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Fachbereich Mathematik

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Nichtlineare Optimierung

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Übungsaufgaben 1

Problem 1. Consider the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ of the form:

$$f(\mathbf{x}) := \frac{1}{2}\langle \mathbf{x}, \boldsymbol{Q}\mathbf{x}\rangle + \langle \mathbf{c}, \mathbf{x}\rangle + \gamma\,,$$

where $\boldsymbol{Q} \in \mathbb{R}^{n\times n}_{\text{sym}}$, $\mathbf{c} \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$. Show that

i) f is convex if and only if $\boldsymbol{Q}$ is positive semi-definite.

ii) f is strictly convex if and only if $\boldsymbol{Q}$ is positive definite.

Problem 2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function such that $\mathbf{x}^* \in \mathbb{R}^n$ is a local minimum of f along every line passing through the point $\mathbf{x}^*$.

i) Show that $\nabla f(\mathbf{x}^*) = 0$.

ii) Can you find an example where $\mathbf{x}^*$ may not be a local minimum of f although it satisfies the property above?

Problem 3. Consider the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(\mathbf{x}) := \frac{1}{2}\langle \mathbf{x}, \boldsymbol{Q}\mathbf{x}\rangle + \langle \mathbf{c}, \mathbf{x}\rangle\,, \tag{1}$$

where $\boldsymbol{Q} \in \mathbb{R}^{n\times n}_{\text{spd}}$ (so it is symmetric, positive definite) and $\mathbf{c} \in \mathbb{R}^n$. Fix $\mathbf{x} \in \mathbb{R}^n$. To this point, we attach a *search direction* $\mathbf{d} \in \mathbb{R}^n$ — which is a direction in which values of f decay (at least locally). The minimum value $\sigma^* \geq 0$ of the following minimization problem

$$f(\mathbf{x} + \sigma^*\mathbf{d}) = \min_{\sigma \geq 0} f(\mathbf{x} + \sigma\mathbf{d})$$

will be referred to as *step size* in the lecture.

i) Discuss why $\sigma^* > 0$.

ii) Show that the choice $\sigma = \sigma^*$ guarantees

$$f(\mathbf{x} + \sigma\mathbf{d}) - f(\mathbf{x}) \leq \sigma\,\gamma\langle \nabla f(\mathbf{x}), \mathbf{d}\rangle \tag{2}$$

for all $\gamma \in (0, \frac{1}{2}]$. The inequality (2) is known as *Armijo's condition*, which will appear in the lecture shortly.

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