Exercise Sheet-4

Stochastic Differential Equations
Dr. Chaudhary

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Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $W = \{W(t), t \geq 0\}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$.

1. Show that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} (\Delta_i^n W)^2 = T \qquad \text{in } L^2(\Omega),$$

   where $\Delta_i^n = W(t_{i+1}^n) - W(t_i^n)$, $t_i = \frac{iT}{n}$.

2. Find the following limits in $L^2(\Omega)$:

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} W(t_i^n)(W(t_{i+1}^n) - W(t_i^n))$$

   and

$$\lim_{n\to\infty} \sum_{j=0}^{n-1} W(t_{i+1}^n)\big(W(t_{i+1}^n) - W(t_i^n)\big).$$

3. Show that each random step process $f \in M^2_{\text{step}}$,

$$I_t(f) = \int_0^t f(s)\mathrm{d}W(s)$$

   is a martingale.

4. Verify the following identities

$$\int_0^T W(t)\mathrm{d}W(t) = \frac{1}{2}W(T)^2 - \frac{1}{2}T, \qquad \int_0^T t\mathrm{d}W(t) = TW(T) - \int_0^T W(t)\mathrm{d}t,$$

   and

$$\int_0^T W(t)^2 dW(t) = \frac{1}{3}W(T)^3 - \int_0^T W(t)\mathrm{d}t.$$

Deadline: 15th Dec 2023, 12:00.