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n)-\mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{e}_\mathbf{v}^n\le&\int_0^1\left\|e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|\nabla_{\Gamma_h^\theta}\cdot\widetilde{e}_x^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)}\left\|e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\mathrm{d}\theta\\ &+\alpha\int_0^1\left\|\nabla_{\Gamma_h^\theta}e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)}\left\|\nabla_{\Gamma_h^\theta}e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\mathrm{d}\theta\\ \le& c\int_0^1\left\|e_v^{n,\theta}\right\|_{H^1(\Gamma_h^\theta)}^2\left\|\widetilde{e}_x^{n,\theta}\right\|_{H^1(\Gamma_h^\theta)}\mathrm{d}\theta.\end{aligned}$$

As before, we use Lemma 3.1 (iii) to bound the integrand using the values at the start of the transformation of the surface ($\theta = 0$) and then use the norm equivalence from Lemma 3.3. Together with the inverse estimate (3.4) we now obtain

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\widetilde{\mathbf{x}}^n) - \mathbf{K}(\widetilde{\mathbf{x}}_*^n))\mathbf{e}_\mathbf{v}^n &\leq c\,\|e_v^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}^2\,\|\widetilde{e}_x^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\ &\leq c\vartheta\,\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)}^2 \leq \frac{1}{24}\,\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 .\end{aligned}$$

The terms (iii) and (iv) are bounded using the same ideas as (i) and (ii).

(iii) Again, with Lemma 3.1 and an $L^2 - L^2 - L^\infty$ estimate, we obtain

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n))\mathbf{x}_*^n &= \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_v^{n,\theta} (D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}) \nabla_{\Gamma_h^\theta} x_*^{n,\theta} \mathrm{d}\theta \\ &\leq c\int_0^1 \left\|\nabla_{\Gamma_h^\theta} e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \left\|D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \left\|\nabla_{\Gamma_h^\theta} x_*^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)} \mathrm{d}\theta \\ &\leq c\,\|e_v^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\,\|\widetilde{e}_x^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\,\|x_*^n\|_{W^{1,\infty}(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} ,\end{aligned}$$

where the last term is similar to (4.10), since $\mathbf{x}_*^n$ is the exact solution, and therefore bounded independently of h and τ. Using the same steps in the same order then yields as before

$$(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n))\mathbf{x}_*^n \leq \frac{1}{48}\,\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + c\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 .$$

(iv) Here we again use the inverse estimate (3.4), as in (ii), to control the $W^{1,\infty}$- error of the position error $\widetilde{\mathbf{e}}_\mathbf{x}^n$. The same structure as before with an $L^2 - L^\infty - L^2$ estimate then yields

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n))\mathbf{e}_\mathbf{x}^n &= \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} e_v^{n,\theta} (D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}) \nabla_{\Gamma_h^\theta} e_x^{n,\theta} \mathrm{d}\theta \\ &\leq c\int_0^1 \left\|\nabla_{\Gamma_h^\theta} e_v^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \left\|D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)} \left\|\nabla_{\Gamma_h^\theta} e_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \mathrm{d}\theta \\ &\leq c\,\|e_v^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\,\vartheta\,\|e_x^n\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\ &\leq \frac{1}{48}\,\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}[\mathbf{x}_*^n]}^2 + c\,\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}[\mathbf{x}_*^n]}^2 .\end{aligned}$$

(v) The Cauchy-Schwarz inequality, combined with Young's inequality now yield

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T\mathbf{A}(\widetilde{\mathbf{x}}_*^n)\mathbf{e}_\mathbf{x}^n &\leq c\,\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}[\widetilde{\mathbf{x}}_*^n]}\,\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}[\widetilde{\mathbf{x}}_*^n]} \\ &\leq \frac{1}{48}\,\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}[\mathbf{x}_*^n]}^2 + c\,\|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}[\mathbf{x}_*^n]}^2 ,\end{aligned}$$

where we also used the norm equivalence from Lemma 3.3.

(vi) The coupling term is the most challenging in this section. We orientate at [16] where the corresponding term in the semi-discrete case is estimated.
We define an intermediate finite element function of $\widetilde{u}_h^n$ and $\widetilde{u}_{*,h}^n$ on the surface $\Gamma_h^\theta = \Gamma_h[\widetilde{\mathbf{x}}_*^n + \theta\widetilde{\mathbf{e}}_x^n]$, i.e.

$$\widetilde{u}_h^{n,\theta} := \sum_{j=1}^{N} (\widetilde{\mathbf{u}}_*^n + \theta\widetilde{\mathbf{e}}_{\mathbf{u}}^n)_j \ \phi_j[\widetilde{\mathbf{x}}_*^n + \theta\widetilde{\mathbf{e}}_x^n] = \widetilde{u}_{*,h}^{n,\theta} + \theta\widetilde{e}_u^{n,\theta} \ \in S_h[\widetilde{\mathbf{x}}_*^n + \theta\widetilde{\mathbf{e}}_x^n].$$

The transport property then yields

$$\partial_\theta^\bullet \widetilde{u}_h^{n,\theta} = \widetilde{e}_u^{n,\theta}. \tag{4.11}$$

We start in the same way as in the proof of Lemma 3.1 by using a intermediate surface $\Gamma_h^\theta = [\theta\widetilde{\mathbf{x}}^n + (1-\theta)\widetilde{\mathbf{x}}_*^n]$, by using the fundamental theorem of calculus. Then we use the Leibniz formula to obtain

$$\begin{aligned}
(\mathbf{e}_{\mathbf{v}}^n)^T(\mathbf{g}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n) - \mathbf{g}(\widetilde{\mathbf{x}}_*^n, \widetilde{\mathbf{u}}_*^n)) &= \int_{\Gamma_h[\widetilde{\mathbf{x}}^n]} g(\widetilde{u}^n, \nabla_{\Gamma_h[\widetilde{\mathbf{x}}^n]}\widetilde{u}^n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}^n]} e_v^{n,\theta} \\
&\quad - \int_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} g(\widetilde{u}_*^n, \nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{u}^n)\nu_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]} e_v^{n,\theta} \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} \int_{\Gamma_h^\theta} g(u_h^{n,\theta}, \nabla_{\Gamma_h^\theta}\widetilde{u}_h^{n,\theta})\nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta \\
&= \int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta}\widetilde{u}_h^{n,\theta})\nu_{\Gamma_h^\theta} e_v^{n,\theta} \right) \\
&\quad + (g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta}\widetilde{u}_h^{n,\theta})\nu_{\Gamma_h^\theta} e_v^{n,\theta} (\nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta})\mathrm{d}\theta,
\end{aligned}$$

where the Leibniz formula was used in the last equality. By the transport property we know that $\partial_\theta^\bullet e_v^{n,\theta} = 0$. Using the product rule twice then yields

$$(\mathbf{e}_{\mathbf{v}}^n)^T(\mathbf{g}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n) - \mathbf{g}(\widetilde{\mathbf{x}}_*^n, \widetilde{\mathbf{u}}_*^n)) = \int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta}\widetilde{u}_h^{n,\theta}) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \tag{4.12}$$

$$+ (g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta}\widetilde{u}_h^{n,\theta}) \left(\partial_\theta^\bullet \nu_{\Gamma_h^\theta} \right) e_v^{n,\theta} \tag{4.13}$$

$$+ (g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta}\widetilde{u}_h^{n,\theta})\nu_{\Gamma_h^\theta} e_v^{n,\theta} (\nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta})\mathrm{d}\theta. \tag{4.14}$$

Now, we estimate each of the summands separately.
The estimate of the extrapolated position error $\widetilde{e}_x^n$ can be extended to $\widetilde{e}_u^n$ by using the exact same steps as in (3.4). Therefore we obtain, for small enough h, τ ,

$$\|\widetilde{e}_u^n\|_{W^{1,\infty}} \leq \gamma,$$

where $\gamma > 0$ is an arbitrary small constant. The finite element function corresponding to $\mathbf{u}_*^n \in \mathbb{R}^N$ is the finite element interpolation (on the exact surface) to u and therefore is arbitrary close to the exact solution (see [5]), for h, τ small enough.

Combined with the Lipschitz continuity of $g(\cdot,\cdot)$ we therefore obtain

$$\left\| g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right\|_{L^\infty(\Gamma_h^\theta)} \leq C. \tag{4.15}$$

With the smoothness of g we extend this estimate to the derivatives of , i.e.

$$\left\| \partial_i g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right\|_{L^\infty(\Gamma_h^\theta)} \leq C. \tag{4.16}$$

Now, we estimate (4.13) with (3.2) in Lemma 3.1 (iv) to arrive at

$$\begin{aligned}
&\int_0^1 \int_{\Gamma_h^\theta} g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \left(\partial_\theta^\bullet \nu_{\Gamma_h^\theta} \right) e_v^{n,\theta} \mathrm{d}\theta \\
&\leq c \int_0^1 \left\| g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right\|_{L^\infty(\Gamma_h^\theta)} \left\| \nabla_{\Gamma_h^0} \widetilde{e}_x^{0,n} \right\|_{L^2(\Gamma_h^0)} \left\| e_v^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\leq c \int_0^1 \left\| \widetilde{e}_x^{0,n} \right\|_{H^1(\Gamma_h^0)} \left\| e_v^{n,\theta} \right\|_{H^1(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\leq c \left\| \widetilde{e}_x^{n} \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \left\| e_v^{n} \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\
&\leq \frac{1}{48} \left\| \mathbf{e}_\mathbf{v}^n \right\|_{\mathbf{K}[\mathbf{x}_*^n]}^2 + c \sum_{j=1}^{p} \left\| \mathbf{e}_\mathbf{x}^{n-j} \right\|_{\mathbf{K}[\mathbf{x}_*^n]}^2,
\end{aligned}$$

where the norm equivalence from Lemma 3.3 was used.
In a similar way we use an $L^\infty - L^\infty - L^2 - L^2$ estimate to bound (4.14) by

$$\begin{aligned}
&\int_0^1 \int_{\Gamma_h^\theta} g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \nu_{\Gamma_h^\theta} e_v^{n,\theta} (\nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta}) \mathrm{d}\theta \\
&\leq \int_0^1 \left\| g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right\|_{L^\infty(\Gamma_h^\theta)} \left\| \nu_{\Gamma_h^\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \left\| e_v^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \left\| \nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\leq c \int_0^1 \left\| e_v^{n,\theta} \right\|_{H^1(\Gamma_h^\theta)} \left\| \widetilde{e}_x^{n,\theta} \right\|_{H^1(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\leq \frac{1}{48} \left\| \mathbf{e}_\mathbf{v}^n \right\|_{\mathbf{K}[\mathbf{x}_*^n]}^2 + c \sum_{j=1}^{p} \left\| \mathbf{e}_\mathbf{x}^{n-j} \right\|_{\mathbf{K}[\mathbf{x}_*^n]}^2.
\end{aligned}$$

We estimate (4.12) by using the chain rule to obtain

$$\begin{aligned}
&\int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta \\
&= \int_0^1 \int_{\Gamma_h^\theta} \left(\partial_1 g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \partial_\theta^\bullet \widetilde{u}_h^{n,\theta} + \partial_2 (g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \partial_\theta^\bullet \left(\nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta} \right) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta \\
&\leq \int_0^1 \int_{\Gamma_h^\theta} \left(c \partial_\theta^\bullet \widetilde{u}_h^{n,\theta} + c \partial_\theta^\bullet \left(\nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta} \right) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta,
\end{aligned}$$

where we used (4.16). We take the following identity from [11, Lemma 2.6] :

$$\partial_\theta^\bullet \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta} = \nabla_{\Gamma_h^\theta} \partial_\theta^\bullet \widetilde{u}_h^{n,\theta} - \left(\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} - \nu_{\Gamma_h^\theta} (\nu_{\Gamma_h^\theta})^T (\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta})^T \right) \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}. \tag{4.17}$$

Together with (4.11) and the definition of $\widetilde{u}_h^{n,\theta}$ we estimate (4.12) further by

$$\begin{aligned}
&\int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta \\
&\leq C \int_0^1 \int_{\Gamma_h^\theta} \left(\widetilde{e}_u^{n,\theta} + \nabla_{\Gamma_h^\theta} \widetilde{e}_u^{n,\theta} - \left(\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} - \nu_{\Gamma_h^\theta} (\nu_{\Gamma_h^\theta})^T (\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta})^T \right) \nabla_{\Gamma_h^\theta} u_h^{*,\theta,n} \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \\
&+ \left(\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} - \nu_{\Gamma_h^\theta} (\nu_{\Gamma_h^\theta})^T (\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta})^T \right) (\nabla_{\Gamma_h^\theta} \theta \widetilde{e}_u^{n}) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta.
\end{aligned}$$

With the triangle inequality it is immediately clear that for $1 \leq p \leq \infty$

$$\begin{aligned}
\left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} - \nu_{\Gamma_h^\theta} (\nu_{\Gamma_h^\theta})^T (\nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta})^T \right\|_{L^p(\Gamma_h^\theta)} &\leq \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^p(\Gamma_h^\theta)} + \left\| \nu_{\Gamma_h^\theta} \right\|_{L^\infty(\Gamma_h^\theta)}^2 \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^p(\Gamma_h^\theta)} \\
&\leq c \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^p(\Gamma_h^\theta)},
\end{aligned}$$

since the unit normal vector $\nu_{\Gamma_h^\theta}$ is bounded in the L^∞- norm. Multiple $L^2 - L^\infty - L^2$ estimates now yield

$$\begin{aligned}
&\int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta \\
&\leq c \int_0^1 \| \widetilde{e}_u^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \left\| \nu_{\Gamma_h^\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} + \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_u^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \left\| \nu_{\Gamma_h^\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \\
&+ \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \left\| \nabla_{\Gamma_h^\theta} u_{*,h}^{\theta,n} \right\|_{L^\infty(\Gamma_h^\theta)} \left\| \nu_{\Gamma_h^\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \\
&+ \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \left\| \nabla_{\Gamma_h^\theta} \theta \widetilde{e}_u^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \left\| \nu_{\Gamma_h^\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \mathrm{d}\theta.
\end{aligned}$$

We use Lemma 3.1(iii) and the estimate for the position error $\widetilde{e}_x^{n,\theta}$ to arrive at

$$\begin{aligned}
&\int_0^1 \int_{\Gamma_h^\theta} \partial_\theta^\bullet \left(g(\widetilde{u}_h^{n,\theta}, \nabla_{\Gamma_h^\theta} \widetilde{u}_h^{n,\theta}) \right) \nu_{\Gamma_h^\theta} e_v^{n,\theta} \mathrm{d}\theta \\
&\leq C \int_0^1 \| \widetilde{e}_u^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} + \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_u^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \\
&+ \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} \left\| \nabla_{\Gamma_h^\theta} u_{*,h}^{\theta,n} \right\|_{L^\infty(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} + \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^\infty(\Gamma_h^\theta)} \| \widetilde{e}_u^{n,\theta} \|_{H^1(\Gamma_h^\theta)} \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\leq C \int_0^1 \| e_v^{n,\theta} \|_{L^2(\Gamma_h^\theta)} (\| \widetilde{e}_u^{n,\theta} \|_{L^2(\Gamma_h^\theta)} + \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_u^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} + \left\| \nabla_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta} \right\|_{L^2(\Gamma_h^\theta)} + \| \widetilde{e}_u^{n,\theta} \|_{H^1(\Gamma_h^\theta)}) \mathrm{d}\theta \\
&\leq C \| e_v^n \|_{H^1(\Gamma[\widetilde{\mathbf{x}}_*^n])} \left(\| \widetilde{e}_x^n \|_{H^1(\Gamma[\widetilde{\mathbf{x}}_*^n])} + \| \widetilde{e}_u^n \|_{H^1(\Gamma[\widetilde{\mathbf{x}}_*^n])} \right) \\
&\leq \frac{1}{48} \| \mathbf{e}_\mathbf{v}^n \|_{\mathbf{K}[\mathbf{x}_*^n]}^2 + C \sum_{j=1}^p \| \mathbf{e}_\mathbf{x}^{n-j} \|_{\mathbf{K}[\mathbf{x}_*^n]}^2 + C \sum_{j=1}^p \| \mathbf{e}_\mathbf{u}^{n-j} \|_{\mathbf{K}[\mathbf{x}_*^n]}^2,
\end{aligned}$$

where the same techniques as before were used to obtain the last bound. The estimates for

(4.12)-(4.14) therefore yield the following estimate for the coupling term

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T(\mathbf{g}(\widetilde{\mathbf{x}}^n,\widetilde{\mathbf{u}}^n)-\mathbf{g}(\widetilde{\mathbf{x}}_*^n,\widetilde{\mathbf{u}}_*^n)) &\leq \frac{3}{48}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}\\ &\quad + C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{u}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}.\end{aligned} \tag{4.18}$$

(vii) The defect term is estimated by using the Cholesky decomposition $\mathbf{K}(\mathbf{x}_*^n)^{1/2}$, defined by $\mathbf{K}(\mathbf{x}_*^n)=\mathbf{K}(\mathbf{x}_*^n)^{1/2}(\mathbf{K}(\mathbf{x}_*^n)^{1/2})^T$ and the Cauchy-Schwarz inequality in the following way:

$$\begin{aligned}(\mathbf{e}_\mathbf{v}^n)^T\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n &= (\mathbf{e}_\mathbf{v}^n)^T\mathbf{K}(\mathbf{x}_*^n)^{1/2}\mathbf{K}(\mathbf{x}_*^n)^{-1/2}\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n\\ &= ((\mathbf{K}(\mathbf{x}_*^n)^{1/2})^T\mathbf{e}_\mathbf{v}^n)^T(\mathbf{K}(\mathbf{x}_*^n)^{-1/2}\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n)\\ &\leq \left\|(\mathbf{K}(\mathbf{x}_*^n)^{1/2})^T\mathbf{e}_\mathbf{v}^n\right\|_2\left\|\mathbf{K}(\mathbf{x}_*^n)^{-1/2}\mathbf{M}(\mathbf{x}_*^n)\mathbf{d}_\mathbf{v}^n\right\|_2\\ &= \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}\|\mathbf{d}_\mathbf{v}^n\|_{*,\mathbf{x}_*^n}\leq\frac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}+c\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n}.\end{aligned} \tag{4.19}$$

Combining the terms (i) to (viii) together with the estimate for the left side yield the inequality

$$\begin{aligned}\frac{1}{2}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} &\leq \frac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|^2_{\mathbf{K}(\mathbf{x}_*^n)} && \text{(i)}\\ &\quad+\frac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} && \text{(ii)}\\ &\quad+\frac{1}{48}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|^2_{\mathbf{K}(\mathbf{x}_*^n)} && \text{(iii)}\\ &\quad+\frac{1}{48}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]} && \text{(iv)}\\ &\quad+\frac{1}{48}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]} && \text{(v)}\\ &\quad+\frac{3}{48}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{u}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]} && \text{(vi)}\\ &\quad+\frac{1}{24}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}+C\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n} && \text{(vii)}\\ &\leq\frac{1}{4}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}+C\|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}[\mathbf{x}_*^n]}\\ &\quad+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{u}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+C\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n}.\end{aligned}$$

Now we absorb the velocity error on the right side and multiply both sides with 4 to obtain

$$\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}\leq C\left(\sum_{j=0}^{p}\left\|\mathbf{e}_\mathbf{x}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+\sum_{j=1}^{p}\left\|\mathbf{e}_\mathbf{u}^{n-j}\right\|^2_{\mathbf{K}[\mathbf{x}_*^n]}+\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n}\right). \tag{4.20}$$

In the following we recall two results that are essential for the stability results arising from the surface PDE. The first one is from Dahlquist's G-theory from [4].

