EBERHARD KARLS UNIVERSITÄT TÜBINGEN

Mathematisch-Naturwissenschaftliche Fakultät

Fachbereich Mathematik

Prof. Dr. Andreas Prohl
Dr. Akash Panda

Stochastische Differentialgleichungen

Sommer-Semester 2022 **Tübingen, 22.06.2022**

Homework 9

Problem 1. Fix $T > 0$, a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and consider the $\mathbb{R}^d$-valued process $\mathbf{X} = \{\mathbf{X}_t;\, 0 \le t \le T\}$ on it. Assume that there exist numbers $\alpha, \beta, C > 0$, such that

$$\mathbb{E}\big[\|\mathbf{X}_t - \mathbf{X}_s\|^\alpha_{\mathbb{R}^d}\big] \le C|t-s|^{1+\beta} \qquad \forall\, s,t \in [0,T]\,. \tag{1}$$

Show that there exists a Hölder continuous modification $\widetilde{\mathbf{X}}$ of $\mathbf{X}$ with exponent $\gamma \in [0, \frac{\beta}{\alpha})$.

Remark & Hint: **1.** This is **Kolmogorov's continuity theorem**, which we *e.g.* applied before for $\mathbf{W}$. **2.** For its proof, apply **Borel-Carelli lemma**, after estimating

$$\mathbb{P}\big[\max_{1\le j\le 2^J} \|\mathbf{X}_{t_j} - \mathbf{X}_{t_{j-1}}\|_{\mathbb{R}^d} \ge 2^{-\gamma J}\big] \qquad \text{along/on a mesh} \quad \mathcal{I}_k = \{t_j\}_{j=0}^{2^J}$$

with the help of Tschebycheff's inequality, and (1).

Problem 2. Fix $T > 0$, and $\mathbf{x} \in \mathbb{R}^d$. In the lecture, we associated the family of maps $\Psi := \{\Psi_{s,t};\, 0 \le s \le t \le T\}$ to the strong solution $\mathbf{X}^{\mathbf{x}} \equiv \{\mathbf{X}^{\mathbf{x}}_t;\, 0 \le t \le T\}$ of the SDE

$$\mathrm{d}\mathbf{X}^{\mathbf{x}}_t = \mathbf{b}(\mathbf{X}^{\mathbf{x}}_t)\mathrm{d}t + \boldsymbol{\sigma}(\mathbf{X}^{\mathbf{x}}_t)\mathrm{d}\mathbf{W}_t \quad \forall\, t \in [0,T]\,, \qquad \mathbf{X}^{\mathbf{x}}_0 = \mathbf{x}\,, \tag{2}$$

where data $(\mathbf{b}, \boldsymbol{\sigma})$ are Lipschitz, and of (sub-)linear asymptotic growth. Show that Ψ is a **Brownian flow.**

Problem 3. Fix $T > 0$, and $\mathbf{x} \in \mathbb{R}^d$. In the lecture, we associated the $\mathbb{R}^{d\times d}$-valued **1st variation process** $\boldsymbol{J}(\mathbf{x}) \equiv \{\boldsymbol{J}_t(\mathbf{x});\, 0 \le t \le T\}$ to $\mathbf{X}^{\mathbf{x}}$ from **SDE (2)**. Verify the related theorem in the lecture which asserts that $\boldsymbol{J}(\mathbf{x})$ solves the linear SDE

$$\boldsymbol{J}_t(\mathbf{x}) = 1_{\mathbb{R}^d} + \int_0^t \nabla_{\mathbf{x}}\mathbf{b}(\mathbf{X}^{\mathbf{x}}_s)\boldsymbol{J}_s(\mathbf{x})\,\mathrm{d}s + \sum_{\ell=1}^d \nabla_{\mathbf{x}}\boldsymbol{\sigma}_\ell(\mathbf{X}^{\mathbf{x}}_s)\boldsymbol{J}_s(\mathbf{x})\mathrm{d}W^\ell_s \qquad \forall\, t \in [0,T]\,. \tag{3}$$

