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Nichtlineare Optimierung

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Übungsaufgaben 8

Problem 1. The **KKT** conditions were shown to hold for cases where the **Abadie CQ** is valid. This assumption is not always easy to verify in practice, which is why the following **Mangasarian-Formovitz CQ** (**MFCQ** for short) is relevant which we define now.

A point

$$\mathbf{x} \in \mathscr{X} \equiv \mathscr{X}_{\mathbf{g},\mathbf{h}} = \Big\{\mathbf{x} \in \mathbb{R}^n : h_j(\mathbf{x}) = 0 \ \ (1 \leq j \leq p), \quad g_i(\mathbf{x}) \leq 0 \ \ (1 \leq i \leq m)\Big\}$$

is said to satisfy **MFCQ** if

(i) $\big\{\nabla h_j(\mathbf{x}) : 1 \leq j \leq p\big\}$ are linearly independent,

(ii) $\exists\, \mathbf{d} \in \mathbb{R}^n : \big(\nabla g_i(\mathbf{x}), \mathbf{d}\big)_{\mathbb{R}^n} < 0 \ (i \in I(\mathbf{x}))$ and $\big(\nabla h_j(\mathbf{x}), \mathbf{d}\big)_{\mathbb{R}^n} = 0 \ (1 \leq j \leq p)$.

(Note that (ii) may be rephrased as: $\exists\, \mathbf{d} \in \mathcal{T}_{\text{strict}}(\mathbf{x})$; see **Problem 3** in **Homework 7**).

a) Show that $\mathbf{x} \in \mathscr{X}$ satisfies **Abadie CQ** if it satisfies **MFCQ**.

b) Consider $\mathscr{X} := \mathscr{X}_{\mathbf{g}} \subset \mathbb{R}^2$, where

$$g_1(x_1, x_2) = x_2 - x_1^2 \quad \text{and} \quad g_2(x_1, x_2) = -x_2\,.$$

Show that $(0, 0)$ does not satisfy **MFCQ** but satisfies **Abadie CQ**.

Hint: for a) We need to show that $\mathbf{e} \in \mathcal{T}_{\text{lin}}(\mathbf{x})$ implies $\mathbf{e} \in \mathcal{T}_{\mathscr{X}}(\mathbf{x})$. Therefore, consider ($k \in \mathbb{N}$)

$$\mathbf{e}_k := \mathbf{e} + \frac{1}{k}\mathbf{d} \qquad \text{where } \mathbf{d} \text{ is from (ii).}$$

By evidence, $\mathbf{e}_k \in \mathcal{T}_{\text{lin}}(\mathbf{x})$ for all $k \in \mathbb{N}$. Similar to **Problem 1** of **Homework 6**, for every k there exists a C^1-path $\mathbf{X}_k : (-\epsilon_k, \epsilon_k) \to \mathbb{R}^n$ such that $\mathbf{h}(\mathbf{X}_k(t)) = \mathbf{0} \ \ \forall |t| < \epsilon_k$, such that $\mathbf{X}_k(0) = \mathbf{x}$ and $\mathbf{X}_k'(0) = \mathbf{e}_k$.

Problem 2. Let $(\mathbf{x}^*, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ be a **KKT**-point of the optimization problem:

$$\min f(\mathbf{x}) \qquad \text{s.t.} \qquad \mathbf{x} \in \mathscr{X} = \mathscr{X}_{\mathbf{g},\mathbf{h}} = \big\{\mathbf{x} \in \mathbb{R}^n : \ \mathbf{g}(\mathbf{x}) \leq \mathbf{0}, \ \mathbf{h}(\mathbf{x}) = \mathbf{0}\big\}, \tag{1}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}, \ \mathbf{g} : \mathbb{R}^n \mapsto \mathbb{R}^m$, and $\mathbf{h} : \mathbb{R}^n \mapsto \mathbb{R}^p$ each are continuously differentiable. Show that $\mathbf{x}^*$ is a stationary point of (1), i.e.,

$$\big(\nabla f(\mathbf{x}^*), \mathbf{d}\big)_{\mathbb{R}^n} \geq 0 \qquad \forall \mathbf{d} \in \mathcal{T}_{\mathscr{X}}(\mathbf{x}^*).$$

Problem 3. Let $C : \mathbb{R} \to \mathbb{R}$ be given as follows:

$$C(y) = \begin{cases} (y-1)^2 & \text{for } y > 1 \\ 0 & \text{for } -1 \le y \le 1 \\ (y+1)^2 & \text{for } y < -1\,. \end{cases}$$

We define $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ as follows,

$$g_i(x_1, x_2) = C(x_1) - x_2\,, \qquad g_2(x_1, x_2) = C(x_1) + x_2\,.$$

Let $f \in C^1(\mathbb{R}^2)$ be convex. Consider the problem:

$$\min f(\mathbf{x}) \qquad \text{s.t.} \qquad \mathbf{x} \in \mathscr{X} \equiv \mathscr{X}_{\mathbf{g}} := \left\{\mathbf{x} \in \mathbb{R}^2 :\ g_i(x_1, x_2) \le 0 \quad (1 \le i \le 2)\right\}.$$

Show that this problem does not satisfy **Slater's condition**, that that it e.g. satisfies **Abadie CQ** at $\mathbf{x} = (0, 0)$.

Abgabe: 12.06.2025.