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Nichtlineare Optimierung

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Übungsaufgaben 2

Problem 1. Let $\boldsymbol{Q} \in \mathbb{R}^{n\times n}_{\mathtt{spd}}$ and $\mathbf{c} \in \mathbb{R}^n$ be given. Consider the quadratic function

$$f(\mathbf{x}) := \frac{1}{2}\langle \mathbf{x}, \boldsymbol{Q}\mathbf{x}\rangle - \langle \mathbf{c}, \mathbf{x}\rangle \,.$$

i) Show that f attains its minimum at $\mathbf{x}^* \in \mathbb{R}^n$ *if and only if* $\boldsymbol{Q}\mathbf{x}^* = \mathbf{c}$.

ii) Let

$$\boldsymbol{Q} = \begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}, \qquad \mathbf{c} = \begin{pmatrix} 2 \\ -8 \end{pmatrix}.$$

We want to use the gradient descent algorithm for the computation of the minimum, starting at $\mathbf{x}_0 = (-2,-2)^\top$. Compute $\{(\mathbf{x}_k, \mathbf{r}_k, \alpha_k) \in [\mathbb{R}^n]^2 \times \mathbb{R};\ 0 \le k \le 4\}$.

Hint: Use **Problem 3** from **Sheet 1** to compute the step size in **ii)**, or the corresponding formula from the lecture.

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Consider the gradient descent method

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \alpha_k \nabla f(\mathbf{x}_k) \qquad (k \in \mathbb{N}_0)\,. \tag{1}$$

i) Show that

$$\|\mathbf{x}_{k+1} - \mathbf{y}\|^2 \le \|\mathbf{x}_k - \mathbf{y}\|^2 - 2\alpha_k\big(f(\mathbf{x}_k) - f(\mathbf{y})\big) + \big(\alpha_k\|\nabla f(\mathbf{x}_k)\|\big)^2 \qquad \forall\, \mathbf{y} \in \mathbb{R}^n\,.$$

ii) Suppose that

$$\sum_{k=0}^{\infty} \alpha_k = \infty \qquad \text{and} \qquad \lim_{k\uparrow\infty} \alpha_k \|\nabla f(\mathbf{x}_k)\|^2 = 0\,.$$

Show that

$$\liminf_{k\uparrow\infty} f(\mathbf{x}_k) = \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x})\,.$$

Hint for ii): Argue by contradiction, using i).

Problem 3.

i) Let $f \in C^1(\mathbb{R}^n)$. Fix $\mathbf{x} \in \mathbb{R}^n$ where $\nabla f(\mathbf{x}) \neq \mathbf{0}$. Let $\mathbf{d}^* \in \mathbb{S}^{n-1} := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 = 1\}$ be a solution of the problem:

$$\min_{\mathbf{d}\in\mathbb{S}^{n-1}} \left\langle \nabla f(\mathbf{x}), \mathbf{d} \right\rangle .$$

Show that $\mathbf{d}^* := -\frac{\nabla f(\mathbf{x})}{\|\nabla f(\mathbf{x})\|_2}$ is the *only* solution of this problem.

ii) Consider the gradient descent method (1), *i.e.*, $\mathbf{r}_k := -\nabla f(\mathbf{x}_k)$. Determine the step size $\alpha_k \in \mathbb{R}^+$ as follows,

$$f\big(\mathbf{x}_k + \alpha_k \mathbf{r}_k\big) := \min_{\sigma\geq 0} f\big(\mathbf{x}_k + \sigma \mathbf{r}_k\big) .$$

Show that $\mathbf{r}_{k+1} \perp \mathbf{r}_k$ for all $k \in \mathbb{N}_0$.

Abgabe: 02.05.2025.