Lemma 4.1 (Dahlquist [4]). *Let $\delta(\xi) = \sum_{j=1}^{p} \delta_j \zeta^j$ and $\mu(\xi) = \sum_{j=1}^{p} \mu_j \zeta^j$ be polynomials of degree at most p (at least one of them of degree p), that have no common divisor. If*

$$\mathrm{Re}\frac{\delta(\zeta)}{\mu(\zeta)} > 0, \qquad \text{for } |\zeta| < 1,$$

then there exists a symmetric positive definite matrix $G = (g_{ij}) \in \mathbb{R}^{p\times p}$ such that for all $\mathbf{w}_0, ..., \mathbf{w}_p \in \mathbb{R}^N$

$$\langle \sum_{i=0}^{p} \delta_i \mathbf{w}_{p-i}, \sum_{i=0}^{p} \mu_i \mathbf{w}_{p-i} \rangle \geq \sum_{i,j=1}^{p} g_{ij} \langle \mathbf{w}_i, \mathbf{w}_j \rangle - \sum_{i,j=1}^{p} g_{ij} \langle \mathbf{w}_{i-1}, \mathbf{w}_{j-1} \rangle.$$

In our case $\delta(\zeta)$ will be the characteristic polynomial of the BDF methods, defined in (2.22). For $p \leq 5$ the following result from the multiplier technique of Nevanlinna and Odeh [19] and gives a linear polynomial $\mu(\zeta)$ that fulfills the conditions for the previous Lemma.

Lemma 4.2 (Nevanlinna & Odeh [19]). *If $p \leq 5$, then there exists $0 \leq \eta < 1$ such that for $\delta(\zeta) = \sum_{l=1}^{p} \frac{1}{l}(1-\zeta)^l$*

$$\mathrm{Re}\frac{\delta(\zeta)}{1-\eta\zeta} > 0, \qquad \text{for } |\zeta| < 1.$$

The smallest possible values of η are found to be $\eta = 0, 0, 0.0836, 0.2878, 0.8160$ for $p = 1, ..., 5$, respectively.

4.2 Estimates for the surface PDE

We recall the error equation (4.2) from the surface PDE and test it with $\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1}$, where $\eta \in [0,1)$ is the coefficient from the multiplier techniques.

$$\begin{aligned}
&(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \frac{1}{\tau} \sum_{j=0}^{p} \delta_j \mathbf{e}_\mathbf{u}^{n-j} + (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{A}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{u}^n \\
&= \frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{M}(\mathbf{x}_*^{n-j}) - \mathbf{M}(\mathbf{x}_*^n)) \mathbf{e}_\mathbf{u}^{n-j} && \text{(i)} \\
&- \frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}) - \mathbf{M}(\mathbf{x}_*^{n-j})) \mathbf{e}_\mathbf{u}^{n-j} && \text{(ii)} \\
&- \frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j})) (\mathbf{u}_*^{n-j} + \mathbf{e}_\mathbf{u}^{n-j}) && \text{(iii)} \\
&- (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{A}(\widetilde{\mathbf{x}}_*^n) - \mathbf{A}(\mathbf{x}_*^n)) \mathbf{e}_\mathbf{u}^n - (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n)) (\mathbf{u}_*^n + \mathbf{e}_\mathbf{u}^n) && \text{(iv)} \\
&+ (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{f}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n) - \mathbf{f}(\widetilde{\mathbf{x}}_*^n, \widetilde{\mathbf{u}}_*^n)) - (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{d}_\mathbf{u}^n. && \text{(v) + (vi)}
\end{aligned}$$

In the following we write in short ρ^n for the right hand side of the error equation, i.e.

$$(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \frac{1}{\tau} \sum_{j=0}^{p} \delta_j \mathbf{e}_\mathbf{u}^{n-j} + (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{A}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{u}^n = \rho^n. \tag{4.21}$$

4.2.1 Estimation of ρ^n

Now, we estimate the terms on the right hand side in order. Most terms are bounded with the same techniques as the terms in Section 4.1 and are therefore only discussed briefly. The main term to estimate here is (iii), which is achieved by Lemma 3.4.

(i) With Lemma (3.2) and a triangle inequality we estimate the first term with

$$\begin{aligned}
&\frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{M}(\mathbf{x}_*^{n-j}) - \mathbf{M}(\mathbf{x}_*^n)) \mathbf{e}_\mathbf{u}^{n-j} \\
&\leq \frac{1}{\tau} \sum_{j=0}^{p} \delta_j C j \tau \left\| \mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \\
&\leq C \sum_{j=0}^{p} \left\| \mathbf{e}_\mathbf{u}^n \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^n)} + \left\| \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \\
&\leq C \sum_{j=0}^{p} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2,
\end{aligned}$$

where Young's inequality and the norm equivalence from (3.2) were used.

(ii) This term is estimated using Lemma 3.3 and the same arguments as before by

$$\begin{aligned}
&\frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}) - \mathbf{M}(\mathbf{x}_*^{n-j})) \mathbf{e}_\mathbf{u}^{n-j} \\
&\leq C \tau^{p-1} \sum_{j=0}^{p} \left\| \mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})} \\
&\leq \gamma \tau^{p-1} \left\| \mathbf{e}_\mathbf{u}^n \right\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 + C \tau^{p-1} \sum_{j=1}^{p} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2 .
\end{aligned}$$

(iii) This term is the most challenging in this section and makes use of Lemma 3.4 . Using bounded coefficients μ_j and the discrete product rule allows us to obtain

$$
\begin{aligned}
&(\mathbf{e}_{\mathbf{u}}^n - \eta \mathbf{e}_{\mathbf{u}}^{n-1})^T \frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}))\mathbf{u}_*^{n-j} \\
&= \sum_{j=0}^{p-1} \frac{\mu_j}{\tau} (\mathbf{e}_{\mathbf{u}}^n - \eta \mathbf{e}_{\mathbf{u}}^{n-1})^T \Big(\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})\mathbf{u}_*^{n-j} - \mathbf{M}(\widetilde{\mathbf{x}}^{n-j-1})\mathbf{u}_*^{n-j-1} - \Big(\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j})\mathbf{u}_*^{n-j} - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j-1})\mathbf{u}_*^{n-j-1} \Big) \Big) \\
&= \sum_{j=0}^{p-1} \mu_j (\mathbf{e}_{\mathbf{u}}^n - \eta \mathbf{e}_{\mathbf{u}}^{n-1})^T \frac{\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}^{n-j-1}) - \Big(\mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j-1}) \Big)}{\tau} \mathbf{u}_*^{n-j-1} \\
&+ \sum_{j=0}^{p-1} \mu_j (\mathbf{e}_{\mathbf{u}}^n - \eta \mathbf{e}_{\mathbf{u}}^{n-1})^T \Big(\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}) \Big) \frac{\mathbf{u}_*^{n-j} - \mathbf{u}_*^{n-j-1}}{\tau} \\
&\le C \sum_{j=0}^{p-1} \Big(\|e_u^n\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}^2 + \|e_u^{n-1}\|_{L^2(\Gamma_h[\mathbf{x}_*^{n-1}])}^2 + \gamma \Big\|\widetilde{E}_V^{n-j}\Big\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} + \|\widetilde{e}_x^{n-j}\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j}])}^2 \Big) \\
&+ \sum_{j=0}^{p-1} \|\widetilde{e}_x^{n-j-1}\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j-1}])}^2 + C \sum_{j=0}^{p-1} \Big(\|e_u^n\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} + \|e_u^{n-1}\|_{L^2(\Gamma_h[\mathbf{x}_*^{n-1}])} \Big) \\
&\cdot \sum_{j=0}^{p-1} \|\widetilde{e}_x^{n-j}\|_{L^2(\Gamma_h[\mathbf{x}_*^{n-j}])} \left\| \frac{u_{*,h}(t_{n-j}) - u_{*,h}(t_{n-j-1})}{\tau} \right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^{n-j}])},
\end{aligned}
$$

where in the last inequality Lemma 3.4 and the norm equivalence from Lemma 3.3 were used. The quotient in the last line is the L^∞-norm of the normal difference quotient of $u_{*,h}$ and therefore bounded independent of h and τ, for τ small enough. This finally leaves us with

$$
\begin{aligned}
&(\mathbf{e}_{\mathbf{u}}^n - \eta \mathbf{e}_{\mathbf{u}}^{n-1})^T \frac{1}{\tau} \sum_{j=0}^{p} \delta_j (\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}))\mathbf{u}_*^{n-j} \\
&\le C \|\mathbf{e}_{\mathbf{u}}^n\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}^2 + C \|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2(\Gamma_h[\mathbf{x}_*^{n-1}])}^2 + \sum_{j=1}^{2p-1} \gamma \Big\|E_V^{n-j}\Big\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j}])}^2 + C \|\mathbf{e}_{\mathbf{x}}^{n-j}\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j}])}^2
\end{aligned}
$$

For the second summand of (iii) the stated restrictions on step and mesh size come into play. Assuming that the error estimate from Theorem 2.1 holds for the past vectors we obtain

$$
\|\mathbf{e}_{\mathbf{x}}^j\|_{\mathbf{K}(\mathbf{x}_*^j)} \le C(h^k + \tau^p) \le C\tau h, \qquad \text{for } j = 1, ..., n-1,
$$

for a constant $C > 0$ independent of τ and h. This allows us to estimate the $W^{1,\infty}$-norm of the extrapolated position error $\widetilde{e}_x^n$ stronger than before, with the same structure as in (3.4). With an inverse estimate we then obtain

$$\begin{aligned}\left\|\nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{e}_x^n\right\|_{L^\infty(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} &\leq Ch^{-1}\left\|\nabla_{\Gamma_h[\widetilde{\mathbf{x}}_*^n]}\widetilde{e}_x^n\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^n])}\\ &\leq Ch^{-1}\left\|\widetilde{\mathbf{e}}_{\mathbf{x}}^n\right\|_{\mathbf{K}(\widetilde{\mathbf{x}}_*^n)} \leq Ch^{-1}\left\|\widetilde{\mathbf{e}}_{\mathbf{x}}^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)}\\ &\leq Ch^{-1}\sum_{j=1}^p\left\|\mathbf{e}_{\mathbf{x}}^{n-j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)}\\ &\leq Ch^{-1}\cdot Ch\tau \leq C\tau.\end{aligned} \tag{4.22}$$

In the following, we write $\Gamma_h^{n-j,\theta} = \Gamma_h[\widetilde{\mathbf{x}}_*^{n-j} + \theta\widetilde{\mathbf{e}}_{\mathbf{x}}^{n-j}]$. We use Lemma 3.1 (i) to deduce

$$\begin{aligned}&(\mathbf{e}_{\mathbf{u}}^n - \eta\mathbf{e}_{\mathbf{u}}^{n-1})^T\frac{1}{\tau}\sum_{j=0}^p\delta_j(\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}))\mathbf{e}_{\mathbf{u}}^{n-j}\\ &\leq \sum_{j=0}^p\frac{\delta_j}{\tau}\int_0^1\int_{\Gamma_h^{n-j,\theta}}\left(e_u^{n,\theta} - \eta e_u^{n-1,\theta}\right)\left(\nabla_{\Gamma_h}\cdot\widetilde{e}_x^{n-j,\theta}\right)e_u^{n-j,\theta}\mathrm{d}\theta\\ &\leq \sum_{j=0}^p\frac{\delta_j}{\tau}\int_0^1\left\|e_u^{n,\theta} - \eta e_u^{n-1,\theta}\right\|_{L^2(\Gamma_h^{n-j,\theta})}\left\|\widetilde{e}_x^{n-j,\theta}\right\|_{W^{1,\infty}(\Gamma_h^{n-j,\theta})}\left\|e_u^{n-j,\theta}\right\|_{L^2(\Gamma_h^{n-j,\theta})}\mathrm{d}\theta\\ &\leq C\sum_{j=0}^p\frac{\delta_j}{\tau}\left(\left\|e_u^n\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-1}])} + \left\|e_u^{n-1}\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-1}])}\right)\left\|\widetilde{e}_x^{n-j}\right\|_{W^{1,\infty}(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-j}])}\left\|e_u^{n-j}\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-j}])},\end{aligned}$$

where we used Lemma 3.1 (ii) and (iii) in the last inequality. Now, we use (4.22) to arrive at

$$\begin{aligned}&C\sum_{j=0}^p\frac{\delta_j}{\tau}\left(\left\|e_u^n\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-1}])} + \left\|e_u^{n-1}\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-1}])}\right)\left\|\widetilde{e}_x^{n-j}\right\|_{W^{1,\infty}(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-j}])}\left\|e_u^{n-j}\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-j}])},\\ &\leq C\sum_{j=0}^p\delta_j\left(\left\|e_u^n\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-1}])} + \left\|e_u^{n-1}\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-1}])}\right)\left\|e_u^{n-j}\right\|_{L^2(\Gamma_h[\widetilde{\mathbf{x}}_*^{n-j}])},\\ &\leq C\sum_{j=0}^p\left\|e_u^{n-j}\right\|_{L^2(\Gamma_h[\mathbf{x}_*^{n-j}])}^2,\end{aligned} \tag{4.23}$$

where in the last line we use Young's inequality and the fact that the δ_j are bounded. By using the definition of the matrix-norms we conclude

$$(\mathbf{e}_{\mathbf{u}}^n - \eta\mathbf{e}_{\mathbf{u}}^{n-1})^T\frac{1}{\tau}\sum_{j=0}^p\delta_j(\mathbf{M}(\widetilde{\mathbf{x}}^{n-j}) - \mathbf{M}(\widetilde{\mathbf{x}}_*^{n-j}))\mathbf{e}_{\mathbf{u}}^{n-j} \leq C\sum_{j=0}^p\left\|\mathbf{e}_{\mathbf{u}}^{n-j}\right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2.$$

(iv) The first summand of this term is bounded in the same way as before by Lemma 3.3 in the following way:

$$\begin{aligned}&(\mathbf{e}_{\mathbf{u}}^n - \eta\mathbf{e}_{\mathbf{u}}^{n-1})^T(\mathbf{A}(\widetilde{\mathbf{x}}_*^n) - \mathbf{A}(\mathbf{x}_*^n))\mathbf{e}_{\mathbf{u}}^n\\ &\leq C\tau^p\left\|\mathbf{e}_{\mathbf{u}}^n - \eta\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^n)}\left\|\mathbf{e}_{\mathbf{u}}^n\right\|_{\mathbf{A}(\mathbf{x}_*^n)}\\ &\leq C\tau^p\left(\left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 + \left\|\mathbf{e}_{\mathbf{u}}^n\right\|_{\mathbf{A}(\mathbf{x}_*^n)}^2\right) \leq C\tau^p\left(\left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 + \left\|\mathbf{e}_{\mathbf{u}}^n\right\|_{\mathbf{A}(\mathbf{x}_*^n)}^2\right).\end{aligned}$$

Using the exact same techniques as in (4.8) (i), yields with an $L^2 - L^2 - L^\infty$ estimate, for the second term of (iv)

$$
\begin{aligned}
&(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n))\mathbf{u}_*^n \\
&= \int_0^1 \int_{\Gamma_h^\theta} \nabla_{\Gamma_h^\theta} (e_u^{n,\theta} - \eta e_u^{\theta,n-1}) (D_{\Gamma_h^\theta} \widetilde{e}_x^{n,\theta}) \nabla_{\Gamma_h^\theta} u_*^{n,\theta} \mathrm{d}\theta \\
&\le \int_0^1 \left\| e_u^{n,\theta} - \eta e_u^{\theta,n-1} \right\|_{H^1(\Gamma_h^\theta)} \left\| \widetilde{e}_x^{n,\theta} \right\|_{H^1(\Gamma_h^\theta)} \left\| u_*^{n,\theta} \right\|_{W^{1,\infty}(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\le C \left\| e_u^n - \eta e_u^{n-1} \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \left\| \widetilde{e}_x^n \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \left\| u_*^n \right\|_{W^{1,\infty}(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\
&\le C \left(\left\| e_u^n \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} + \eta \left\| e_u^{n-1} \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \right) \sum_{j=0}^{p-1} \left\| e_x^{n-1-j} \right\|_{H^1(\Gamma_h[\widetilde{\mathbf{x}}_*^n])} \\
&\le C \left(\gamma \left\| \mathbf{e}_\mathbf{u}^n \right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \gamma \left\| \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1})}^2 + \sum_{j=0}^{p-1} \left\| \mathbf{e}_\mathbf{x}^{n-1-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1-j})}^2 \right).
\end{aligned}
$$

With the same structure, using a $L^2 - L^\infty - L^2$ estimate we see,

$$
\begin{aligned}
&(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{A}(\widetilde{\mathbf{x}}^n) - \mathbf{A}(\widetilde{\mathbf{x}}_*^n))\mathbf{e}_\mathbf{u}^n \\
&\le \gamma \left(\left\| \mathbf{e}_\mathbf{u}^n \right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \left\| \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1})}^2 \right),
\end{aligned}
$$

where we used the $W^{1,\infty}$ estimate of the position error from (3.4).

The nonlinear term (v) can be estimated exactly like the coupling term before, which was bounded in (4.18). The only differences are the test function and the fact that the normal vector does not appear here. However, the missing of the normal vector here only simplifies the computation, all steps that do not concern with the normal vector can just be repeated in the exact same way to obtain, for an arbitrary small $\gamma > 0$

$$
(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T (\mathbf{f}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n) - \mathbf{f}(\widetilde{\mathbf{x}}_*^n, \widetilde{\mathbf{u}}_*^n)) \le \gamma \sum_{j=0}^{p} \left\| \mathbf{e}_\mathbf{u}^{n-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-j})}^2 + C \sum_{j=0}^{p} \left\| \mathbf{e}_\mathbf{x}^{n-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-j})}^2 .
$$

The defect term arising from $\mathbf{d}_\mathbf{u}$ are estimated analogous to the bound of the defect $\mathbf{d}_\mathbf{v}$, that was bounded in (4.19). Using the Cauchy-Schwartz inequality, combined with Young's inequality yields

$$
\begin{aligned}
(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{d}_\mathbf{u}^n &= (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{K}(\mathbf{x}_*^n)^{1/2} \mathbf{K}(\mathbf{x}_*^n)^{-1/2} \mathbf{M}(\mathbf{x}_*^n) \mathbf{d}_\mathbf{u}^n \\
&= \left(\left(\mathbf{K}(\mathbf{x}_*^n)^{1/2} \right)^T (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1}) \right) \cdot \left(\mathbf{K}(\mathbf{x}_*^n)^{-1/2} \mathbf{M}(\mathbf{x}_*^n) \mathbf{d}_\mathbf{u}^n \right) \\
&\le \left\| \left(\mathbf{K}(\mathbf{x}_*^n)^{1/2} \right)^T (\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1}) \right\|_2 \left\| \mathbf{K}(\mathbf{x}_*^n)^{-1/2} \mathbf{M}(\mathbf{x}_*^n) \mathbf{d}_\mathbf{u}^n \right\|_2 \\
&= \left\| \mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{K}(\mathbf{x}_*^n)} \left\| \mathbf{d}_\mathbf{u}^n \right\|_{*,\mathbf{x}_*^n} \\
&\le \gamma \left\| \mathbf{e}_\mathbf{u}^n \right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \gamma \left\| \mathbf{e}_\mathbf{u}^{n-1} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1})}^2 + C \left\| \mathbf{d}_\mathbf{u}^n \right\|_{*,\mathbf{x}_*^n}^2
\end{aligned}
$$

