Exercise Sheet-3

Stochastic Differential Equations
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1 Exercise: General Properties of Brownian motion

Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $W = \{W(t), t \geq 0\}$ be a Brownian motion on $(\Omega, \mathcal{F}, P)$.

1. Show that

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi t}} \exp(\frac{-x^2}{2t}) \ \forall \ x \in \mathbb{R}, t > 0$$

   is probability density of $W(t)$ and find the expectation and variance of $W(t)$.

2. Show that

$$E[W_s W_t] = \min\{s, t\} \ \forall \ s, t \geq 0.$$

3. show that

$$\mathbb{E}\big[|W(t) - W(s)|^2\big] = |t - s| \ \forall \, s, t \geq 0.$$

4. Let $t > 0$. Compute the characteristic function

$$\phi_{W(t)}(\lambda) := \mathbb{E}(\exp(i\lambda W(t))) \ \forall \ \lambda \in \mathbb{R}.$$

5. Compute $\mathbb{E}\big[|W(t)|^4\big]$ with the help of part (4).

Deadline: 17th Nov 2023, 12:00.