Remark & Hint: **1.** Instead of (3), we may consider the following SDE

$$\mathrm{d}\boldsymbol{\eta}^{\mathbf{h}}_t = \nabla_{\mathbf{x}}\mathbf{b}(\mathbf{X}^{\mathbf{x}}_t)\boldsymbol{\eta}^{\mathbf{h}}_t\mathrm{d}t + \sum_{\ell=1}^d \nabla_{\mathbf{x}}\boldsymbol{\sigma}_\ell(\mathbf{X}^{\mathbf{x}}_t)\boldsymbol{\eta}^{\mathbf{h}}_t\mathrm{d}W^\ell_t \qquad \forall\, t \in [0,T]\,, \qquad \boldsymbol{\eta}^{\mathbf{h}}_0 \equiv \boldsymbol{\eta}^{\mathbf{h}}_0(\mathbf{x}) = \mathbf{h}\,, \tag{4}$$

for arbitrary $\mathbf{h} \in \mathbb{R}^d$.

2. Now verify ($p \geq 1$)

$$\Big(\mathbb{E}\Big[\sup_{0\leq t\leq T} \|\mathbf{X}_t^{\mathbf{x}+\mathbf{h}} - \mathbf{X}_t^{\mathbf{x}} - \boldsymbol{\eta}_t^{\mathbf{h}}\|_{\mathbb{R}^d}^p\Big]\Big)^{1/p} \leq C_p \|\mathbf{h}\|_{\mathbb{R}^d}^2 \qquad \forall\, \mathbf{h} \in \mathbb{R}^d .$$

Now use **BDG**-inequality, and **mean-value theorem** to show this estimate, and then conclude from it that indeed $\boldsymbol{J}_t(\mathbf{x}) = \nabla_{\mathbf{x}} \Psi_{0,t}(\mathbf{x})$.

Problem 4. Consider the following equation on $\mathbb{R}^d$ $(d \geq 1)$:

$$\begin{cases} \partial_t u(t,\mathbf{x}) & = \frac{1}{2}\Delta u(t,\mathbf{x}), \\ u(0,\mathbf{x}) & = f(\mathbf{x}), \end{cases} \tag{5}$$

where $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function. Then we have the well-known Feynman-Kac formula for the solution of the above equation

$$u(t,\mathbf{x}) = \mathbb{E}\big[f\big(\mathbf{x} + \mathbf{W}_t\big)\big] ,$$

where $\mathbf{W}$ is an $\mathbb{R}^d$-valued Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{I}_{k^J} = \{t_j^J\}_{j=0}^J \subset [0,1]$ be an equi-distant mesh of size k^J.

(i) Let $d = 1$ and $f(x) = \sin(\pi x)$. Consider points $x_1 = -10, x_2 = -8, x_3 = -6, \cdots, x_{10} = 8, x_{11} = 10$, and fix $J = 100$ and $\mathtt{M} = 3000$. For each $j = 0, \cdots, J$, compute

$$u(t_j^J, x_i) \approx \mathbb{E}_{\mathtt{M}}\Big[f\big(x_i + \sum_{j=0}^{J-1} \Delta_j^J W\big)\Big] \quad (i = 1, \cdots, 10) ,$$

with Wiener increments $\Delta_j^J W := W_{t_{j+1}^J} - W_{t_j^J} \sim \sqrt{k^J}\mathcal{N}(0,1)$.

(ii) Let $d = 4$. For $\mathbf{x} \in \mathbb{R}^4$, take $f(\mathbf{x}) = \prod_{\ell=1}^4 \sin(\pi \mathbf{x}_\ell)$, where

$$\mathbf{x}_1 = (1,0,0,0)^\top \quad \mathbf{x}_2 = (0,1,0,0)^\top \quad \mathbf{x}_3 = (0,0,1,0)^\top \quad \mathbf{x}_4 = (0,0,0,1)^\top .$$

Fix $J = 100$ and $\mathtt{M} = 3000$. For $i = 1, \cdots, 4$, plot the temporal evolution

$$j \mapsto u(t_j^J, \mathbf{x}_i) \approx \mathbb{E}_{\mathtt{M}}\Big[f\big(\mathbf{x}_i + \sum_{\ell=0}^{J-1} \Delta_\ell^J \mathbf{W}\big)\Big] .$$

Date of Submission: 29.06.2022.