Collecting all the bounds for the summands on the right hand side now gives

$$
\begin{aligned}
\rho^n \leq\, & C \sum_{j=0}^{p} \left\| \mathbf{e}_{\mathbf{u}}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2 && \text{(i)} \\
& + \gamma \tau^{p-1} \left\| \mathbf{e}_{\mathbf{u}}^{n} \right\|_{\mathbf{M}(\mathbf{x}_*^{n})}^2 + C\tau^{p-1} \sum_{j=1}^{p} \left\| \mathbf{e}_{\mathbf{u}}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2 && \text{(ii)} \\
& + C \left\| e_{\mathbf{u}}^{n} \right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}^2 + C \left\| e_{\mathbf{u}}^{n-1} \right\|_{L^2(\Gamma_h[\mathbf{x}_*^{n-1}])}^2 && \text{(iii)} \\
& + \sum_{j=1}^{2p-1} \left(\gamma \left\| E_V^{n-j} \right\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j}])}^2 + C \left\| e_{\mathbf{x}}^{n-j} \right\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j}])}^2 \right) + C \sum_{j=0}^{p} \left\| \mathbf{e}_{\mathbf{u}}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2 && \text{(iii)} \\
& + C\tau^p \left(\left\| \mathbf{e}_{\mathbf{u}}^{n-1} \right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 + \left\| \mathbf{e}_{\mathbf{u}}^{n} \right\|_{\mathbf{A}(\mathbf{x}_*^{n})}^2 \right) && \text{(iv)} \\
& + C \left(\gamma \left\| \mathbf{e}_{\mathbf{u}}^{n} \right\|_{\mathbf{K}(\mathbf{x}_*^{n})}^2 + \gamma \left\| \mathbf{e}_{\mathbf{u}}^{n-1} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1})}^2 + \sum_{j=0}^{p-1} \left\| \mathbf{e}_{\mathbf{x}}^{n-1-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1-j})}^2 \right) && \text{(iv)} \\
& + \gamma \sum_{j=0}^{p} \left\| \mathbf{e}_{\mathbf{u}}^{n-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-j})}^2 + C \sum_{j=0}^{p} \left\| \mathbf{e}_{\mathbf{x}}^{n-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-j})}^2 && \text{(v)} \\
& + \gamma \left\| \mathbf{e}_{\mathbf{u}}^{n} \right\|_{\mathbf{K}(\mathbf{x}_*^{n})}^2 + \gamma \left\| \mathbf{e}_{\mathbf{u}}^{n-1} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-1})}^2 + C \left\| \mathbf{d}_{\mathbf{u}}^{n} \right\|_{*,\mathbf{x}_*^n}^2 && \text{(vi)} \\
\leq\, & C \sum_{j=0}^{p} \left\| \mathbf{e}_{\mathbf{u}}^{n-j} \right\|_{\mathbf{M}(\mathbf{x}_*^{n-j})}^2 + \gamma \left\| \mathbf{e}_{\mathbf{u}}^{n-j} \right\|_{\mathbf{A}(\mathbf{x}_*^{n-j})}^2 \\
& + \sum_{j=1}^{2p-1} \left(C \left\| \mathbf{e}_{\mathbf{x}}^{n-j} \right\|_{\mathbf{K}(\mathbf{x}_*^{n-j})}^2 + \gamma \left\| E_V^{n-j} \right\|_{H^1(\Gamma_h[\mathbf{x}_*^{n-j}])}^2 \right) + C \left\| \mathbf{d}_{\mathbf{u}}^{n} \right\|_{*,\mathbf{x}_*^n}^2
\end{aligned}
$$

Adding up the estimate for $j = 1, \dots, n$ now yields

$$
\sum_{j=1}^{n} \rho^j \leq C \sum_{j=1}^{n} \left(\left\| \mathbf{e}_{\mathbf{u}}^{j} \right\|_{\mathbf{M}(\mathbf{x}_*^{j})}^2 + \gamma \left\| \mathbf{e}_{\mathbf{u}}^{j} \right\|_{\mathbf{A}(\mathbf{x}_*^{j})}^2 + \left\| \mathbf{e}_{\mathbf{x}}^{j} \right\|_{\mathbf{K}(\mathbf{x}_*^{j})}^2 + \gamma \left\| E_V^{j} \right\|_{H^1(\Gamma_h[\mathbf{x}_*^{j}])}^2 + \left\| \mathbf{d}_{\mathbf{u}}^{j} \right\|_{*,\mathbf{x}_*^j}^2 \right)
$$

for an arbitrary small constant $\gamma > 0$.
In the following, we estimate differential quotients E_V^j in terms of velocity errors e_v^j. The main tool to achieve this is an identity from [1] which relates both in the following way:

$$
E_V^n = \frac{e_x^n - e_x^{n-1}}{\tau} = C \sum_{j=0}^{n-p} \chi_j \left(e_v^{n-j} + d_x^{n-j} \right), \tag{4.24}
$$

with $|\chi_j| \leq \vartheta^j$ for a constant $0 < \vartheta < 1$.
We introduce a new notation to denote Fourier series. In the following we write

$$
\widehat{\mathbf{E}}_{\mathbf{V}}(\xi) = \sum_{j=-\infty}^{\infty} \mathbf{E}_{\mathbf{V}}^j e^{ij\xi}, \qquad \widehat{\mathbf{e}}_{\mathbf{v}}(\xi) = \sum_{j=-\infty}^{\infty} \left(\mathbf{e}_{\mathbf{v}}^j + \mathbf{d}_x^j \right) e^{ij\xi}, \qquad \widehat{\chi}(\xi) = \sum_{j=-\infty}^{\infty} \chi_j e^{ij\xi}, \tag{4.25}
$$

where i denotes the imaginary unit. Here the coefficients are meant to be zero for $j \notin \{1,...,N\}$. By using coefficient comparison and (4.24) we see

$$\widehat{\mathbf{E}}_{\mathbf{V}}(\xi) = \widehat{\chi}(\xi)\widehat{\mathbf{e}}_{\mathbf{v}}(\xi).$$

Now we use Lemma 3.2 and Parseval's theorem to obtain

$$\begin{aligned}\sum_{j=0}^{n} \left\|\mathbf{E}_{\mathbf{V}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 \leq C\sum_{j=0}^{n} \left\|\mathbf{E}_{\mathbf{V}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 = \sum_{j=-\infty}^{\infty} \left\|\mathbf{E}_{\mathbf{V}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 &= \int_{-\pi}^{\pi} \|\widehat{\mathbf{E}}_{\mathbf{V}}(\xi)\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \,\mathrm{d}\xi \\ &= \int_{-\pi}^{\pi} |\widehat{\chi}(\xi)|^2 \,\|\widehat{\mathbf{e}}_{\mathbf{v}}(\xi)\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \,\mathrm{d}\xi \\ &\leq \max_{\xi\in[-\pi,\pi]} |\widehat{\chi}(\xi)|^2 \int_{-\pi}^{\pi} \|\widehat{\mathbf{e}}_{\mathbf{v}}(\xi)\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \,\mathrm{d}\xi \\ &\leq C\sum_{j=0}^{n} \|\mathbf{e}_{\mathbf{v}}^{j}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \leq C\sum_{j=0}^{n} \|\mathbf{e}_{\mathbf{v}}^{j}\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 ,\end{aligned} \tag{4.26}$$

where we used again Parseval's theorem in the last line, combined with the bound of $\widehat{\chi}(\xi)$. The last inequality above then follows from Lemma 3.2. This now finally leaves us with

$$\sum_{j=1}^{n} \rho^j \leq C\sum_{j=1}^{n} \left(\|\mathbf{e}_{\mathbf{u}}^{j}\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \|\mathbf{e}_{\mathbf{x}}^{j}\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \gamma C \left\|\mathbf{e}_{\mathbf{v}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \left\|\mathbf{d}_{\mathbf{u}}^{j}\right\|_{*,\mathbf{x}_*^j}^2 \right), \tag{4.27}$$

with an arbitrary small constant $\gamma > 0$.

4.2.2 Estimation of the left hand side

We now estimate the term with the stiffness matrix from below using the Cauchy-Schwartz and Young's inequalities

$$\begin{aligned}(\mathbf{e}_{\mathbf{u}}^{n} - \eta\mathbf{e}_{\mathbf{u}}^{n-1})^T\mathbf{A}(\mathbf{x}_*^n)\mathbf{e}_{\mathbf{u}}^{n} &\geq \|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \eta \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^n)} \|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{A}(\mathbf{x}_*^n)} \\ &\geq \|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \frac{1}{2}\eta \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \frac{1}{2}\eta \|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 .\end{aligned}$$

With Lemma 3.2 we obtain

$$\begin{aligned}\left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 &= \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 + \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 \\ &= \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 + (\mathbf{e}_{\mathbf{u}}^{n-1})^T \left(\mathbf{A}(\mathbf{x}_*^n) - \mathbf{A}(\mathbf{x}_*^{n-1})\right) \mathbf{e}_{\mathbf{u}}^{n-1} \\ &\leq (1 + c\tau) \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 .\end{aligned} \tag{4.28}$$

Using this norm equivalence, combined with the inequality above we conclude

$$(\mathbf{e}_{\mathbf{u}}^{n} - \eta\mathbf{e}_{\mathbf{u}}^{n-1})^T\mathbf{A}(\mathbf{x}_*^n)\mathbf{e}_{\mathbf{u}}^{n} \geq \left(1 - \frac{1}{2}\eta\right) \|\mathbf{e}_{\mathbf{u}}^{n}\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \left(\frac{1}{2}\eta + c\tau\right) \left\|\mathbf{e}_{\mathbf{u}}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 .$$

For the term with the mass matrix $\mathbf{M}(\mathbf{x}_*^n)$ we use the following notation:

$$\mathbf{E}_\mathbf{u}^n = (\mathbf{e}_\mathbf{u}^{n-p+1}, ..., \mathbf{e}_\mathbf{u}^{n-1}, \mathbf{e}_\mathbf{u}^n),$$

with the past error vectors of $\mathbf{u}$, and the G-generated norm

$$|\mathbf{E}_\mathbf{u}^n|_{G,\mathbf{x}_*^n}^2 = \sum_{i,j=1}^p g_{ij} (\mathbf{e}_\mathbf{u}^{n-p+i})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{u}^{n-p+j}.$$

This family of matrices $\mathbf{E}_\mathbf{u}^k$ is extended for $k < p$ by defining error vectors like $\mathbf{e}_\mathbf{u}^{-1}$ before the initial time as zero vectors. Since G is positive definite we know that the eigenvalues of G are real, positive and can be ordered $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_p$. Therefore we have the following norm equivalence

$$\lambda_1 \sum_{j=1}^p \left\| \mathbf{e}_\mathbf{u}^{n-p+j} \right\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 \leq \sum_{i,j=1}^p g_{ij} (\mathbf{e}_\mathbf{u}^{n-p+i})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{u}^{n-p+j} \leq \lambda_p \sum_{j=1}^p \left\| \mathbf{e}_\mathbf{u}^{n-p+j} \right\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 . \tag{4.29}$$

With the definition of the matrix in Lemma 4.1 we now have

$$\begin{aligned}(\mathbf{e}_\mathbf{u}^n - \eta \mathbf{e}_\mathbf{u}^{n-1})^T \mathbf{M}(\mathbf{x}_*^n) \sum_{j=0}^p \delta_j \mathbf{e}_\mathbf{u}^{n-j} &\geq \sum_{i,j=1}^p g_{ij} (\mathbf{e}_\mathbf{u}^{n-p+i})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{u}^{n-p+j} \\ &\quad - \sum_{i,j=1}^p g_{ij} (\mathbf{e}_\mathbf{u}^{n-p+i-1})^T \mathbf{M}(\mathbf{x}_*^n) \mathbf{e}_\mathbf{u}^{n-p+j-1} \\ &= |\mathbf{E}_\mathbf{u}^n|_{G,\mathbf{x}_*^n}^2 - |\mathbf{E}_\mathbf{u}^{n-1}|_{G,\mathbf{x}_*^n}^2 .\end{aligned}$$

With the same argument as in (4.28) we see by Lemma 3.2 that

$$|\mathbf{E}_\mathbf{u}^{n-1}|_{G,\mathbf{x}_*^n}^2 \leq (1 + c\tau) |\mathbf{E}_\mathbf{u}^{n-1}|_{G,\mathbf{x}_*^{n-1}}^2 ,$$

Combining both estimates from below for the left-hand side of the error equation (4.21) and applying the above norm estimate now yields

$$|\mathbf{E}_\mathbf{u}^n|_{G,\mathbf{x}_*^n}^2 - (1 + c\tau) |\mathbf{E}_\mathbf{u}^{n-1}|_{G,\mathbf{x}_*^{n-1}}^2 + \tau \left(1 - \frac{1}{2}\eta\right) \|\mathbf{e}_\mathbf{u}^n\|_{\mathbf{A}(\mathbf{x}_*^n)}^2 - \tau \left(\frac{1}{2}\eta + c\tau\right) \left\|\mathbf{e}_\mathbf{u}^{n-1}\right\|_{\mathbf{A}(\mathbf{x}_*^{n-1})}^2 \leq \tau \rho^n .$$

We now use this inequality for $j = 1, ..., n$.
For a short notation let T_n denote the left hand side of the above equation. For small enough step size $1 + c\tau > 0$ and therefore we take a weighted sum from the inequalities above to create a telescope sum, that cancels out the terms containing $\mathbf{E}_\mathbf{u}^j$ for $0 < j < n$. With that we deduce

$$\sum_{j=1}^n (1 + c\tau)^{n-j} T_j \leq \tau \sum_{j=1}^n (1 + c\tau)^{n-j} \rho^j \leq \tau \sum_{j=1}^n e^{c\tau(n-j)} \rho^j \leq \tau e^{c\tau n} \sum_{j=1}^n \rho^j .$$

For the left hand side we obtain by expanding the T_j

$$
\begin{aligned}
&|\mathbf{E}_{\mathbf{u}}^n|^2_{G,\mathbf{x}_*^n} - (1+c\tau)^n|\mathbf{E}_{\mathbf{u}}^0|^2_{G,\mathbf{x}_*^0} + \tau(1-\frac{1}{2}\eta)\sum_{j=1}^n (1+c\tau)^{n-j}\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)} \\
&- \tau(\frac{1}{2}\eta + c\tau)\sum_{j=1}^n (1+c\tau)^{n-j}\left\|\mathbf{e}_{\mathbf{u}}^{j-1}\right\|^2_{\mathbf{A}(\mathbf{x}_*^{j-1})} \\
&\geq |\mathbf{E}_{\mathbf{u}}^n|^2_{G,\mathbf{x}_*^n} + \tau\left(1-\frac{1}{2}\eta\right)\|\mathbf{e}_{\mathbf{u}}^n\|^2_{\mathbf{A}(\mathbf{x}_*^n)} + \tau\sum_{j=1}^{n-1}(1+c\tau)^{n-j}(1-\frac{1}{2}\eta - (\frac{1}{2}\eta + c\tau)(1+c\tau))\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)},
\end{aligned}
$$

where we used an index shift on the right sum to combine both. We now take a look at one of the factors of the coefficient from the sum above. Multiplying out yields, for an arbitrary small $\gamma > 0$,

$$
\begin{aligned}
1-\frac{1}{2}\eta - \left(\frac{1}{2}\eta + c\tau\right)(1+c\tau) &= 1-\frac{1}{2}\eta - \frac{1}{2}\eta + c\tau - c\tau\left(\frac{1}{2}\eta + c\tau\right) \\
&= 1-\eta + c\tau\left(1-\frac{1}{2}\eta - c\tau\right) \geq 1-\eta + c\tau \geq \gamma,
\end{aligned}
$$

for a small enough step size τ. We point attention to the fact that $1-\frac{1}{2}\eta \geq 1-\frac{1}{2}\eta(\geq \gamma)$ for τ small enough. This now yields

$$
|\mathbf{E}_{\mathbf{u}}^n|^2_{G,\mathbf{x}_*^n} + \tau\sum_{j=1}^n (1+c\tau)^{n-j}\gamma\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)} \leq e^{c\tau n}|\mathbf{E}_{\mathbf{u}}^0|^2_{G,\mathbf{x}_*^0} + \tau e^{c\tau n}\sum_{j=1}^n \rho^j.
$$

Using the norm equivalence from (4.29), multiplying on both sides with a positive factor then gives for $n\tau \leq T$

$$
\sum_{j=1}^p \left\|\mathbf{e}_{\mathbf{u}}^{n-p+j}\right\|^2_{\mathbf{M}(\mathbf{x}_*^n)} + \tau\sum_{j=1}^n \left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)} \leq C\sum_{j=0}^{p-1}\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{M}(\mathbf{x}_*^j)} + \tau C\sum_{j=1}^n \rho^j.
$$

We plug in the estimate (4.27) we obtained for $\sum_{j=1}^n \rho^j$ to arrive at

$$
\begin{aligned}
\sum_{j=1}^p \left\|\mathbf{e}_{\mathbf{u}}^{n-p+j}\right\|^2_{\mathbf{M}(\mathbf{x}_*^n)} + \tau\sum_{j=1}^n \left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)} &\leq C\sum_{j=0}^{p-1}\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{M}(\mathbf{x}_*^j)} + C\tau\sum_{j=1}^n\left(\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{M}(\mathbf{x}_*^j)} + \gamma\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)} + \left\|\mathbf{e}_{\mathbf{x}}^j\right\|^2_{\mathbf{K}(\mathbf{x}_*^j)}\right) \\
&+ C\tau\sum_{j=1}^n\left(\gamma\left\|\mathbf{e}_{\mathbf{v}}^j\right\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \left\|\mathbf{d}_{\mathbf{u}}^j\right\|^2_{*,\mathbf{x}_*^j}\right).
\end{aligned}
$$

We absorb the terms with $\mathbf{e}_{\mathbf{u}}^j$ in the $\mathbf{A}$-norm on the right-hand side and then use the Gronwall inequality to obtain

$$
\begin{aligned}
\sum_{j=1}^p \left\|\mathbf{e}_{\mathbf{u}}^{n-p+j}\right\|^2_{\mathbf{M}(\mathbf{x}_*^n)} + \tau\sum_{j=1}^n \left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)} &\leq C\sum_{j=0}^{p-1}\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{M}(\mathbf{x}_*^j)} + C\tau\sum_{j=1}^{n-p}\left(\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{M}(\mathbf{x}_*^j)} + \tau\sum_{j=1}^n\left\|\mathbf{e}_{\mathbf{u}}^j\right\|^2_{\mathbf{A}(\mathbf{x}_*^j)}\right) \\
&+ C\tau\sum_{j=1}^n\left(\left\|\mathbf{e}_{\mathbf{x}}^j\right\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \gamma\left\|\mathbf{e}_{\mathbf{v}}^j\right\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \left\|\mathbf{d}_{\mathbf{u}}^j\right\|^2_{*,\mathbf{x}_*^j}\right).
\end{aligned}
$$

Gronwall's inequality now yields

$$\sum_{j=1}^{p} \left\|\mathbf{e}_{\mathbf{u}}^{n-p+j}\right\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 + \tau \sum_{j=1}^{n} \left\|\mathbf{e}_{\mathbf{u}}^{j}\right\|_{\mathbf{A}(\mathbf{x}_*^j)}^2 \le C\tau \sum_{j=1}^{n} \left(\left\|\mathbf{e}_{\mathbf{x}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \gamma \left\|\mathbf{e}_{\mathbf{v}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \left\|\mathbf{d}_{\mathbf{u}}^{j}\right\|_{*,\mathbf{x}_*^j}^2 \right) + C \sum_{j=0}^{p-1} \left\|\mathbf{e}_{\mathbf{u}}^{j}\right\|_{\mathbf{M}(\mathbf{x}_*^j)}^2 . \tag{4.30}$$

Now we add up both sides for $j = 1, ..., n$ and multiply it with τ to obtain furthermore

$$\tau \sum_{j=1}^{n} \left\|\mathbf{e}_{\mathbf{u}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 \le C\tau \sum_{j=1}^{n} \left(\left\|\mathbf{e}_{\mathbf{x}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \gamma \left\|\mathbf{e}_{\mathbf{v}}^{j}\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \left\|\mathbf{d}_{\mathbf{u}}^{j}\right\|_{*,\mathbf{x}_*^j}^2 \right) + C \sum_{j=0}^{p-1} \left\|\mathbf{e}_{\mathbf{u}}^{j}\right\|_{\mathbf{M}(\mathbf{x}_*^j)}^2 . \tag{4.31}$$

4.3 Estimates for the ODE

This part of the proof is mainly taken from [17].
We recall the error equation (4.4) and expand the sum by adding $\delta_j = 0$ for $j > p$ and then write

$$\frac{1}{\tau} \sum_{j=0}^{n} \delta_{n-j} \mathbf{e}_{\mathbf{x}}^{j} = \mathbf{e}_{\mathbf{v}}^{n} - \mathbf{d}_{\mathbf{x}}^{n}.$$

