Exercise Sheet-1

Stochastic Differential Equations
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November 6, 2023

1 Exercise: General Properties of Conditional Expectation

Consider a probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$.

1. **Linearity:** For any random variables X and Y and constants $a, b \in \mathbb{R}$, prove that $\mathbb{E}[aX + bY \mid \mathcal{G}] = a\mathbb{E}[X \mid \mathcal{G}] + b\mathbb{E}[Y \mid \mathcal{G}]$.

2. **Taking Out What's Known:** Show that if Y is $\mathcal{G}$-measurable, then $\mathbb{E}[XY \mid \mathcal{G}] = Y \cdot \mathbb{E}[X \mid \mathcal{G}]$.

3. **Tower Property:** Prove that for sub-sigma-algebras $\mathcal{H} \subset \mathcal{G} \subset \mathcal{F}$, $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}] \mid \mathcal{H}] = \mathbb{E}[X \mid \mathcal{H}]$.

4. **Independence:** If X is independent of $\mathcal{G}$, show that $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$.

5. **Positivity:** Prove that if X is a non-negative random variable, then $\mathbb{E}[X \mid \mathcal{G}] \geq 0$.

Deadline: 10th Nov 2023, 12:00.