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Fachbereich Mathematik

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Nichtlineare Optimierung

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Übungsaufgaben 4

Problem 1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be strictly convex, two times differentiable. Consider the iteration ($t \in \mathbb{N}_0$)

$$\mathbf{x}^{(t+1)} = \mathbf{x}^{(t)} + \mathbf{r}^{(t)}\,, \qquad \text{where} \qquad \mathbf{r}^{(t)} = -\boldsymbol{D}^{(t)}\nabla f(\mathbf{x}^{(t)})\,, \quad \text{with} \quad \boldsymbol{D}^{(t)} = [\nabla^2 f(\mathbf{x}^{(t)})]^{-1}\,. \tag{1}$$

a) Show that $\mathbf{r}^{(t)} \in \mathbb{R}^d$ is descent direction.

b) Show that scheme (1) is *Newton's method* to compute a *stationary point* $\mathbf{x}^*$ of f. In particular for every $t \in \mathbb{N}_0$, the Newton iterate $\mathbf{x}^{(t+1)} \in \mathbb{R}^d$ from (1) may me interpreted as minimizer of the *quadratic approximation* $f^{(t)} : \mathbb{R}^d \to \mathbb{R}$ *of* f *in the neighborhood* f *of* $\mathbf{x}^{(t)}$*, where*

$$\mathbf{x} \mapsto f^{(t)}(\mathbf{x}) := f(\mathbf{x}^{(t)}) + \big(\nabla f(\mathbf{x}^{(t)}), \mathbf{x} - \mathbf{x}^{(t)}\big) + \frac{1}{2}\Big(\mathbf{x} - \mathbf{x}^{(t)}, \nabla^2 f(\mathbf{x}^{(t)})[\mathbf{x} - \mathbf{x}^{(t)}]\Big)\,.$$

Problem 2. Let $-\infty < K \leq f \in C^2(\mathbb{R}^n)$ for some $K \in \mathbb{R}$, which is strictly convex and coercive.

a) Let $\{\mathbf{x}^{(t)}\}_{t\in\mathbb{N}_0} \subset \mathbb{R}^n$ be the iterates of the *globalized Newton method*. Show that the whole sequence $\{\mathbf{x}^{(t)}\}_{t\in\mathbb{N}_0}$ converges to the minimum $\mathbf{x}^* \in \mathbb{R}^n$.

b) Choose $\gamma \in (0, \frac{1}{2})$. For $\varepsilon > 0$, let $\mathcal{B}_\varepsilon(\mathbf{x}^*) \subset \mathbb{R}^n$ denote the ball of radius ε around the minimum $\mathbf{x}^*$ of f. Show that there exists $\varepsilon > 0$ sufficiently small, such that for all $\mathbf{x} \in \mathcal{B}_\varepsilon(\mathbf{x}^*) \setminus \{\mathbf{x}^*\}$, and $\mathbf{r}$ from $(1)_2$ in **Problem 1**, thre holds for all $\alpha \in (0, 1]$:

$$f(\mathbf{x} + \alpha\mathbf{r}) - f(\mathbf{x}) \leq \alpha\gamma\big(\nabla f(\mathbf{x}), \mathbf{r}\big)\,.$$

Abgabe: 15.05.2025.