Subtracting the first p terms from the left side then yields

$$\sum_{j=p}^{n} \delta_{n-j} \mathbf{e}_{\mathbf{x}}^{j} = \tau (\mathbf{e}_{\mathbf{v}}^{n} - \widehat{\mathbf{d}_{\mathbf{x}}}^{n}), \tag{4.32}$$

with

$$\widehat{\mathbf{d}_{\mathbf{x}}}^{n} = \mathbf{d}_{\mathbf{x}}^{n} + \frac{1}{\tau} \sum_{j=0}^{p-1} \delta_{n-j} \mathbf{e}_{\mathbf{x}}^{j}.$$

We introduce multiple power series as a technical tool. As coefficients we use $\mathbf{e}_{\mathbf{x}}^{n}$, δ_n and the right hand side of (4.32), which then gives three power series denoted by

$$e(\zeta) := \sum_{n=p}^{\infty} \mathbf{e}_{\mathbf{x}}^{n} \zeta^n, \qquad \delta(\zeta) := \sum_{n=0}^{\infty} \delta_n \zeta^n, \qquad d(\zeta) := \sum_{n=p}^{\infty} \tau (\mathbf{e}_{\mathbf{v}}^{n} - \widehat{\mathbf{d}_{\mathbf{x}}}^{n}) \zeta^n.$$

With the Cauchy product of $e(\zeta)$ and $\delta(\zeta)$ and (4.32) we now arrive at

$$e(\zeta)\delta(\zeta) = \sum_{n=p}^{\infty} \left(\sum_{j=p}^{n} \delta_{n-j} \mathbf{e}_{\mathbf{x}}^{j} \right) \zeta^n = \sum_{n=p}^{\infty} \tau \left(\mathbf{e}_{\mathbf{v}}^{n} - \widehat{\mathbf{d}_{\mathbf{x}}}^{n} \right) \zeta^n = d(\zeta). \tag{4.33}$$

We further introduce the power series of the inverse $\delta(\zeta)$. Since all zeros of $\delta(\zeta)$ are outside of the unit circle, there are bounded coefficients $\mu_n \in \mathbb{R}$, such that

$$\mu(\zeta) := \sum_{n=0}^{\infty} \mu_n \zeta^n = \frac{1}{\delta(\zeta)}.$$

Now (4.33) and again the Cauchy product imply

$$e(\zeta) = e(\zeta)\delta(\zeta)\mu(\zeta) = \left(\sum_{n=p}^{\infty} \tau(\mathbf{e}_\mathbf{v}^n - \widehat{\mathbf{d}_\mathbf{x}}^n\right)\zeta^n\mu(\zeta)$$
$$= \sum_{n=p}^{\infty} \tau\left(\sum_{j=p}^{n} \mu_{n-j}(\mathbf{e}_\mathbf{v}^j - \widehat{\mathbf{d}_\mathbf{x}}^j)\right)\zeta^n.$$

A coefficient comparison then yields directly

$$\mathbf{e}_\mathbf{x}^n = \tau \sum_{j=p}^{n} \mu_{n-j}(\mathbf{e}_\mathbf{v}^j - \widehat{\mathbf{d}_\mathbf{x}}^j).$$

The BDF method is zero-stable for $p \leq 6$, therefore all zeros of $\delta(\zeta)$ are outside the unit circle except for the simple zero at $\zeta = 1$. Therefore the coefficients of $\mu(\zeta)$ are bounded for all $n \in \mathbb{N}$ by a $\mu_n \leq c \leq \mathbb{R}$.
Taking the $\mathbf{K}(\mathbf{x}_*^n)$ norm on both sides now yields

$$\begin{aligned} \|\mathbf{e}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 &\leq C\tau \sum_{j=p}^{n} \left\|\mathbf{e}_\mathbf{v}^j - \widehat{\mathbf{d}_\mathbf{x}}^j\right\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \\ &\leq C\tau \sum_{j=p}^{n} \left\|\mathbf{e}_\mathbf{v}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + C\tau \sum_{j=p}^{n} \left\|\mathbf{d}_\mathbf{x}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + C \sum_{j=0}^{p-1} \left\|\mathbf{e}_\mathbf{x}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}. \end{aligned} \tag{4.34}$$

4.4 Combination of stability estimates

Here we recall the obtained stability results, transform them using the discrete Gronwall Lemma and then combine them. We recall estimate from the velocity law (4.20) :

$$\|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \leq C\left(\sum_{i=n-p}^{n} \left\|\mathbf{e}_\mathbf{x}^i\right\|_{\mathbf{K}(\mathbf{x}_*^i)}^2 + \sum_{i=1}^{p} \left\|\mathbf{e}_\mathbf{u}^{n-i}\right\|_{\mathbf{K}(\mathbf{x}_*^{n-i})}^2 + \|\mathbf{d}_\mathbf{v}^n\|_{*,\mathbf{x}_*^n}^2\right). \tag{4.35}$$

and insert the stability estimate from the ODE (4.34), for $i = n-p, ..., n$, to obtain

$$\begin{aligned} \|\mathbf{e}_\mathbf{v}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 \leq{} & C \sum_{i=n-p}^{n} \left(C\tau \sum_{j=p}^{i} \left\|\mathbf{e}_\mathbf{v}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + C\tau \sum_{j=p}^{i} \left\|\mathbf{d}_\mathbf{x}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + C \sum_{j=0}^{p-1} \left\|\mathbf{e}_\mathbf{x}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}\right) \\ & + \sum_{i=1}^{p} \left\|\mathbf{e}_\mathbf{u}^{n-i}\right\|_{\mathbf{K}(\mathbf{x}_*^{n-i})}^2 + \|\mathbf{d}_\mathbf{v}^n\|_{*,\mathbf{x}_*^n}^2 \\ \leq{} & C\tau \sum_{j=p}^{n} \left\|\mathbf{e}_\mathbf{v}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + C\tau \sum_{j=p}^{n} \left\|\mathbf{d}_\mathbf{x}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 \\ & + C \sum_{j=0}^{p-1} \left\|\mathbf{e}_\mathbf{x}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + C \sum_{i=1}^{p} \left\|\mathbf{e}_\mathbf{u}^{n-i}\right\|_{\mathbf{K}[\mathbf{x}_*^{n-i}]}^2 + C\, \|\mathbf{d}_\mathbf{v}^n\|_{*,\mathbf{x}_*^n}^2 . \end{aligned}$$

Addition of (4.34) yields

$$\begin{aligned}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} \le C\tau \sum_{j=p}^{n} \Big(\big\|\mathbf{e}_\mathbf{v}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} \Big) + C\tau \sum_{j=p}^{n} \big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} \\ + C\sum_{j=0}^{p-1} \big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + C\sum_{i=1}^{p} \big\|\mathbf{e}_\mathbf{u}^{n-i}\big\|^2_{\mathbf{K}[\mathbf{x}_*^{n-i}]} + C\,\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n},\end{aligned}$$

where we added the $\mathbf{K}(\mathbf{x}_*^i)$-norm of the position errors on the right-hand side, so that we are in the position of the discrete Gronwall Lemma. For $\tau < \tau_0$ we can absorb $C\tau\Big(\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)}\Big)$ on the right-hand side. Using the discrete Gronwall inequality then yields

$$\begin{aligned}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} &\le C\tau \sum_{k=p}^{n-1} \Bigg(\tau\sum_{j=p}^{k} \big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \sum_{j=0}^{p-1} \big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \sum_{i=1}^{p} \big\|\mathbf{e}_\mathbf{u}^{k-i}\big\|^2_{\mathbf{K}[\mathbf{x}_*^{k-i}]} + \big\|\mathbf{d}_\mathbf{v}^k\big\|^2_{*,\mathbf{x}_*^n} \Bigg) \\ &\quad + C\tau \sum_{j=p}^{n} \big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + C\sum_{j=0}^{p-1} \big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + C\sum_{i=1}^{p} \big\|\mathbf{e}_\mathbf{u}^{n-i}\big\|^2_{\mathbf{K}[\mathbf{x}_*^{n-i}]} + C\,\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n} \\ &\le C\tau \sum_{j=p}^{n} \big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + C\sum_{j=0}^{p-1} \big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} \\ &\quad + C\tau \sum_{j=0}^{n-1} \big\|\mathbf{e}_\mathbf{u}^j\big\|^2_{\mathbf{K}[\mathbf{x}_*^j]} + C\tau \sum_{j=p}^{n-1} \big\|\mathbf{d}_\mathbf{v}^j\big\|^2_{*,\mathbf{x}_*^j} + C\,\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n},\end{aligned}$$

where we used that $e^{C\tau n} \le e^{CT}$ is bounded independent of h and τ. We insert (4.31), to bound the errors in u and obtain

$$\begin{aligned}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} \le C\tau \sum_{j=p}^{n} \big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + C\sum_{j=0}^{p-1} \big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + C\tau \sum_{j=p}^{n-1} \big\|\mathbf{d}_\mathbf{v}^j\big\|^2_{*,\mathbf{x}_*^j} + C\,\|\mathbf{d}_\mathbf{v}^n\|^2_{*,\mathbf{x}_*^n} \\ + C\tau \sum_{j=1}^{n-1} \Big(\big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{e}_\mathbf{v}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{d}_\mathbf{u}^j\big\|^2_{*,\mathbf{x}_*^j} \Big) + C\sum_{j=0}^{p-1} \big\|\mathbf{e}_\mathbf{u}^j\big\|^2_{\mathbf{M}(\mathbf{x}_*^j)}\end{aligned}$$

We use Gronwall's inequality again to arrive at

$$\begin{aligned}\|\mathbf{e}_\mathbf{v}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_\mathbf{x}^n\|^2_{\mathbf{K}(\mathbf{x}_*^n)} \le C\tau \sum_{j=p}^{n} \Big(\big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{d}_\mathbf{v}^j\big\|^2_{*,\mathbf{x}_*^j} + \big\|\mathbf{d}_\mathbf{u}^j\big\|^2_{*,\mathbf{x}_*^j} \Big) \\ + C\sum_{j=0}^{p-1} \Big(\big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{e}_\mathbf{u}^j\big\|^2_{\mathbf{M}(\mathbf{x}_*^j)} \Big)\end{aligned} \tag{4.36}$$

We now use the estimates above and insert them into (4.30) to obtain

$$\begin{aligned}\sum_{j=1}^{p} \big\|\mathbf{e}_\mathbf{u}^{n-p+j}\big\|^2_{\mathbf{M}(\mathbf{x}_*^n)} + \tau\sum_{j=1}^{n} \big\|\mathbf{e}_\mathbf{u}^j\big\|^2_{\mathbf{A}(\mathbf{x}_*^j)} \le C\tau \sum_{j=p}^{n} \Big(\big\|\mathbf{d}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{d}_\mathbf{v}^j\big\|^2_{*,\mathbf{x}_*^j} + \big\|\mathbf{d}_\mathbf{u}^j\big\|^2_{*,\mathbf{x}_*^j} \Big) \\ + C\sum_{j=0}^{p-1} \Big(\big\|\mathbf{e}_\mathbf{x}^j\big\|^2_{\mathbf{K}(\mathbf{x}_*^j)} + \big\|\mathbf{e}_\mathbf{u}^j\big\|^2_{\mathbf{M}(\mathbf{x}_*^j)} \Big).\end{aligned} \tag{4.37}$$

Summation of the stability estimates (4.36) and (4.37) finally allows us to conclude

$$\|\mathbf{e}_{\mathbf{x}}^{n}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \|\mathbf{e}_{\mathbf{v}}^{n}\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \sum_{j=1}^{p} \left\|\mathbf{e}_{\mathbf{u}}^{n-p+j}\right\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 + \tau \sum_{j=1}^{n} \left\|\mathbf{e}_{\mathbf{u}}^{j}\right\|_{\mathbf{A}(\mathbf{x}_*^j)}^2$$
$$\leq C\tau \sum_{j=p}^{n} \Big(\|\mathbf{d}_{\mathbf{x}}^{j}\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \|\mathbf{d}_{\mathbf{v}}^{j}\|_{*,\mathbf{x}_*^j}^2 + \|\mathbf{d}_{\mathbf{u}}^{j}\|_{*,\mathbf{x}_*^j}^2 \Big) + C \sum_{j=0}^{p-1} \Big(\|\mathbf{e}_{\mathbf{x}}^{j}\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \|\mathbf{e}_{\mathbf{u}}^{j}\|_{\mathbf{M}(\mathbf{x}_*^j)}^2 \Big),$$

which is the stated result. □

5 Geometric estimates

In this section we collect results that will allow us to estimate the consistency errors. Now the geometry, that did not enter in the stability analysis, will enter into the estimation of the bounds of the defects (4.1).

Remark 3. *In the following, if used as an integrand, we write ∇_Γ for the surface gradient of the exact surface and ∇_{Γ_h} for the piecewise surface gradient on a discrete surface. It is then clear by the domain of the integral which surface is meant by the gradient. The same convention is used for the exact and discrete normal vectors $\nu_{\Gamma(X)}$ and $\nu_{\Gamma_h(X_h)}$.*

5.1 Approximation results

Since we compare functions on different surfaces, we need to make sure that the lifting process of a function does not create an error that decreases the order of convergence. However, this is guaranteed by a result from [6] and [5].
We collect norm equivalences of the lift in the following Lemma.

Lemma 5.1. *Let η_h and its lift be as before. Then the L^p- and $W^{1,p}$-norms on the discrete and continuous surfaces are equivalent for $1 \leq p \leq \infty$, uniformly in the mesh size $h \leq h_0$ (for sufficiently small $h_0 > 0$) and in $t \in [0,T]$.*

As a special case of the above Lemma for the spaces $L^2(\Gamma(X))$ and $H^1(\Gamma(X))$ there is a constant C, for $h \leq h_0$ and $t \in [0,T]$, such that

$$C^{-1} \|\eta_h\|_{L^2(\Gamma_h(X_h^*))} \leq \left\|\eta_h^l\right\|_{L^2(\Gamma(X))} \leq C \|\eta_h\|_{L^2(\Gamma_h(X_h^*))},$$
$$C^{-1} \|\eta_h\|_{H^1(\Gamma_h(X_h^*))} \leq \left\|\eta_h^l\right\|_{H^1(\Gamma(X))} \leq C \|\eta_h\|_{H^1(\Gamma_h(X_h^*))}.$$

In the next Lemma from [20], the distance function and the difference of the normal vectors on the discrete and continuous surfaces are estimated.

Lemma 5.2. *Let $\Gamma(X(.,t))$ and $\Gamma_h(X_h^*(.,t))$ be as before. Then, for $h < h_0$ the following estimates hold:*

$$\|d\|_{L^\infty(\Gamma_h(X_h^*))} \leq Ch^{k+1}, \quad \left\|\nu_{\Gamma(X)} - \nu_{\Gamma_h(X_h^*)}^l\right\|_{L^\infty(\Gamma(X))} \leq Ch^k,$$

with constants $C > 0$ independent of $h < h_0$ and $t \in [0,T]$.

5.2 Bilinear Forms

As in the literature, we define bilinear forms (e.g. see [16]) on the Sobolev spaces and the finite element spaces as in [7]. We denote again the exact surface with $\Gamma(X)$, the interpolated surface with $\Gamma(X_h^*)$ and the exact and discrete velocities with v and v_h^*. Let $z, \varphi \in H^1(\Gamma(X))$, and $Z_h, \phi_h \in S_h(\Gamma_h((X_h^*)))$. Then we define on the exact surface $\Gamma(X)$

$$\begin{aligned} m(X; z, \varphi) &= \int_{\Gamma(X)} z\varphi, & m(X_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} Z_h \phi_h, \\ a(X; z, \varphi) &= \int_{\Gamma(X)} \nabla_\Gamma z \cdot \nabla_\Gamma \varphi, & a(X_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} \nabla_{\Gamma_h} Z_h \cdot \nabla_{\Gamma_h} \phi_h, \\ q(X; v; z, \varphi) &= \int_{\Gamma(X)} (\nabla_\Gamma \cdot v) z\varphi, & q(X_h^*; v_h^*; Z_h, \phi_h) &= \int_{\Gamma(X_h^*)} (\nabla_{\Gamma_h} \cdot v_h^*) Z_h \phi_h. \end{aligned}$$

5.3 Interpolations

For a function $u : \mathcal{G}_T \to \mathbb{R}$ we define its finite element interpolation $\widetilde{I}_h u \in S_h[\mathbf{x}_*^n]$ by

$$\widetilde{I}_h u = \sum_{j=1}^N u_j \phi[\mathbf{x}_*^n],$$

where the u_j are the elements of the nodal vector $\mathbf{u} \in \mathbb{R}^N$, which we recall as

$$u_j(t) = u(X(q_j, t)).$$

The interpolation of three dimensional functions $v : \mathcal{G}_T \to \mathbb{R}^3$ is defined in the same way.

Remark 4. *Since the nodal vector is used to define the interpolation, we can immediately expand the definition to an arbitrary discrete surface. In this case, we define, if* $\mathbf{x}$ *is a nodal vector*

$$\widetilde{I}_h u[\mathbf{x}] = \sum_{j=1}^N u_j \phi_j[\mathbf{x}].$$

Since, in the following, it is always clear from the context on what domain the function is, we drop the nodal vector $\mathbf{x}$ *as an argument.*
In other words, $\widetilde{I}_h u$ *denotes the finite element function that corresponds to the nodal vector* $\mathbf{u}$*, on an arbitrary discrete surface corresponding to a nodal vector* $\mathbf{x}$*.*

We define, as an approximation to u on the exact surface $\Gamma(X)$,

$$I_h u = (\widetilde{I}_h u)^l.$$

For the further consistency analysis we need to control the interpolation error of functions on surfaces. The following result is given by [7, Proposition 2.7].

Lemma 5.3. *There exists a constant* $c > 0$ *independent of* $h < h_0$*, with a sufficiently small* $h_0 > 0$*, and* t *such that for* $u(\cdot, t) \in H^{k+1}(\Gamma(t))$*, for* $t \in [0, T]$*,*

$$\begin{aligned} \|u - I_h u\|_{L^2(\Gamma(X))} &\le ch^{k+1} \|u\|_{H^{k+1}(\Gamma(X))}, \\ \|\nabla_\Gamma (u - I_h u)\|_{L^2(\Gamma(X))} &\le ch^k \|u\|_{H^{k+1}(\Gamma(X))}. \end{aligned}$$

As done in [7] we define a discrete velocity of the exact surface. We define $\hat{v}_h : \mathcal{G}_T \to \mathbb{R}^3$ by

$$\widehat{v}_h((X_h^*)^l(\cdot,t),t) = \frac{\mathrm{d}}{\mathrm{dt}}(X_h^*)^l(\cdot,t).$$

This yields a discrete material derivative for functions that operate on the exact surface $\Gamma(X)$, which we define by

$$\widehat{\partial}_h^\bullet u(x,t) = \partial_t u(x,t) + \nabla u(x,t)\cdot \widehat{v}_h(x,t), \qquad \text{for } x \in \Gamma(X),\ t \in [0,T], \tag{5.1}$$

provided the quantities on the right-hand side exist. Together with the interpolation error this immediately yields, as in Corollary 5.7 in [7]

$$\left\|\partial^\bullet u(x,t) - \widehat{\partial}_h^\bullet u(x,t)\right\|_{L^2(\Gamma(X))} = \|\nabla u\cdot(\widehat{v}_h - v)\|_{L^2(\Gamma(X))} \tag{5.2}$$

$$\le C\,\|u\|_{H^1(\Gamma(X))}\left(\left\|\widehat{v}_h - \widetilde{I}_h v\right\|_{L^2(\Gamma(X))} + \left\|\widetilde{I}_h v - v\right\|_{L^2(\Gamma(X))}\right) \tag{5.3}$$

$$\le C h^{k+1}\,\|u\|_{H^1(\Gamma(X))}\,\|v\|_{L^2(\Gamma(X))} \tag{5.4}$$

We recall geometric estimates from [13, Lemma 5.6], that are essential for the consistency analysis later.

