EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Mathematisch-Naturwissenschaftliche Fakultät

Fachbereich Mathematik

Prof. Dr. Andreas Prohl
Benita Tscheppe

Nichtlineare Optimierung

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Übungsaufgaben 6

Problem 1. Let $h_j \in C^1(\mathbb{R}^n)$, for $1 \leq j \leq p$. Consider the regular point

$$\mathbf{x} \in \mathscr{X} = \left\{\mathbf{z} \in \mathbb{R}^n;\ h_j(\mathbf{z}) = 0\ (1 \leq j \leq p)\right\}.$$

Show that for every tangential vector $\mathbf{y} \in \mathscr{T}_{\mathscr{X}}(\mathbf{x})$ there exists a path $\psi : (-\varepsilon, \varepsilon) \to \mathscr{X}$ such that

$$\psi(0) = \mathbf{x}, \qquad \psi'(0) = \mathbf{y}.$$

Hint: Use the implicit function theorem.

Problem 2. For $1 \leq j \leq p$, let $h_j \in C^2(\mathbb{R}^n)$, and $f \in C^2(\mathbb{R}^n)$. Let the regular point $\mathbf{x}^* \in \mathscr{X}$ be a local minimum of f on

$$\mathbf{x} \in \mathscr{X} = \{\mathbf{x} \in \mathbb{R}^n;\ h_j(\mathbf{x}) = 0\ (1 \leq j \leq p)\}.$$

Then show that

$$\big(\mathbf{y}, \nabla^2 \mathscr{L}^2(\mathbf{x}^*, \boldsymbol{\lambda})\mathbf{y}\big) \geq 0 \qquad \forall\, \mathbf{y} \in \mathscr{T}_{\mathscr{X}}(\mathbf{x}^*),$$

where $\mathscr{L} : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is the Lagrange function.

Hint: Use again the path $\psi : (-\varepsilon, \varepsilon) \to \mathscr{X}$ such that $\psi(0) = \mathbf{x}^*$ and $\psi'(0) = \mathbf{y}$.

Problem 3. a) Let

$$f(x_1, x_2) = x_1 + x_2 \qquad \textit{and} \qquad h(x_1, x_2) = x_1^2 + x_2^2 - 2.$$

Consider the problem:

$$\min f(x_1, x_2) \qquad \text{subject to} \qquad (x_1, x_2) \in \mathscr{X} = \{\mathbf{x} \in \mathbb{R}^2;\ h(\mathbf{x}) = \mathbf{0}\}.$$

Use the optimality conditions to compute $\mathbf{x}^* \in \mathscr{X}$.

b) Let $h_1, h_2 : \mathbb{R}^2 \to \mathbb{R}$ be two maps, with

$$h_1(x_1, x_2) = (x_1 - 1)^2 + x_2^2 - 1, \qquad h_2(x_1, x_2) = (x_1 - 2)^2 + x_2^2 - 4.$$

Consider the constrained optimization problem:

$$\min f(x_1, x_2) \qquad \text{subject to} \qquad (x_1, x_2) \in \mathscr{X} = \{\mathbf{x} \in \mathbb{R}^2;\ h_1(\mathbf{x}) = h_2(\mathbf{x}) = \mathbf{0}\},$$

where f is defined as in part **a)**. Argue that the problem has a local minimum at $\mathbf{x}^* = \mathbf{0}$, but that there does *not* exist a Lagrange multiplier for this local minimum.

Abgabe: 29.05.2025.