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Tübingen, den 27.06.2018

8. Exercise sheet for Numerik für Differentialgleichungen auf Oberflächen

Exercise 19. Assume that the surface and its evolution is sufficiently smooth. Prove the following geometric estimates:

$$\|d\|_{L^\infty(\Gamma_h(t))} \le ch^2, \qquad \|1-\delta_h\|_{L^\infty(\Gamma_h(t))} \le ch^2,$$
$$\|\nu - \nu_h\|_{L^\infty(\Gamma_h(t))} \le ch, \qquad \text{and}$$
$$\|(\partial_h^\bullet)^{(\ell)} d\|_{L^\infty(\Gamma_h(t))} \le ch^2, \qquad \|(\partial_h^\bullet)^{(\ell)} \delta_h\|_{L^\infty(\Gamma_h(t))} \le ch^2.$$

Hint. For the last estimate use the formula

$$\delta_h(x) = \nu(x) \cdot \nu_h(x) \prod_{j=1}^m (1 - d(x,t)K_j(x)) \qquad (x \in \Gamma_h(t)),$$

where $K_j(x) = \frac{\kappa_j(y(x,t))}{1+d(x,t)\kappa_j(y(x,t))}$ with κ_j being the principle curvatures.

** You can try to prove this expression.

Exercise 20. Consider the lifted material points

$$y(t) = y(x(t),t) \in \Gamma(t),$$

with $x(t) \in \Gamma_h(t)$, for $0 \le t \le T$.

The velocity of the lifted material points is, counter-intuitively, not the lift of the discrete velocity V_h. Derive an expression for this material velocity, denoted by v_h, which gives the ODE for the material points:

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = v_h(y(t),t)$$
$$y(0) = y^0 \in \Gamma^0.$$

Hint. Use the definition of the lift:

$$y(t) = x(t) - \nu(y(t),t) d(x,t).$$

Exercise *. Let $1 \le k \le 5$, and consider the values

$$\boldsymbol{u}^{n-1}, \boldsymbol{u}^{n-2}, \dots, \boldsymbol{u}^{n-k}$$

be given, and which approximate $u(t_{n-j})$, on equidistant steps with $\tau > 0$. Find an approximation to the unknown

$$\boldsymbol{u}(t_n).$$

Hint. Try the special cases $k = 1$ and/or $k = 2$.

Programming exercise 3. Consider the parabolic problem on the evolving surface $\Gamma(t)$, with velocity v given by `func_v.m` (bouncing–ellipsoid example),

$$\begin{aligned} \partial^\bullet u + u\nabla_{\Gamma(t)} \cdot v - \Delta_{\Gamma(t)} u &= f && \text{on} \quad \Gamma(t) \quad \text{for} \quad [0,T], \\ u(\cdot,0) &= u^0 && \text{on} \quad \Gamma^0 = \{|x| = 1\}. \end{aligned}$$

(a) Assume that the exact solution is given to be, with $x = (x_1, x_2, x_3)$,

$$u(x,t) = e^{-6t} x_1 x_2.$$

The functions `func_sol.m` and `func_f.m` are given.

(b) Approximate the above problem using surface finite elements as a space discretisations, combined with a k-step BDF method.

To assemble the mass and stiffness matrices use the already implemented function `[A,M]=surface_assembly(Elements,Nodes)` from PA1.

Evolve the discrete surface as discussed in A17.

(c) Use the 3-step BDF method to solve the corresponding fully discretised until with $T = 1$. Using all meshes from PA1

`Sphere_elements_j.txt`
`Sphere_nodes_j.txt` $\quad j = 0, \ldots, 5,$

and all the time step sizes $\tau = 0.2, 0.1, 0.05, 0.025, 0.0125$.

Compute the following errors of the numerical solution, when compared to the exact solution (given by `func_sol.m` from above).

As an output generate two convergence plots* using the errors measured in the L^2 norm and the H^1 semi-norm at step N (with $N\tau = 1$):

$$\begin{aligned} \|(u_h^N)^\ell - u(\cdot,T)\|^2_{L^2(\Gamma(T))} &\approx \|\boldsymbol{e}^N\|^2_{\boldsymbol{M}} = (\boldsymbol{e}^N)^T \boldsymbol{M} \boldsymbol{e}^N, \\ \|\nabla((u_h^N)^\ell - u(\cdot,T))\|^2_{L^2(\Gamma(T))} &\approx \|\boldsymbol{e}^N\|^2_{\boldsymbol{A}} = (\boldsymbol{e}^N)^T \boldsymbol{A} \boldsymbol{e}^N. \end{aligned}$$

- In the first figure plot the error curves for each mesh against the step size.
- In the second figure plot the error curves for each time step size against the mesh width.

* As discussed before. Also see the example files!

Bonus: Explain the behaviour of the convergence plots. Does the "flatting out" of the curves contradict the convergence theory?

Bonus: Do the same using an implicit Runge–Kutta methods (Radau IIA $s = 2$).

The functions and the grid arrays can be found at `https://na.uni-tuebingen.de/ex/surfPDE_ss18/PA3.zip`.

Discussed on the tutorials on 03.07.2018. The programming exercise is due on 11.07.2018, 12 s.t.