Lemma 5.4. (Geometric estimates)
Let X be a surface and $\mathbf{x}^*$ a nodal vector with exact nodal values such that $\Gamma_h(\mathbf{x})$ is a piecewise polynomial discretization of order k of $\Gamma(X)$ as before. For arbitrary $Z_h, \varphi_h \in S_h(\mathbf{x}^*)$ we then have for $h \le h_0$, with $h_0 > 0$ sufficiently small

$$|m(X; Z_h^l, \varphi_h^l) - m(\mathbf{x}^*; Z_h, \varphi_h)| \le ch^{k+1}\left\|Z_h^l\right\|_{L^2(\Gamma(X))}\left\|\varphi_h^l\right\|_{L^2(\Gamma(X))} \tag{5.5}$$

$$|a(X; Z_h^l, \varphi_h^l) - a(\mathbf{x}^*; Z_h, \varphi_h)| \le ch^{k+1}\left\|\nabla_{\Gamma_h} Z_h^l\right\|_{L^2(\Gamma(X))}\left\|\nabla_{\Gamma_h}\varphi_h^l\right\|_{L^2(\Gamma(X))} \tag{5.6}$$

$$|q(X; \hat{v}_h; Z_h^l, \varphi_h^l) - q(\mathbf{x}^*; v_h^*; Z_h, \varphi_h)| \le ch^{k+1}\left\|Z_h^l\right\|_{L^2(\Gamma(X))}\left\|\varphi_h^l\right\|_{L^2(\Gamma(X))}. \tag{5.7}$$

The constant c on the right hand side is not depending on h.

6 Defect bounds

Now we have all the necessary tools to estimate the defects defined in (4.1).
A combination with the stability result (4.7) will then allow us to obtain the full convergence result. We formulate the defect bounds in the following Lemma.

Lemma 6.1. *Consider the full discretized problem (2.19-2.21) with the conditions conditions of Theorem 2.1. Then there exist $h_0, \tau_0 > 0$, such that for all $h < h_0, \tau < \tau_0$ the following consistency result holds*

$$\|\mathbf{d}_\mathbf{u}^n\|_{\star,\mathbf{x}_*^n} = \|d_u^n\|_{H_h^{-1}(\Gamma_h[\mathbf{x}_*^n])} \le C(\tau^p + h^k),$$

$$\|\mathbf{d}_\mathbf{v}^n\|_{\star,\mathbf{x}_*^n} = \|d_v^n\|_{H_h^{-1}(\Gamma_h[\mathbf{x}_*^n])} \le C(\tau^p + h^k),$$

$$\|\mathbf{d}_\mathbf{x}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} = \|d_x^n\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \le C\tau^p.$$

Proof. All finite element functions $\psi_h \in (S_h[\mathbf{x}_*^n])^3$ have exactly one corresponding $\bar{\psi}_h \in S_h[\widetilde{\mathbf{x}}_*^n]$ with the same nodal vector. We therefore drop the bar above and instead always write ψ_h, which is then the finite element function on the same domain as the integral.

6.1 Defect of v

By definition of the defects (4.1) we know that the corresponding finite element function $d_v^n \in S_h[\widetilde{\mathbf{x}}_*^n]$ fulfills the following problem:
For all finite element functions $\psi_h \in (S_h[\mathbf{x}_*^n])^3$ we have

$$\begin{aligned} m(X_h^*(t_n); d_v^n, \psi_h) = {}& m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) + \alpha a(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) \\ & + \beta a(\widetilde{X}_h^*(t_n); \widetilde{I}_h X(\cdot, t_n), \psi_h) - m(\widetilde{X}_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h) \end{aligned}$$

Consider in comparison the rearranged weak form (2.13) at the time t_n

$$\begin{aligned} 0 = & - m(X(t_n); v(\cdot, t_n), \psi_h^l) - \alpha a(X(t_n); v(\cdot, t_n), \psi_h^l) \\ & - \beta a(X(t_n); X(\cdot, t_n), \psi_h^l) + m(X(t_n); g(u, \nabla_{\Gamma_h} u)\nu_\Gamma, \psi_h^l). \end{aligned}$$

Naturally every summand on the right-hand side of both equations has a corresponding term on the other equation. We add both of those equations and rewrite the differences of each of those terms. This allows us to obtain

$$\begin{aligned} m(X_h^*(t_n); d_v^n, \psi_h) = {}& m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - m(X(t_n); v(\cdot, t_n), \psi_h^l) \\ & + \alpha a(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - \alpha a(X(t_n); v(\cdot, t_n), \psi_h^l) \\ & + \beta a(\widetilde{X}_h^*(t_n); \widetilde{I}_h X(\cdot, t_n), \psi_h) - \beta a(X(t_n); X(\cdot, t_n), \psi_h^l) \\ & - m(\widetilde{X}_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h) + m(X(t_n); g(u, \nabla_{\Gamma_h} u)\nu_\Gamma, \psi_h^l). \end{aligned}$$

Now we want to separate the defect into a purely spatial part which was estimated in [16] and a part depending on the extrapolation. This now yields for the first summands

$$\begin{aligned} & m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - m(X(t_n); v(\cdot, t_n), \psi_h^l) \\ & = m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) \\ & + m(X_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - m(X(t_n); v(\cdot, t_n), \psi_h^l). \end{aligned}$$

Adding such intermediate terms for all summands and using the defect $d_{h,v}$ from Section 8 of [16], which collects all purely spatial terms from above we obtain

$$\begin{aligned} m(X_h^*(t_n); d_v^n, \psi) = {}& m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) \\ & + \alpha(a(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h) - a(X_h^*(t_n); \widetilde{I}_h v(\cdot, t_n), \psi_h)) \\ & + \beta(a(\widetilde{X}_h^*(t_n); \widetilde{I}_h X(\cdot, t_n), \psi_h) - a(X_h^*(t_n); \widetilde{I}_h X(\cdot, t_n), \psi_h)) \\ & - m(\widetilde{X}_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h) + m(X_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h) \end{aligned} \tag{6.1}$$

$$+ m(X_h^*(t_n); d_{h,v}(\cdot, t_n), \psi_h). \tag{6.2}$$

Now we estimate the spatial defect (6.2) in the last line and the other terms separately. We start with the time related parts (6.1).

6.1.1 Time related defect of v

Now we return to the estimates of the time related parts of the defect. We start with the right hand side of the first line of (6.1) and use, a similar argument as in Lemma 3.1. With an intermediate surface $\Gamma_h^\theta = \Gamma[\mathbf{x}_*^n + \theta(\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n)]$ and the fundamental theorem of calculus we obtain

$$\begin{aligned}
&|m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot,t_n), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h v(\cdot,t_n), \psi_h)| \\
&= \left| \int_0^1 \int_{\Gamma_h^\theta} \psi_h^\theta (\nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta}) v_{*,h}^{n,\theta} \mathrm{d}\theta \right| \\
&\leq \int_0^1 \left\|\psi_h^\theta\right\|_{L^2(\Gamma_h^\theta)} \left\|\nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \left\|v_{*,h}^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)} \mathrm{d}\theta,
\end{aligned}$$

where we used an $L^2 - L^2 - L^\infty$ estimate in the last step. Then with Lemma 3.1 and the triangle inequality we deduce

$$\begin{aligned}
&\int_0^1 \left\|\psi_h^\theta\right\|_{L^2(\Gamma_h^\theta)} \left\|\nabla_{\Gamma_h^\theta} \cdot \widetilde{e}_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \left\|v_{*,h}^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\leq \left\|\psi_h^0\right\|_{L^2(\Gamma_h^0)} \left\|\widetilde{e}_x^{n,0}\right\|_{H^1(\Gamma_h^0)} \left\|v_{*,h}^{n,0}\right\|_{L^\infty(\Gamma_h^0)} \\
&\leq c \left\|\psi_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \\
&\cdot \Big(\left\|v_*(\cdot,t_n)\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])} + \left\|v_{*,h}(\cdot,t_n) - v_*(\cdot,t_n)\right\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])} \Big).
\end{aligned}$$

The $v_*(\cdot,t_n)$ is to be understood as the exact velocity on the exact surface lifted on the discrete surface with exact nodes $\Gamma_h(X_h^*(t_n))$. However, since the lift does not change the values in the nodes it is clear that $v_{*,h}$ and v_* coincide in the nodes $\mathbf{x}_*^n$. We use an interpolation estimate ([5] Proposition 2.7) for the last term to obtain

$$\begin{aligned}
&|m(\widetilde{X}_h^*(t_n); \widetilde{I}_h v(\cdot,t_n), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h v(\cdot,t_n), \psi_h)| \\
&\leq C \left\|\psi_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} (1 + ch^2) \left\|v_*(\cdot,t_n)\right\|_{W^{q,\infty}(\Gamma_h[\mathbf{x}_*^n])} \\
&\leq C \left\|\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)} \left\|\psi_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} \\
&\leq C\tau^p \left\|\psi_h\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}.
\end{aligned}$$

The other terms of (6.1) are all estimated in the same way, the calculations and the corresponding estimates are the rest of this section.

The second line of (6.1) is again estimated by Lemma 3.1 to get with an $L^\infty - L^2 - L^2$ estimate

$$\begin{aligned}
&\left|a(\widetilde{X}_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - a(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h)\right| \\
&= \left|\int_0^1 \int_{\Gamma_h^\theta} (\nabla_{\Gamma_h^\theta} v_{*,h}^{n,\theta})(D_{\Gamma_h^\theta}\widetilde{e}_h^{n,\theta})(\nabla_{\Gamma_h^\theta}\psi_h^\theta)\mathrm{d}\theta\right| \\
&\le \int_0^1 \left\|\nabla_{\Gamma_h^\theta} v_{*,h}^{n,\theta}\right\|_{L^\infty(\Gamma_h^\theta)} \left\|D_{\Gamma_h^\theta}\widetilde{e}_x^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} \left\|\nabla_{\Gamma_h^\theta}\psi_h^\theta\right\|_{L^2(\Gamma_h^\theta)} \mathrm{d}\theta \\
&\le C\left(\left\|\nabla_{\Gamma_h^0} v_{*,h}^{n,0}\right\|_{L^\infty(\Gamma_h^0)} \left\|D_{\Gamma_h^0}\widetilde{e}_x^{n,0}\right\|_{L^2(\Gamma_h^0)} \left\|\nabla_{\Gamma_h^0}\psi_h^0\right\|_{L^2(\Gamma_h^0)}\right) \\
&\le C\left(\left\|v_{*,h}^n\right\|_{W^{1,\infty}(\Gamma[\mathbf{x}_*^n])} \left\|\widetilde{e}_x^n\right\|_{H^1(\Gamma[\mathbf{x}_*^n])} \left\|\psi_h\right\|_{H^1(\Gamma[\mathbf{x}_*^n])}\right) \\
&\le C\left\|\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n\right\|_{\mathbf{K}(\mathbf{x}_*^n)} \left\|\psi_h\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])} \\
&\le C\tau^p \left\|\psi_h\right\|_{H^1(\Gamma_h[\mathbf{x}_*^n])},
\end{aligned}$$

where we used the same steps as in the term before. The estimate in the last step is estimated with Taylor's theorem, see [1].
The last term of the time related defect is bounded by

$$\begin{aligned}
&\left|m(\widetilde{X}_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h) - m(X_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h)\right| \\
&\le \left|m(\widetilde{X}_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h} - \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h)\right| \\
&+ \left|m(\widetilde{X}_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h)\right| \\
&+ \left|m(X_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}) - g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}, \psi_h)\right| \\
&\le Ch^k \left\|\varphi_h\right\|_{L^2(\Gamma[\mathbf{x}_*^n])} \\
&+ \left|m(\widetilde{X}_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h)\right| \\
&+ Ch^k \left\|\varphi_h\right\|_{L^2(\Gamma[\mathbf{x}_*^n])},
\end{aligned}$$

where we used Cauchy Schwartz and an interpolation error estimate in the last inequality. Since both finite element functions coincide in the last remaining summand, we are in the position to use Lemma 3.3, which yields

$$\begin{aligned}
&\left|m(\widetilde{X}_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h) - m(X_h^*(t_n); \widetilde{I}_h(g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h}), \psi_h)\right| \\
&\qquad \le C\tau^p \left\|\psi_h\right\|_{\mathbf{M}(\mathbf{x}_*^n)}.
\end{aligned}$$

6.1.2 Spatial defect of v

For the spatial term we obtain

$$\begin{aligned}
m(X_h^*(t_n); d_{h,v}(\cdot,t_n),\psi_h) &= m(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - m(X(t_n); v(\cdot,t_n),\psi_h^l) \\
&+ \alpha(a(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - a(X(t_n); v(\cdot,t_n),\psi_h^l)) \\
&+ \beta(a(X_h^*(t_n);\widetilde{I}_h X(\cdot,t_n),\psi_h) - a(X(t_n); X(\cdot,t_n),\psi_h^l)) \\
&- m(X_h^*(t_n); g(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h},\psi_h) + m(X(t_n); g(u,\nabla_\Gamma u)\nu_\Gamma,\psi_h^l).
\end{aligned}$$

These pairs are now bounded pairwise. For the first difference we obtain, by using Lemma (5.4) and the Cauchy-Schwartz inequality

$$\begin{aligned}
&|m(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - m(X(t_n);v(\cdot,t_n),\psi_h^l)|\\
&\le |m(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - m(X(t_n);I_h v(\cdot,t_n),\psi_h^l)|\\
&+|m(X(t_n);I_h v(\cdot,t_n) - v(\cdot,t_n),\psi_h^l)|\\
&\le ch^{k+1}\left\|\psi_h^l\right\|_{L^2(\Gamma(X))}\|v(\cdot,t_n)\|_{L^2(\Gamma(X))}.
\end{aligned}$$

For the estimation of the second summand we use the interpolation bound from Lemma 5.3. We estimate the second difference with the exact same techniques by

$$\begin{aligned}
&|a(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - a(X(t_n);v(\cdot,t_n),\psi_h^l)|\\
&\le |a(X_h^*(t_n);\widetilde{I}_h v(\cdot,t_n),\psi_h) - a(X(t_n);I_h v(\cdot,t_n),\psi_h^l)|\\
&+|a(X(t_n);I_h v(\cdot,t_n) - v(\cdot,t_n),\psi_h^l)|\\
&\le ch^{k}\left\|\nabla_{\Gamma(X)}\psi_h^l\right\|_{L^2(\Gamma(X))}\left\|\nabla_{\Gamma(X)}v(\cdot,t_n)\right\|_{L^2(\Gamma(X))}.
\end{aligned}$$

Notice the order of the interpolation error is k instead of $k+1$, hence the order of the whole estimate drops by one to k.
For the difference of the nonlinear parts we estimate

$$\begin{aligned}
&|m(X_h^*(t_n);g(\widetilde{I}_h u,\nabla_{\Gamma_h}\widetilde{I}_h u)\nu_{\Gamma_h},\psi_h) - m(X(t_n);g(u,\nabla_\Gamma u)\nu_\Gamma,\psi_h^l)|\\
&\le |m(X_h^*(t_n);(g(\widetilde{I}_h u,\nabla_{\Gamma_h}\widetilde{I}_h u) - g(u,\nabla_\Gamma u)^{-l})\nu_{\Gamma_h},\psi_h)|\\
&+|m(X_h^*(t_n);g(u,\nabla_\Gamma u)^{-l}\nu_{\Gamma_h},\psi_h) - m(X(t_n);g(u,\nabla_\Gamma u)\nu_\Gamma,\psi_h^l)|\\
&\le |m(X_h^*(t_n);(g(\widetilde{I}_h u,\nabla_{\Gamma_h}\widetilde{I}_h u) - g(u,\nabla_\Gamma u)^{-l})\nu_{\Gamma_h},\psi_h)|\\
&+|m(X_h^*(t_n);g(u,\nabla_\Gamma u)^{-l}(\nu_{\Gamma_h}-\nu_\Gamma^{-l}),\psi_h)|\\
&+|m(X_h^*(t_n);g(u,\nabla_\Gamma u)^{-l}\nu_\Gamma^{-l},\psi_h) - m(X(t_n);g(u,\nabla_\Gamma u)\nu_\Gamma,\psi_h^l)|.
\end{aligned}$$

The first summand can now be bounded by a $L^2 - L^\infty - L^2$ estimate and using the local Lipschitz continuity of g

$$\begin{aligned}
&|m(X_h^*(t_n);(g(\widetilde{I}_h u,\nabla_{\Gamma_h}\widetilde{I}_h u) - g(u^{-l},(\nabla_\Gamma u)^{-l}))\nu_{\Gamma_h},\psi_h)|\\
&\le \left\|g(\widetilde{I}_h u,\nabla_{\Gamma_h}\widetilde{I}_h u) - g(u^{-l},(\nabla_\Gamma u)^{-l})\right\|_{L^2[\mathbf{x}_*^n]}\|\nu_{\Gamma_h}\|_{L^\infty(\Gamma_h[\mathbf{x}_*^n])}\|\psi_h\|_{L^2[\mathbf{x}_*^n]}\\
&\le C\left(\left\|\widetilde{I}_h u - u^{-l}\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} + \left\|\nabla_{\Gamma_h}\widetilde{I}_h u - (\nabla_\Gamma u)^{-l}\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\right)\|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\\
&\le C\left(\left\|(I_h u - u)^{-l}\right\|_{L^2[\mathbf{x}_*^n]} + \left\|\nabla_{\Gamma_h}(I_h u - u)^{-l}\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])} + \left\|\nabla_{\Gamma_h}u^{-l} - (\nabla_\Gamma u)^{-l}\right\|_{L^2[\mathbf{x}_*^n]}\right)\|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\\
&\le C\left(h^k + \left\|\nabla_{\Gamma_h}u^{-l} - (\nabla_\Gamma u)^{-l}\right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}\right)\|\psi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])},
\end{aligned}$$

where we used the Lemma 5.1and 5.3 in the last inequality.
With the norm equivalence from Lemma (5.1) we then obtain

$$|m(X_h^*(t_n);(g(\widetilde{I}_h u,\nabla_{\Gamma_h}\widetilde{I}_h u) - g(u^{-l},(\nabla_\Gamma u)^{-l}))\nu_{\Gamma_h},\psi_h)| \le ch^k\left\|\psi_h^l\right\|_{L^2(\Gamma(X))},$$

since we have

$$\left\|(\nabla_{\Gamma_h} u^{-l})^l - \nabla_\Gamma u\right\|_{L^2(\Gamma(X))} \leq ch^k. \tag{6.3}$$

Furthermore, we use an $L^2 - L^\infty - L^2$ estimate and Lemma 5.2 to obtain

$$\begin{aligned}
&|m(X_h^*(t_n); g(u, \nabla_\Gamma u)^{-l}(\nu_{\Gamma_h} - \nu_\Gamma^{-l}), \psi_h)| \\
\leq &\left\|g(u, \nabla_\Gamma u)^{-l}\right\|_{L^2(\Gamma[\mathbf{x}_*^n])} \left\|\nu_{\Gamma_h} - \nu_\Gamma^{-l}\right\|_{L^\infty(\Gamma[\mathbf{x}_*^n])} \|\psi_h\|_{L^2(\Gamma[\mathbf{x}_*^n])} \\
\leq &ch^k \left\|\psi_h^l\right\|_{L^2(\Gamma(X))},
\end{aligned}$$

where we also used the norm equivalence for the lifted functions in the last step.
The final summand is now estimated directly with Lemma 5.4

$$\begin{aligned}
|m(X_h^*(t_n); g(u, \nabla_\Gamma u)^{-l}\nu_\Gamma^{-l}, \psi_h) - m(X(t_n); g(u, \nabla_\Gamma u)\nu_\Gamma, \psi_h^l)| \\
\leq ch^{k+1} \|g(u, \nabla_\Gamma u)\nu_\Gamma\|_{L^2(\Gamma(X))} \left\|\psi_h^l\right\|_{L^2(\Gamma(X))}.
\end{aligned}$$

This now yields for the complete spatial defect

$$|m(X_h^*(t_n); d_{h,v}(\cdot, t_n), \psi_h)| \leq ch^k \left\|\psi_h^l\right\|_{L^2(\Gamma(X))}.$$

6.2 The defect of u

We continue by estimating the defect d_u^n. The definition (4.1) reads in bilinear notation as

$$\begin{aligned}
m(X_h^*(t_n); d_u^n, \varphi_h) = &\frac{1}{\tau}\sum_{j=0}^{p} \delta_j m(\widetilde{X}_h^*(t_{n-j}); \widetilde{I}_h u(\cdot, t_{n-j}), \varphi_h) + a(\widetilde{X}_h^*(t_n); \widetilde{I}_h u(\cdot, t_n), \varphi_h) \\
&- m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u(\cdot, t_n), \nabla_{\Gamma_h} \widetilde{I}_h u(\cdot, t_n)), \varphi_h)
\end{aligned}$$

Here $\widetilde{I}_h u(\cdot, t_{n-j})$ is understood as the finite element function on $\Gamma(\widetilde{\mathbf{x}}_*^{n-j})$ corresponding with the nodal vector $\mathbf{u}_*^{n-j}$, which collects the evaluation of $u(\cdot, t_{n-j})$ in the nodes $\mathbf{x}_*^{n-j}$. Similar to the defect from the velocity law we rewrite the weak formulation of the surface PDE to obtain

$$\begin{aligned}
0 = &-\frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n); u(\cdot, t_n), \varphi_h^l) - a(X(t_n); u(\cdot, t_n), \varphi_h^l) \\
&+ m(X(t_n); f(u(\cdot, t_n), \nabla_{\Gamma_h} u(\cdot, t_n)), \varphi_h^l).
\end{aligned}$$

Adding these two equations now gives

$$\begin{aligned}
m(X_h^*(t_n); d_u^n, \psi_h) = &\frac{1}{\tau}\sum_{j=0}^{p} \delta_j m(\widetilde{X}_h^*(t_{n-j}); \widetilde{I}_h u(\cdot, t_{n-j}), \varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n); u(\cdot, t_n), \varphi_h^l) && \text{(i)} \\
&+ a(\widetilde{X}_h^*(t_n); \widetilde{I}_h u(\cdot, t_n), \varphi_h) - a(X(t_n); u(\cdot, t_n), \varphi_h^l) && \text{(ii)} \\
&+ m(X(t_n); f(u(\cdot, t_n), \nabla_{\Gamma_h} u(\cdot, t_n)), \varphi_h^l) && \text{(iii)} \\
&- m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u(\cdot, t_n), \nabla_{\Gamma_h} \widetilde{I}_h u(\cdot, t_n)), \varphi_h), && \text{(iii)}
\end{aligned}$$

where the terms were rearranged in a way that all differences will turn out small. In the following we estimate the three lines above, in order.

(i) We use the extrapolated surface $\widetilde{X}_h^*$ to rewrite

$$\begin{aligned}
&\frac{1}{\tau}\sum_{j=0}^{p}\delta_j m(\widetilde{X}_h^*(t_{n-j});\widetilde{I}_h u(\cdot,t_{n-j}),\varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n);u(\cdot,t_n),\varphi_h^l)\\
&= \frac{1}{\tau}\sum_{j=0}^{p}\delta_j m(\widetilde{X}_h^*(t_{n-j});\widetilde{I}_h u(\cdot,t_{n-j}),\varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(\widetilde{X}_h^*(t_n);\widetilde{I}_h u,\varphi_h)\\
&+ \frac{\mathrm{d}}{\mathrm{dt}} m(\widetilde{X}_h^*(t_n);\widetilde{I}_h u,\varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n);u,\varphi_h^l)
\end{aligned}$$

With Taylor's expansion we can estimate the first difference to be of order τ^p with Peano kernels, as in [1]. We derive the second difference with the Leibniz rule to arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dt}} m(\widetilde{X}_h^*(t_n);\widetilde{I}_h u,\varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n);u,\varphi_h^l)\\
&= m(\widetilde{X}_h^*(t_n);\widetilde{\partial}_h^\bullet \widetilde{I}_h u,\varphi_h) - m(X(t_n);\partial^\bullet u,\varphi_h^l) - m(X(t_n);u,\partial^\bullet \varphi_h^l) \qquad (6.4)\\
&+ q(\widetilde{X}_h^*(t_n);\widetilde{v}_h^*;\widetilde{I}_h u,\varphi_h) - q(X(t_n);v;u,\varphi_h^l).
\end{aligned}$$

Using the discrete material derivative from (5.1) immediately yields, together with the property (5.2), for (6.4)

$$\begin{aligned}
\left|m(X(t_n);\partial^\bullet u,\varphi_h^l) + m(X(t_n);u,\partial^\bullet\varphi_h^l)\right| &\le m(X(t_n);\widehat{\partial}_h^\bullet u,\varphi_h^l) + \left\|\widehat{\partial}_h^\bullet u - \partial^\bullet u\right\|_{L^2(\Gamma(X))}\left\|\varphi_h^l\right\|_{L^2(\Gamma(X))}\\
&+ \|u\|_{L^2(\Gamma(X))}\left\|\partial^\bullet\varphi_h^l - \widehat{\partial}_h^\bullet\varphi_h^l\right\|_{L^2(\Gamma(X))}\\
&\le m(X(t_n);\widehat{\partial}_h^\bullet u,\varphi_h^l) + Ch^{k+1}\left\|\varphi_h^l\right\|_{L^2(\Gamma(X))},
\end{aligned}$$

where we also used the transport property $\widehat{\partial}_h^\bullet\varphi_h^l = 0$. This leaves us with

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{dt}} m(\widetilde{X}_h^*(t_n);\widetilde{I}_h u,\varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n);u,\varphi_h^l)\right| \qquad (6.5)\\
&= \left|m(\widetilde{X}_h^*(t_n);\widetilde{\partial}_h^\bullet\widetilde{I}_h u,\varphi_h) - m(X(t_n);\widehat{\partial}_h^\bullet u,\varphi_h^l)\right| \qquad (6.6)\\
&+ \left|q(\widetilde{X}_h^*(t_n);\widetilde{v}_h^*;\widetilde{I}_h u,\varphi_h) - q(X(t_n);v;u,\varphi_h^l)\right| \qquad (6.7)\\
&+ Ch^{k+1}\left\|\varphi_h^l\right\|_{L^2(\Gamma(X))}.
\end{aligned}$$

In the following we take a closer look at (6.6). Using intermediate terms yields

$$\begin{aligned}
&\left|m(\widetilde{X}_h^*(t_n); \widetilde{\partial}_h^\bullet \widetilde{I}_h u, \varphi_h) - m(X(t_n); \widehat{\partial}_h^\bullet u, \varphi_h^l)\right| \\
&\leq \left|m(\widetilde{X}_h^*(t_n); \widetilde{\partial}_h^\bullet \widetilde{I}_h u, \varphi_h) - m(X_h^*(t_n); \widetilde{I}_h \widehat{\partial}_h^\bullet u, \varphi_h^l)\right| \\
&+ \left|m(X_h^*(t_n); \widetilde{I}_h \widehat{\partial}_h^\bullet u - \left(\widehat{\partial}_h^\bullet u\right)^{-l}, \varphi_h^l)\right| \\
&+ \left|m(X_h^*(t_n); \left(\widehat{\partial}_h^\bullet u\right)^{-l}, \varphi_h) - m(X(t_n); \widehat{\partial}_h^\bullet u, \varphi_h^l)\right| \\
&\leq \left|m(\widetilde{X}_h^*(t_n); \widetilde{\partial}_h^\bullet \widetilde{I}_h u, \varphi_h) - m(X_h^*(t_n); \widetilde{I}_h \widehat{\partial}_h^\bullet u, \varphi_h^l)\right| \\
&+ C \left\| \widetilde{I}_h \widehat{\partial}_h^\bullet u - \left(\widehat{\partial}_h^\bullet u\right)^{-l} \right\|_{L^2(\Gamma(X_h^*))} \left\|\varphi_h^l\right\|_{L^2(\Gamma(X_h^*))} \\
&+ C h^{k+1} \left\|\widehat{\partial}_h^\bullet u\right\|_{L^2(\Gamma(X))} \left\|\varphi_h\right\|_{L^2(\Gamma(X))},
\end{aligned}$$

where the geometric estimates from Lemma 5.4 and the Cauchy–Schwartz inequality were used in the last step. Now the norm equivalences for lifts (5.1) and the interpolation error estimate from Lemma 5.3 lead us to

$$\begin{aligned}
&\left|m(\widetilde{X}_h^*(t_n); \widetilde{\partial}_h^\bullet \widetilde{I}_h u, \varphi_h) - m(X(t_n); \widehat{\partial}_h^\bullet u, \varphi_h^l)\right| && (6.8)\\
&\leq \left|m(\widetilde{X}_h^*(t_n); \widetilde{\partial}_h^\bullet \widetilde{I}_h u, \varphi_h) - m(X_h^*(t_n); \widetilde{I}_h \widehat{\partial}_h^\bullet u, \varphi_h^l)\right| && (6.9)\\
&+ C\left(h^k + \tau^p\right) \left\|\widehat{\partial}_h^\bullet u\right\|_{L^2(\Gamma(X))} \left\|\varphi_h\right\|_{L^2(\Gamma(X))} && (6.10)
\end{aligned}$$

The transport property ensures that $\widetilde{I}_h$ and $\partial_h^\bullet$ commute, since

$$\begin{aligned}
\partial_h^\bullet \widetilde{I}_h u = \partial_h^\bullet \sum_{j=1}^N u(X_h^*(q_j,t),t)\phi_j[X_h^*] &= \sum_{j=1}^N \left(\frac{\mathrm{d}}{\mathrm{dt}} u(X_h^*(q_j,t),t)\right)\phi_j[X_h^*] && (6.11)\\
&= \sum_{j=1}^N \left(\frac{\mathrm{d}}{\mathrm{dt}} u(x_j(t),t)\right)\phi_j[X_h^*] && (6.12)\\
&= \widetilde{I}_h \partial^\bullet u = \widetilde{I}_h \widehat{\partial}_h^\bullet u, && (6.13)
\end{aligned}$$

where in the last line was used that the discrete material derivative (defined in (5.1)) and the material derivative coincide in the nodes $x_j(t)$. We also recall that on $\widetilde{X}_h^*$ the material derivative of the interpolation $\widetilde{\partial}_h^\bullet \widetilde{I}_h u$ is due to the transport property given by

$$\widetilde{\partial}_h^\bullet \widetilde{I}_h u = \sum_{j=1}^N \left(\widetilde{\partial}_h^\bullet u_j(t)\right) \widetilde{\phi}_j[\widetilde{X}_h^*] = \sum_{j=1}^N \left(\frac{\mathrm{d}}{\mathrm{dt}} u(x_j(t),t)\right) \widetilde{\phi}_j[\widetilde{X}_h^*]. \tag{6.14}$$

The last equality holds, since $u_j(t) = u(x_j(t),t)$ does not depend on the extrapolated surface. We define the nodal vector $\dot{\mathbf{u}} \in \mathbb{R}^N$ with

$$(\dot{\mathbf{u}}_*^n)_j = \frac{\mathrm{d}}{\mathrm{dt}} u(x_j(t),t), \text{ for } 1 \leq j \leq N.$$

This nodal vector now corresponds with (6.14) to the finite element function $\widetilde{\partial}^{\bullet}_h \widetilde{I}_h u$ on $\Gamma(\widetilde{X}_h^*)$ and with (6.11) to $\partial^{\bullet}_h \widetilde{I}_h u$ on $\Gamma(X_h^*)$. We denote the nodal vector corresponding to φ_h with $\boldsymbol{\varphi} \in \mathbb{R}^N$. This allows us to rewrite (6.9), to obtain

$$\begin{aligned}
&\left| m(\widetilde{X}_h^*(t_n); \widetilde{\partial}^{\bullet}_h \widetilde{I}_h u, \varphi_h) - m(X_h^*(t_n); \widetilde{I}_h \widehat{\partial}^{\bullet}_h u, \varphi_h^l) \right| \\
&= \left| \dot{\mathbf{u}}_*^n \left(\mathbf{M}(\widetilde{\mathbf{x}}_*^n) - \mathbf{M}(\mathbf{x}_*^n) \right) \boldsymbol{\varphi} \right| \\
&\le C\tau^p \left\| \dot{\mathbf{u}}_*^n \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \left\| \boldsymbol{\varphi} \right\|_{\mathbf{M}(\mathbf{x}_*^n)} \\
&\le C\tau^p \left\| \widetilde{I}_h \partial^{\bullet}_h u \right\|_{L^2(X_h^*(t_n))} \left\| \varphi_h \right\|_{L^2(X_h^*(t_n))} \\
&\le C\tau^p (1 + Ch^k) \left\| \partial^{\bullet} u \right\|_{L^2(X)} \left\| \varphi_h^l \right\|_{L^2(X)},
\end{aligned}$$

where Lemma 3.3 and the usual norm equivalences were used. In the last step an interpolation estimate from Lemma 5.3 ensures the estimate with the norm of $\partial^{\bullet}_h u$ instead of its interpolation. Combining the steps above we can now estimate (6.6) with

$$\left| m(\widetilde{X}_h^*(t_n); \widetilde{\partial}^{\bullet}_h \widetilde{I}_h u, \varphi_h) - m(X(t_n); \widehat{\partial}^{\bullet}_h u, \varphi_h^l) \right| \le C(h^k + \tau^p) \left\| \partial^{\bullet} u \right\|_{L^2(\Gamma(X))} \left\| \varphi_h \right\|_{L^2(\Gamma(X))},$$

where we used the norm equivalence for lifted functions from Lemma 5.1 .
Now we turn to the second sum of the same term, i.e. (6.7):

$$\begin{aligned}
&\left| q(\widetilde{X}_h^*(t_n); \widetilde{v}_h^*; \widetilde{I}_h u, \varphi_h) - q(X(t_n); v; u, \varphi_h^l) \right| && (6.15)\\
&= \left| q(\widetilde{X}_h^*(t_n); \widetilde{v}_h^*; \widetilde{I}_h u, \varphi_h) - q(\widetilde{X}_h^*(t_n); \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) \right| && (6.16)\\
&+ \left| q(\widetilde{X}_h^*(t_n); \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) - q(X_h^*(t_n); \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) \right| && (6.17)\\
&+ \left| q(X_h^*(t_n); \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) - q(X(t_n); \hat{v}_h; I_h u, \varphi_h^l) \right| && (6.18)\\
&+ \left| q(X(t_n); \hat{v}_h; I_h u, \varphi_h^l) - q(X(t_n); v; I_h u, \varphi_h^l) \right| && (6.19)\\
&+ \left| q(X(t_n); v; I_h u, \varphi_h^l) - q(X(t_n); v; u, \varphi_h^l) \right|. && (6.20)
\end{aligned}$$

Now (6.16) and (6.19) are bounded with an $L^2 - L^\infty - L^2$ estimate, and (6.20) can be controlled with an $L^\infty - L^2 - L^2$ estimate. Furthermore, (6.18) can be bounded with the last geometric estimate from Lemma 5.4. This leaves us with

$$\begin{aligned}
&\left| q(\widetilde{X}_h^*(t_n); \widetilde{v}_h^*; \widetilde{I}_h u, \varphi_h) - q(X(t_n); v; u, \varphi_h^l) \right| \\
&\le \left\| \nabla_{\Gamma_h} \left(\widetilde{v}_h^* - \widetilde{I}_h v \right) \right\|_{L^2(\widetilde{X}_h^*)} \left\| \widetilde{I}_h u \right\|_{L^\infty(\widetilde{X}_h^*)} \left\| \varphi_h \right\|_{L^2(\widetilde{X}_h^*)} \\
&+ \left| q(\widetilde{X}_h^*; \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) - q(X_h^*; \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) \right| \\
&+ Ch^{k+1} \left\| I_h u \right\|_{L^2(\Gamma(X))} \left\| \varphi_h^l \right\|_{L^2(\Gamma(X))} \\
&+ C \Big(\left\| \nabla_\Gamma \left(\hat{v}_h - I_h v \right) \right\|_{L^2(\Gamma(X))} + \left\| \nabla_\Gamma \left(I_h v - v \right) \right\|_{L^2(\Gamma(X))} \Big) \left\| I_h u \right\|_{L^\infty(\Gamma(X))} \left\| \varphi_h^l \right\|_{L^2(\Gamma(X))} \\
&+ \left\| \nabla_\Gamma v \right\|_{L^\infty(\Gamma(X))} \left\| I_h u - u \right\|_{L^2(\Gamma(X))} \left\| \varphi_h^l \right\|_{L^2(\Gamma(X))}.
\end{aligned}$$

Since $\hat{v}_h$ and v agree on the nodes, their interpolations coincide. Using the interpolation estimates from Lemma 5.3 and norm equivalences for lifts now allows us to obtain

$$\begin{aligned}
\left|q(\widetilde{X}_h^*(t_n);\widetilde{v}_h^*;\widetilde{I}_h u,\varphi_h) - q(X(t_n);v;u,\varphi_h^l)\right| &\leq C\,\|\widetilde{\mathbf{v}}_*^n - \mathbf{v}_*^n\|_{\mathbf{K}(\mathbf{x}_*^n)}\,\|\boldsymbol{\varphi}\|_{\mathbf{M}(\mathbf{x}_*^n)} \\
&\quad + \left|q(\widetilde{X}_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h) - q(X_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h)\right| \\
&\quad + Ch^{k+1}\,\|I_h u\|_{L^2(\Gamma(X))}\,\|\varphi_h^l\|_{L^2(\Gamma(X))} \\
&\quad + Ch^k\left(\|\hat{v}_h\|_{L^2(\Gamma(X))} + \|v\|_{L^2(\Gamma(X))}\right)\|\varphi_h^l\|_{L^2(\Gamma(X))} \\
&\quad + Ch^{k+1}\,\|u\|_{L^2(\Gamma(X))}\,\|\varphi_h^l\|_{L^2(\Gamma(X))} \\
&\leq C\left(\tau^p + h^k\right)\|\varphi_h^l\|_{L^2(\Gamma(X))} \\
&\quad + \left|q(\widetilde{X}_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h) - q(X_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h)\right|,
\end{aligned}$$

where an estimate for the extrapolation was used in the last inequality. The difference with the extrapolation is, as before, bounded with Taylor's theorem and of order τ^p. The last remaining summand now is estimated with the techniques from Lemma 3.1. We use an intermediate surface $\Gamma_h^\theta = \Gamma[\mathbf{x}_*^n + \theta(\widetilde{\mathbf{x}}_*^n - \mathbf{x}_*^n)]$ and the fundamental theorem of calculus to obtain

$$\begin{aligned}
&\left|q(\widetilde{X}_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h) - q(X_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h)\right| \\
&= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta}\int_{\Gamma_h^\theta}\left(\nabla_{\Gamma_h^\theta}\cdot\widetilde{I}_h^\theta v\right)\left(\widetilde{I}_h^\theta u\right)\varphi_h^\theta \mathrm{d}\theta\right| \\
&\leq C\left|\int_0^1\int_{\Gamma_h^\theta}\partial_\theta^\bullet\left(\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v\right)\left(\widetilde{I}_h^\theta u\right)\varphi_h^\theta + \left(\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v\right)\left(\widetilde{I}_h^\theta u\right)\varphi_h^\theta\left(\nabla_{\Gamma_h^\theta}\cdot\widetilde{e}_{x,*}^\theta\right)\mathrm{d}\theta\right|,
\end{aligned}$$

where the Leibniz rule was used as before. The transport property ensures that the material derivative of a finite element function with constant coefficient vanishes (i.e. $\partial_\theta^\bullet\varphi_h^\theta{=}0$). We can use this together with the product rule to see that it is sufficient to take the material derivative of the first factor of the product. We remind ourselves of (4.17), which yields

$$\partial_\theta^\bullet\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v = \nabla_{\Gamma_h^\theta}\partial_\theta^\bullet\widetilde{I}_h^\theta v - \left(\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta} - \nu_{\Gamma_h^\theta}(\nu_{\Gamma_h^\theta})^T(\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta})^T\right)\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v.$$

The first summand on the right side vanishes because of the transport property. We insert this identity into the right-hand side above to obtain

$$\begin{aligned}
&\left|q(\widetilde{X}_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h) - q(X_h^*;\widetilde{I}_h v;\widetilde{I}_h u,\varphi_h)\right| \\
&\leq C\int_0^1\int_{\Gamma_h^\theta}\left|\left(\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta} - \nu_{\Gamma_h^\theta}(\nu_{\Gamma_h^\theta})^T(\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta})^T\right)\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v\left(\widetilde{I}_h^\theta u\right)\varphi_h^\theta\right| \\
&\qquad + \left|\left(\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v\right)\left(\widetilde{I}_h^\theta u\right)\varphi_h^\theta\left(\nabla_{\Gamma_h^\theta}\cdot\widetilde{e}_{x,*}^\theta\right)\right|\mathrm{d}\theta \\
&\leq C\int_0^1\left(\left\|\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)} + \left\|\nu_{\Gamma_h^\theta}\right\|_{L^\infty(\Gamma_h^\theta)}^2\left\|\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\right)\left\|\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v\right\|_{L^\infty(\Gamma_h^\theta)}\left\|\widetilde{I}_h^\theta u\right\|_{L^\infty(\Gamma_h^\theta)}\left\|\varphi_h^\theta\right\|_{L^2(\Gamma_h^\theta)} \\
&\qquad + \left\|\nabla_{\Gamma_h^\theta}\widetilde{I}_h^\theta v\right\|_{L^\infty(\Gamma_h^\theta)}\left\|\widetilde{I}_h^\theta u\right\|_{L^\infty(\Gamma_h^\theta)}\left\|\varphi_h^\theta\right\|_{L^2(\Gamma_h^\theta)}\left\|\nabla_{\Gamma_h^\theta}\cdot\widetilde{e}_{x,*}^\theta\right\|_{L^2(\Gamma_h^\theta)}\mathrm{d}\theta \\
&\leq C\int_0^1\left\|\nabla_{\Gamma_h^\theta}\widetilde{e}_{x,*}^{n,\theta}\right\|_{L^2(\Gamma_h^\theta)}\left\|\varphi_h^\theta\right\|_{L^2(\Gamma_h^\theta)}\mathrm{d}\theta.
\end{aligned}$$

The last inequality is due to the fact that the L^∞-norm of interpolations (and their gradients) of functions is bounded, for example we can deduce

$$\begin{aligned}\|I_h u\|_{L^\infty(\Gamma(X))} + &\leq \|u\|_{L^\infty(\Gamma(X))} + \|u - I_h u\|_{L^\infty(\Gamma(X))} \\ &\leq \|u\|_{L^\infty(\Gamma(X))} + Ch^2 \left(\|u\|_{W^{1,\infty}(\Gamma(X))} + h \|u\|_{W^{2,\infty}(\Gamma(X))} \right) \leq C,\end{aligned}$$

with common L^∞-interpolation estimates, like [14, Lemma 3.8].
Now, we use Lemma 3.1 and an estimate to conclude

$$\left| q(\widetilde{X}_h^*; \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) - q(X_h^*; \widetilde{I}_h v; \widetilde{I}_h u, \varphi_h) \right| \leq C\tau^p \|\varphi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}$$

Combining this result with the others above now yields for (i)

$$\left| \frac{1}{\tau} \sum_{j=0}^{p} \delta_j m(\widetilde{X}_h^*(t_{n-j}); \widetilde{I}_h u(\cdot, t_{n-j}), \varphi_h) - \frac{\mathrm{d}}{\mathrm{dt}} m(X(t_n); u(\cdot, t_n), \varphi_h^l) \right| \leq C(h^k + \tau^p) \|\varphi_h\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}$$

(ii) The second term is estimated by using multiple zeros, i.e.

$$\begin{aligned}&\left| a(\widetilde{X}_h^*(t_n); \widetilde{I}_h u, \varphi_h) - a(X(t_n); u, \varphi_h^l) \right| \\ &\leq \left| a(\widetilde{X}_h^*(t_n); \widetilde{I}_h u, \varphi_h) - a(\mathbf{x}_*^n; \widetilde{I}_h u, \varphi_h) \right| \\ &+ \left| a(X_h^*(t_n); \widetilde{I}_h u, \varphi_h) - a(X(t_n); I_h u, \varphi_h^l) \right| \\ &+ \left| a(X(t_n); I_h u, \varphi_h^l) - a(X(t_n); u, \varphi_h^l) \right| \\ &\leq C \left(h^k + \tau^p \right) \|u\|_{L^2(\Gamma(X))} \left\|\varphi_h^l\right\|_{L^2(\Gamma(X))},\end{aligned}$$

where we used Lemma 3.1 for the first term, estimate the second summand with a geometric estimate from Lemma 5.4 and the last summand with the Cauchy-Schwartz inequality and the interpolation bound from Lemma 5.3.

(iii) For the last summand of the defect in u we use an intermediate term as before, to obtain

$$|m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X(t_n); f(u, \nabla_{\Gamma_h} u), \varphi_h^l)| \tag{6.21}$$
$$\leq |m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h)| \tag{6.22}$$
$$+ |m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X(t_n); f(u, \nabla_{\Gamma_h} u), \varphi_h^l)|. \tag{6.23}$$

Using the interpolation estimates from Lemma 5.3 and Lemma 3.3 allows us to estimate (6.22) by

$$\begin{aligned}&|m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h)| \\ &\leq |m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(\widetilde{X}_h^*(t_n); \widetilde{I}_h f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) \\ &+ |m(\widetilde{X}_h^*(t_n); \widetilde{I}_h f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X_h^*(t_n); \widetilde{I}_h f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h)| \\ &+ |m(X_h^*(t_n); \widetilde{I}_h f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h)| \\ &\leq Ch^{k+1} \left\| f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) \right\|_{L^2(\Gamma[\widetilde{\mathbf{x}}_*^n])} \\ &+ |m(\widetilde{X}_h^*(t_n); \widetilde{I}_h f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h) - m(X_h^*(t_n); \widetilde{I}_h f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u), \varphi_h)| \\ &+ Ch^{k+1} \left\| f(\widetilde{I}_h u, \nabla_{\Gamma_h} \widetilde{I}_h u) \right\|_{L^2(\Gamma_h[\mathbf{x}_*^n])}.\end{aligned}$$

Since the finite element functions in the last pair that remains correspond to the same nodal vector, we conclude

$$|m(\widetilde{X}_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u), \varphi_h) - m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u), \varphi_h)| \le C(h^k + \tau^p).$$

Now, we continue by estimating (6.23). By using $f(u, \nabla_\Gamma u)^{-l} = f(u^{-l}, (\nabla u)^{-l})$ and the first geometric estimate from Lemma 5.4, we obtain

$$\begin{aligned}
&|m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u), \varphi_h) - m(X(t_n); f(u, \nabla_\Gamma u), \varphi_h^l)| \\
&\le |m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u) - f(u^{-l}, (\nabla_\Gamma u)^{-l}), \varphi_h) \\
&+ |m(X_h^*(t_n); f(u, (\nabla_\Gamma u))^{-l}, \varphi_h) - m(X(t_n); f(u, \nabla_\Gamma u), \varphi_h^l)| \\
&\le |m(X_h^*(t_n); f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u) - f(u^{-l}, (\nabla_{\Gamma_h} u)^{-l}), \varphi_h) \\
&+ ch^{k+1} \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} . \\
&\le \left\| f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u) - f(u^{-l}, (\nabla_{\Gamma_h} u)^{-l}) \right\|_{L^2[\mathbf{x}_*^n]} \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} \\
&+ ch^{k+1} \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} ,
\end{aligned}$$

where we used the Cauchy Schwartz inequality in the last estimate. Using the local lipschitz continuity of f and the usual interpolation estimates (Lemma 5.3) now yields for the first term of the sum above

$$\begin{aligned}
&\left\| f(\widetilde{I}_h u, \nabla_{\Gamma_h}\widetilde{I}_h u) - f(u^{-l}, (\nabla_{\Gamma_h} u)^{-l}) \right\|_{L^2[\mathbf{x}_*^n]} \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} \\
\le& C\left(\left\| \widetilde{I}_h u - u^{-l} \right\|_{L^2[\mathbf{x}_*^n]} + \left\| \nabla_{\Gamma_h}\widetilde{I}_h u - (\nabla_{\Gamma_h} u)^{-l} \right\|_{L^2[\mathbf{x}_*^n]} \right) \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} \\
\le& C\left(Ch^{k+1} + \left\| \nabla_{\Gamma_h}\left(\widetilde{I}_h u - u^{-l}\right) + \nabla_{\Gamma_h} u^{-l} - (\nabla_{\Gamma_h} u)^{-l} \right\|_{L^2[\mathbf{x}_*^n]} \right) \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} \\
\le& C\left(Ch^{k+1} + Ch^k + \left\| \nabla_{\Gamma_h} u^{-l} - (\nabla_{\Gamma_h} u)^{-l} \right\|_{L^2[\mathbf{x}_*^n]} \right) \|\varphi_h\|_{L^2[\mathbf{x}_*^n]} .
\end{aligned}$$

With (6.3) and the norm equivalence for lifts in Lemma 5.1 we can estimate the final term with

$$\left\| \nabla_{\Gamma_h} u^{-l} - (\nabla_\Gamma u)^{-l} \right\|_{L^2[\mathbf{x}_*^n]} \le C \left\| \left(\nabla_{\Gamma_h} u^{-l}\right)^l - \nabla_\Gamma u \right\|_{L^2(\Gamma(X))} \le Ch^k.$$

Combining all estimates for (i),(ii) and (iii) now yields

$$m(X_h^*(t_n); d_u, \varphi) \le C(h^k + \tau^p).$$

6.3 The defect of x

The defect $\mathbf{d}_\mathbf{x}^n \in \mathbb{R}^{3N}$ is given by

$$\mathbf{d}_\mathbf{x}^n = \frac{1}{\tau}\sum_{j=0}^{p} \delta_j \mathbf{x}_*(t_{n-j}) - \dot{\mathbf{x}}_*(t_n).$$

This defect solely arises due to the time discretization. Using Taylor's theorem and Peano kernels, as done in ([1], Lemma 6.1) gives the estimate

$$\|\mathbf{d}_{\mathbf{x}}^n\|_{\mathbf{K}[\mathbf{x}_*^n]} \leq C\tau^p.$$

□

Now, the main statement for the errors follows from inserting the defect bounds into the stability results. In the following, we now give the proof of Theorem 2.1.

Proof. Inserting the defect bounds from Lemma 6.1 into the stability result from Proposition 1 yields

$$\begin{aligned}
&\|\mathbf{e}_{\mathbf{x}}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \|\mathbf{e}_{\mathbf{v}}^n\|_{\mathbf{K}(\mathbf{x}_*^n)}^2 + \sum_{j=1}^{p} \left\|\mathbf{e}_{\mathbf{u}}^{n-p+j}\right\|_{\mathbf{M}(\mathbf{x}_*^n)}^2 + \tau \sum_{j=1}^{n} \left\|\mathbf{e}_{\mathbf{u}}^j\right\|_{\mathbf{A}(\mathbf{x}_*^j)}^2 \\
&\leq C\tau \sum_{j=p}^{n} \left(\left\|\mathbf{d}_{\mathbf{x}}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \left\|\mathbf{d}_{\mathbf{v}}^j\right\|_{*,\mathbf{x}_*^j}^2 + \left\|\mathbf{d}_{\mathbf{u}}^j\right\|_{*,\mathbf{x}_*^j}^2 \right) + C \sum_{j=0}^{p-1} \left(\left\|\mathbf{e}_{\mathbf{x}}^j\right\|_{\mathbf{K}(\mathbf{x}_*^j)}^2 + \left\|\mathbf{e}_{\mathbf{u}}^j\right\|_{\mathbf{M}(\mathbf{x}_*^j)}^2 \right), \qquad (6.24) \\
&\leq C\tau \sum_{j=p}^{n} C\left(h^k + \tau^p\right)^2 + C \sum_{j=0}^{p-1} C(h^k+\tau^p)^2 \leq C(h^k+\tau^p)^2
\end{aligned}$$

For arbitrary $a, b > 0$, Young's inequality yields

$$\left(\sqrt{a} + \sqrt{b}\right) = \sqrt{\left(\sqrt{a} + \sqrt{b}\right)^2} = \sqrt{a + 2\sqrt{a}\sqrt{b} + b} \leq \sqrt{2\,(a+b)}.$$

Taking the square root of both sides of (6.24) and then using the estimate from below multiple times allows us to obtain

$$\|\mathbf{e}_{\mathbf{x}}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_{\mathbf{v}}^n\|_{\mathbf{K}(\mathbf{x}_*^n)} + \|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{M}(\mathbf{x}_*^n)} + \sqrt{\tau \sum_{j=1}^{n} \left\|\mathbf{e}_{\mathbf{u}}^j\right\|_{\mathbf{A}(\mathbf{x}_*^j)}^2} \leq C(h^k + \tau^p),$$

where we dropped the past errors in u. As before in (2.23) we write $\widehat{u}_h^n$ for the finite element function corresponding to $\mathbf{u}^n$ on the discrete surface with exact nodes. Using the interpolation estimate from Lemma 5.3, together with Lemma 5.1 we obtain

$$\begin{aligned}
\left\|(u_h^n)^L - u(\cdot,t_n)\right\|_{L^2(\Gamma(t_n))} &\leq \left\|(u_h^n)^L - I_h u(\cdot,t_n) + I_h u(\cdot,t_n) - u(\cdot,t_n)\right\|_{L^2(\Gamma(t_n))} \\
&\leq C\left\|\widehat{u_h^n} - \widetilde{I}_h u(\cdot,t_n)\right\|_{L^2(\Gamma_h(X_h^*(t_n)))} + Ch^k \\
&\leq C\|\mathbf{e}_{\mathbf{u}}^n\|_{\mathbf{M}[\mathbf{x}_*^n]} + Ch^{k+1} \leq C(h^k+\tau^p).
\end{aligned}$$

Using this argument structure in the same way for the other errors yields the stated result. □

7 Numerical experiments

The main obstacle to the implementation is the matrix assembly, i.e. the calculation of the finite element matrices **A** and **M**. We use an element-by-element implementation and the usual reference element techniques to calculate local FEM matrices and insert these into the global mass/stiffness matrix.

From Theorem 2.1 we can not conclude convergence of linear finite elements, we therefore implement second order finite elements. This proves to be more challenging at a number of points, mainly in the matrix assembly and the mesh generation, which we discuss in the following.

7.1 Generation of a second order triangulation

For this implementation, we used the distmesh [20] package for MATLAB to generate a linear discretization of the initial surface with a given distance function. We then transform every linear element by creating additional points in the middle of each edge and projecting these new points to the boundary. These points characterize the element, however for computational purposes we need a parametrization of the element.
We define our reference element $\widehat{E}$ as the convex hull of the origin, $(1,0,0)$ and $(0,1,0)$, with nodes at

$$\widehat{\text{Nodes}} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}.$$

In the following, everything related to the reference element is denoted with a hat. For a second order element $E \subset \mathbb{R}^3$ we choose the parametrization $\Phi_E : \widehat{E} \to E$ to be the polynomial of second degree that interpolates the nodes of E. (In a loose notation we demand $\Phi_E(\widehat{\text{Nodes}}) = \text{Nodes}_E$.) Furthermore, we write $\widehat{\phi}_j$ for the second order basis functions of $\widehat{E}$ and $\phi_j = \widehat{\phi}_j \circ \Phi_E^{-1}$ for the basis functions on E.

Understood as a map $\Phi_E : \mathbb{R}^3 \to \mathbb{R}^3$ is singular, since it maps the points above and below the reference element on the surface as well.
To get around this we demand that the differential $D\Phi_E$ pushes forward the normal vector on the reference Element $\nu_{\widehat{E}} = (0,0,1)^t$ on the normal vector on the Element ν_E i.e. $D\Phi_E \nu_{\widehat{E}} = \nu_E$.
To understand why we define Φ_E in a neighborhood of the reference element $\widehat{E}$ like this, we observe the following calculation for the surface gradient of a basis function on an arbitrary element, :

$$\begin{aligned} \nabla_E \phi_j &= \nabla(\widehat{\phi}_j \circ \Phi_E^{-1}) - \nu_E \nu_E^T \nabla(\widehat{\phi}_j \circ \Phi_E^{-1}) \\ &= (D\Phi_E^{-1})^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1} - \nu_E \nu_E^T (D\Phi_E^{-1})^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1} \\ &= (D\Phi_E^{-1})^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1} - \nu_E (D\Phi_E^{-1} \nu_E)^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1} \\ &= (D\Phi_E^{-1})^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1} - \nu_E (\nu_{\widehat{E}})^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1} \\ &= (D\Phi_E^{-1})^T (\nabla\widehat{\phi}_j) \circ \Phi_E^{-1}. \end{aligned} \tag{7.1}$$

The second term now vanishes, since the gradient of basis functions on the reference element is zero in the last entry and the normal vector on the reference element is just $(0,0,1)$. The property above will be useful later in the matrix assembly of the stiffness matrix.

7.2 Matrix Assembly

The computational complexity mainly comes from the matrix assembly, since they have to be recalculated at any time step due to the fact that the surface is evolving.

Using the integral transformation theorem gives us a formula for the local mass matrix

$$\mathbf{M}_{ij}^{loc} = \int_E \phi_j \phi_i = \int_E \widehat{\phi_j} \circ \Phi_E^{-1} \widehat{\phi_i} \circ \Phi_E^{-1} = \int_{E_0} \widehat{\phi_j}\widehat{\phi_i} \left|\det(D\Phi_E)\right|.$$

With the same structure, combined with (7.1) we obtain

$$\begin{aligned}\mathbf{A}_{ij}^{loc} &= \int_E \nabla_E \phi_j \cdot \nabla_E \phi_i \\ &= \int_E \left(D\Phi_E^{-T}\left((\nabla\widehat{\phi_j}) \circ \Phi_E^{-1}\right)\right) \cdot \left(D\Phi_E^{-T}\left((\nabla\widehat{\phi_i}) \circ \Phi_E^{-1}\right)\right) \\ &= \int_{\widehat{E}} \left(D\Phi_E^{-T}\nabla\widehat{\phi_j}\right) \cdot \left(D\Phi_E^{-T}\nabla\widehat{\phi_i}\right) \left|\det D\Phi_E\right|,\end{aligned}$$

where the differential $D\Phi_E^{-T}$ is evaluated at a point $\Phi_E(\xi)$, for $\xi \in \widehat{E}$.

7.3 Constructing numerical experiments

We test the discretization by choosing the evolution $X(q,t) = r(t)q$ with a function $r : [0,1] \to \mathbb{R}_+$ and the initial surface Γ^0 chosen as the unit sphere. The exact surface $\Gamma(t)$ is therefore given by a sphere with Radius $r(t)$. The exact velocity is then given by

$$v(X(q,t),t) = \frac{\mathrm{d}}{\mathrm{dt}}X(q,t) = \frac{\mathrm{d}}{\mathrm{dt}}r(t)q = \dot{r}(t)q = \dot{r}(t)\frac{X(q,t)}{\|X(q,t)\|} = \frac{\dot{r}(t)}{r(t)}X(q,t).$$

The divergence of the velocity is now obtained by

$$\nabla_{\Gamma(X)} \cdot v = \frac{\dot{r}(t)}{r(t)}\nabla_{\Gamma(X)} \cdot X = 2\frac{\dot{r}(t)}{r(t)},$$

and the Laplace–Beltrami operator by

$$\Delta_{\Gamma(X)} v(x,t) = \frac{\dot{r}(t)}{r(t)}\Delta_{\Gamma(X)} X = -\frac{\dot{r}(t)}{r(t)}\frac{2}{r(t)^2}X = -2\frac{\dot{r}(t)}{r(t)^3}X$$

On the surface, we choose $u = a(t)xyz$ to be an eigenfunction of the Laplace-Beltrami operator on a sphere, multiplied with an arbitrary function a only depending on time. A long, however straightforward computation then yields

$$\Delta_{\Gamma(X)} u = -\frac{12}{r(t)^2} u.$$

The material derivative is given by

$$\begin{aligned}\partial^\bullet u(x,t) &= \frac{\mathrm{d}}{\mathrm{dt}} u(X(q,t),t) = \frac{\mathrm{d}}{\mathrm{dt}} a(t) r(t)^3 q_1 q_2 q_3 = a(t) 3 r(t)^2 \dot{r}(t) q_1 q_2 q_3 + \dot{a}(t) r(t)^3 q_1 q_2 q_3 \\ &= 3\frac{\dot{r}(t)}{r(t)} a(t) x_1 x_2 x_3 + \dot{a}(t) x_1 x_2 x_3 = \left(3\frac{\dot{r}(t)}{r(t)} + \frac{\dot{a}(t)}{a(t)}\right) u.\end{aligned}$$

With (2.7), we obtain for the right hand side f (with $\alpha = \beta = 1$)

$$\begin{aligned}f(u, \nabla_{\Gamma(X)} u) &= \partial^\bullet u + u \nabla_{\Gamma(X)} \cdot v - \Delta_{\Gamma(X)} u \\ &= \left(3\frac{\dot{r}(t)}{r(t)} + \frac{\dot{a}(t)}{a(t)}\right) u + 2\frac{\dot{r}(t)}{r(t)} u + \frac{12}{r(t)^2} u = \left(5\frac{\dot{r}(t)}{r(t)} + \frac{\dot{a}(t)}{a(t)} + \frac{12}{r(t)^2}\right) u.\end{aligned}$$

Plugging in u and v in the velocity law (2.8) now yields for g

$$\begin{aligned}g(u, \nabla_{\Gamma(X)} u) \nu_{\Gamma(X)} &= v - \alpha \Delta_{\Gamma(X)} v + \beta H_{\Gamma(X)} \nu_{\Gamma(X)} \\ &= \left(\frac{\dot{r}(t)}{r(t)} + 2\alpha \frac{\dot{r}(t)}{r(t)^3} + \beta \frac{2}{r(t)^2}\right) X \\ &= \left(\dot{r}(t) + 2\alpha \frac{\dot{r}(t)}{r(t)^2} + \beta \frac{2}{r(t)}\right) \nu_{\Gamma(X)} \\ &= \frac{u}{a(t) x_1 x_2 x_3} \left(\dot{r}(t) + 2\alpha \frac{\dot{r}(t)}{r(t)^2} + \beta \frac{2}{r(t)}\right) \nu_{\Gamma(X)},\end{aligned}$$

where the additional factor that was created in the last step serves as an artificial coupling of the velocity law to the solution of the surface PDE.

7.4 Convergence Plots

Setting $a(t) = \exp(-t^2)$ and $r(t) = 1 + \frac{1}{2}\sin(\pi t)$, we now calculate approximations of u and X at a time $T = 1$, for various h and τ.
We then plot the L^2- norm and the H^1-seminorm of the errors e_u^n and e_x^n, with double logarithmic scale. We fix the step size and mesh size in turn to observe space and time convergence rates.
In the following, we present plots for the quadratic ESFEM / BDF3 discretization. The time convergence rate ($\mathcal{O}(\tau^p)$) is observed both for x_h^n and u_h^n and for both norms.
The space convergence rate for the L^2-norm is observed to be of order $\mathcal{O}(h^3)$, where we were only able to prove the rate $\mathcal{O}(h^2)$, since we do not have a natural access to the Ritz-projection.
The space convergence rate for the H^1-norm of $\mathcal{O}(h^2)$ is observed for both u_h^n and x_h^n.
All numerical experiments were conducted using $\alpha = \beta = 1$.

In the following, we present convergence plots of $\left\|\left(x_h^n\right)^L - x_{\Gamma(X)}\right\|$, for the L^2-norm and the H^1-seminorm.

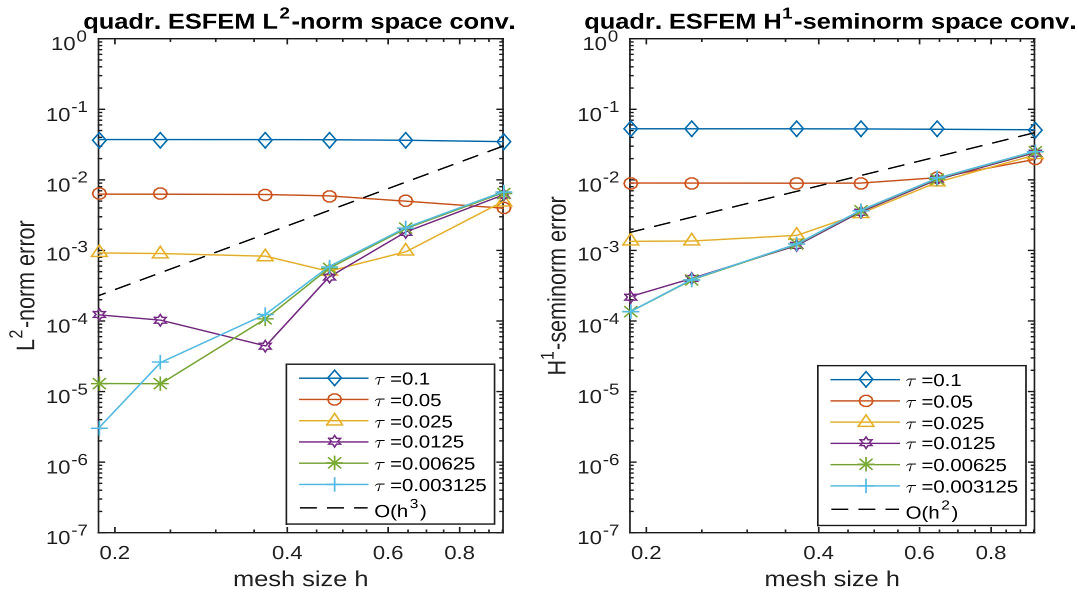

Figure 1: Time convergence of x_h^n, with quadratic ESFEM/ BDF3, for (2.7)-(2.9)

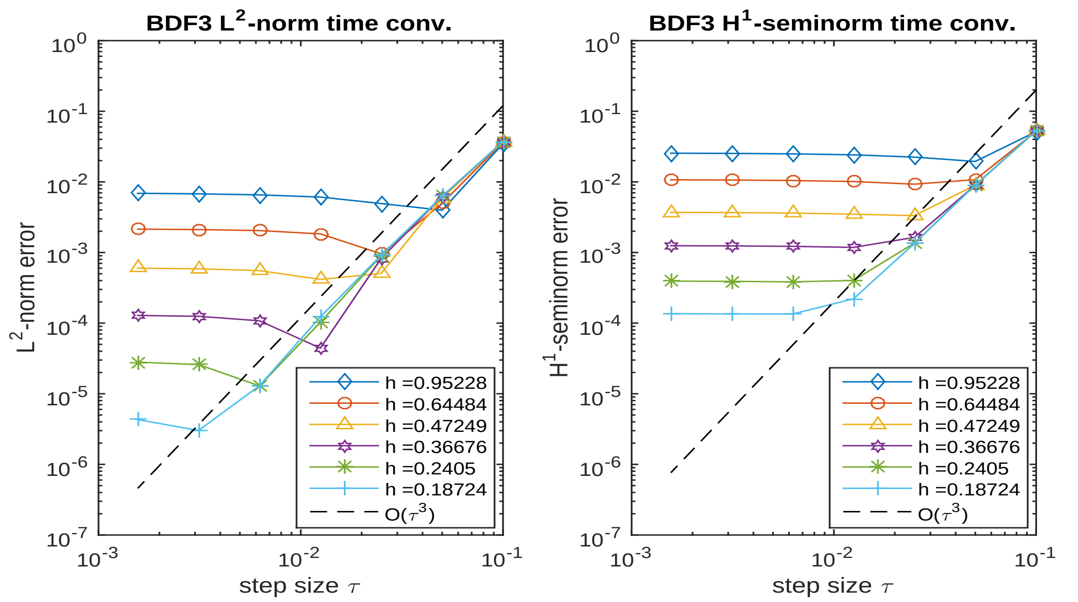

Figure 2: Space convergence of x_h^n, with quadratic ESFEM/ BDF3, for (2.7)-(2.9)

In the following, we present convergence plots of $\left\|\left(u_h^n\right)^L - u\right\|$, for the L^2-norm and the H^1-seminorm.

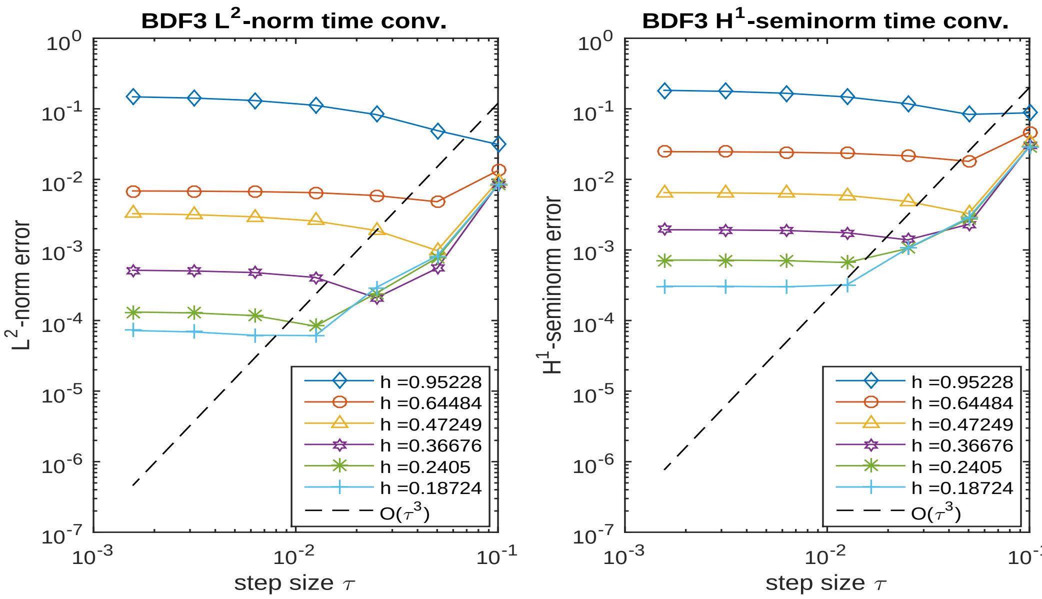

Figure 3: Time convergence of u_h^n, with quadratic ESFEM/ BDF3, for (2.7)-(2.9)

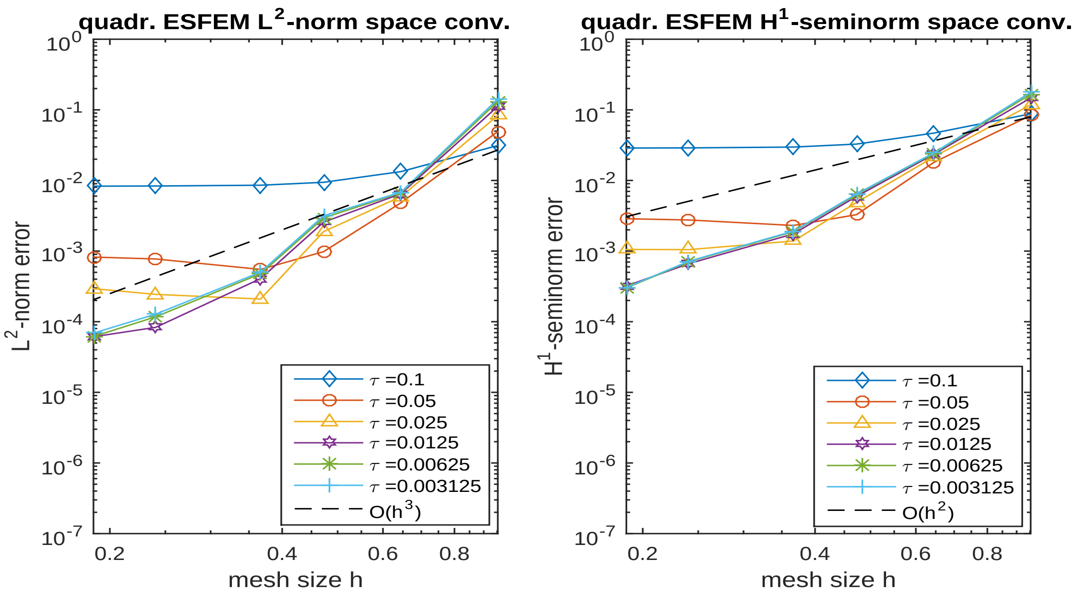

Figure 4: Space convergence of u_h^n, with quadratic ESFEM/ BDF3, for (2.7)-(2.9)

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In the full discretization of the surface PDE (2.19) appear mass matrices on surfaces corresponding to extrapolated nodal vectors, i.e. $\mathbf{M}(\widetilde{\mathbf{x}}^{n-j})$, where we could also reassemble at the numerical approximations $\mathbf{x}^{n-j}$. This corresponds of exchanging (2.19) with

$$\frac{1}{\tau}\left(\delta_0\mathbf{M}(\widetilde{\mathbf{x}}^n)\mathbf{u}^n + \sum_{j=1}^{p}\delta_j\mathbf{M}(\mathbf{x}^{n-j})\mathbf{u}^{n-j}\right) + \mathbf{A}(\widetilde{\mathbf{x}}^n)\mathbf{u}^n = \mathbf{f}(\widetilde{\mathbf{x}}^n, \widetilde{\mathbf{u}}^n). \tag{7.2}$$

We note that this change does come at a price, since we previously did compute $\mathbf{M}(\mathbf{x}^{n-j})$. In our numerical experiments this additional matrix assembly increases the runtime by approximately 40%.

Convergence plots, using (2.19) or (7.2), are basically identical, which implies that both schemes work equally fine for this experiment. To visualize the difference of both discretizations, we compute a numerical solutions $u_h^{n,1}$ using (2.19) and a numerical solution $u_h^{n,2}$, using (7.2). We then plot, in the same format as before, the space and time convergence of $\|u_h^{n,1} - u_h^{n,2}\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}$.

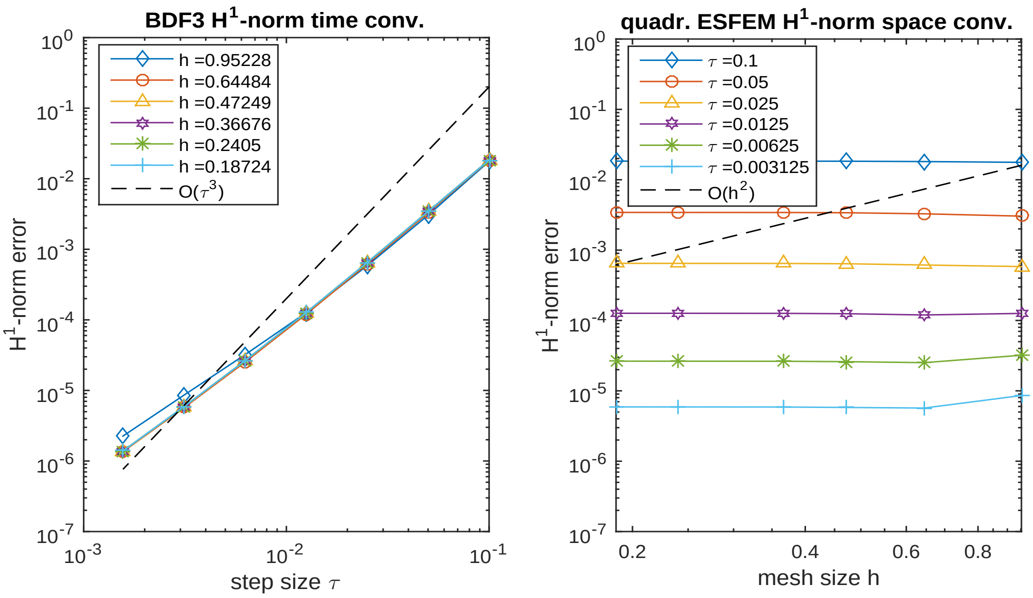

Figure 5: Convergence plots for $\|u_h^{n,1} - u_h^{n,2}\|_{H^1(\Gamma_h[\mathbf{x}_*^n])}$

We observe that the H^1-norm of the difference of both numerical solutions is much smaller than the error and completely dominated by the time discretization error. Therefore, reassembling the mass matrices in the numerical solutions $\mathbf{x}^j$ (at a high computational cost) and using (7.2) did not improve the accuracy of the numerical solution in any relevant way, compared to using (2.19).

Zusammenfassung

In dieser Arbeit wird ein gekoppeltes System geometrischer Evolutionsgleichung auf zweidimensionalen bewegten Oberflächen $\Gamma(t) \subset R^3$ volldiskretisiert und numerisch analysiert. Die Evolution der Oberfläche selbst ist dabei nicht vorgegeben, sondern selbst abhängig von einer Lösung einer partiellen Differentialgleichung (PDE) auf der bewegten Oberfläche.
Die Evolution löst einen regularisierten mittleren Krümmungsfluss, welcher gekoppelt ist an eine parabolische partielle Differentialgleichung auf der Oberfläche. In [16] wurde bereits die Konvergenz einer Semidiskretisierung, mithilfe finiter Elemente von Ordnung $k \geq 2$, des gekoppelten Systems nachgewiesen. Daraufhin wurde in [17] eine linear implizite BDF-Zeitdiskretisierung der semidiskreten PDE analysiert und für BDF-Verfahren der Ordnung $p \leq 5$ Konvergenz bewiesen. Diese Arbeit erweitert nun die numerische Analyse dieses Systems und enthält eine vollständige Konvergenzanalyse der Volldiskretisierung des gekoppelten Systems.

Im ersten Abschnitt wird eine kurze Einleitung in das Thema gegeben. Bisherige Ergebnisse werden referiert und die Struktur der Arbeit wird beschrieben.

Der zweite Abschnitt führt die grundlegende Notation eingeführt und wichtige Eigenschaften von bewegten Oberflächen werden wiederholt. Anschließend wird eine schwache Formulierung hergeleitet und finite Elemente Räume werden eingeführt. Mithilfe von BDF-Verfahren wird eine Volldiskretisierung des Problems aufgestellt. Abschließend wird das Hauptresultat formuliert und in einen Satz gefasst.

Im dritten Abschnitt werden Hilfsresultate für die aus [16] und [17] wiederholt, die eine zentrale Rolle in der Stabilitätsanalyse spielen. Ein neues Resultat wurde hinzugefügt, welches einen Differentenquotienten der Massematrizen abschätzt.

Der vierte Abschnitt beinhaltet die Stabilitätsanalyse. Zu Beginn werden Fehlergleichungen hergeleitet. Hierfür wird die Interpolation der exakten Lösung in das diskretisierte System eingesetzt. Diese lösen das numerische System nicht exakt, sondern nur bis auf einen Restterm, der sogenannte Defekt. Subtraktion der numerischen Lösung eingesetzt in das numerische System liefert dann Fehlergleichungen. Nun werden die Fehlergleichungen nacheinander getestet, anschließend werden die Terme auf der rechten Seite nacheinander abgeschätzt und absorbiert. Eine wichtige Rolle spielen dabei die Multiplier-Techniken aus [19]. Das Stabilitätsresultat schätzt nun die numerischen Fehler in Termen der eingeführten Defekte ab.

Abschnitt 5 gibt wichtige geometrische Abschätzungen an, die später in der Konsistenzanalyse verwendet werden.

Die Konsistenzanalyse befindet sich im sechsten Abschnitt. Hier werden die in der Stabilitätsanalyse eingeführten Defekte, mithilfe der geometrischen Abschätzungen aus der Sektion zuvor, beschränkt. Gemeinsam mit dem Stabilitätsresultat aus dem vierten Abschnitt wird nun das Hauptresultat, in Form des Satzes aus dem vorherigen Abschnitt, bewiesen.

Im siebten Abschnitt werden numerische Experimente beschrieben. Die zentralen Ideen der Implementierung von finiten Elementen zweiter Ordnung auf gekrümmten Oberflächen werden erläutert. Ein simpler Weg zur Erstellung numerischer Experimente wird dargestellt. Die bewiesenen Konvergenzraten werden anhand Fehlerplots numerischer Experimente illustriert.

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Eigenständigkeitserklärung

Hiermit erkläre ich, dass ich die vorliegende Masterarbeit selbstständig und nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertigt habe. Die aus fremden Quellen direkt oder indirekt übernommenen Stellen sind als solche kenntlich gemacht.

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