EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Mathematisch-Naturwissenschaftliche Fakultät

Fachbereich Mathematik

Prof. Dr. Andreas Prohl
Dr. Akash Panda

Stochastische Differentialgleichungen

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Homework 8

Problem 1. Fix $T > 0$. To construct a strong solution for an SDE with data $(\mathbf{b}, \boldsymbol{\sigma})$ which satisfy **Assumption 1**, the first step of the proof was to establish solvability (in $\mathbb{Y} := L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^d))$) for its truncation $n \in \mathbb{N}$, *i.e.*,

$$\mathrm{d}\mathbf{X}^n_t = \mathbf{b}(\mathbf{X}^n_t)\mathrm{d}t + \boldsymbol{\sigma}_n(\mathbf{X}^n_t)\mathrm{d}\mathbf{W}_t \quad (0 \le t \le T)\,, \qquad \mathbf{X}^n_0 = \mathbf{x} \in \mathbb{R}^d\,. \tag{1}$$

To accomplish this goal, we introduced und studied in class the mapping $\Gamma_n : \mathbb{Y} \to \mathbb{Y}$, via $\boldsymbol{\zeta} = \Gamma_n(\boldsymbol{\eta})$, where

$$\mathrm{d}\boldsymbol{\zeta}_t = \mathbf{b}(\boldsymbol{\zeta}_t)\mathrm{d}t + \boldsymbol{\sigma}_n(\boldsymbol{\eta})\mathrm{d}\mathbf{W}_t \quad (0 \le t \le T)\,, \qquad \boldsymbol{\zeta}_0 = \mathbf{x} \in \mathbb{R}^d\,, \tag{2}$$

and used its re-formulation as *randomized ODE* to first construct *local*, and then *global* solutions via **Assumption 1.**

Then, we discussed that Γ_n is a contraction on $\widetilde{\mathbb{Y}} := C_{\mathbb{F}}\big([0,T]; L^2(\Omega;\mathbb{R}^d)\big)$ for small times $0 \le T \le T_0 < \infty$, by using Ito's formula, **Assumption 1**, Lipschitz property of $\boldsymbol{\sigma}_n$, and Gronwall's inequality.

Now show that the same property holds for Γ_n on $\mathbb{Y}$.

Hint: Proceed similarly as before, and use the results from the lecture, and **BDG-inequality**.

Problem 2. For every $n \in \mathbb{N}$, we have now established the existence (and uniqueness) of a solution $\mathbf{X}^n \in \mathbb{Y}$ of (1) by Banach FP theorem through (2), and a continuation argument. There remains to verify convergence of $\{\mathbf{X}^n\}_n \subset \mathbb{Y}$ to $\mathbf{X} \in \mathbb{Y}$, and the identification of the limit as strong solution of

$$\mathrm{d}\mathbf{X}_t = \mathbf{b}(\mathbf{X}_t)\mathrm{d}t + \boldsymbol{\sigma}(\mathbf{X}_t)\mathrm{d}\mathbf{W}_t \quad (0 \le t \le T)\,, \qquad \mathbf{X}_0 = \mathbf{x} \in \mathbb{R}^d\,. \tag{3}$$

In class, we therefore verified uniform bounds (in n) for solutions $\{\mathbf{X}^n\}_{n\in\mathbb{N}}$ of (1) **for the particular case** $q = 2$, *i.e.*:

$$\exists\, C > 0 : \qquad \sup_{n\in\mathbb{N}} \mathbb{E}\big[\|\mathbf{X}^n_{t\wedge \ell_{n,r}}\|^q_{\mathbb{R}^d}\big] \le C\Big(1 + \|\mathbf{x}\|^q_{\mathbb{R}^d}\Big)\exp\big(C_q t\big) \qquad \forall\, t \in [0,T]\,, \tag{4}$$

where, for a given $r > 0$, we introduced the stopping time

$$\ell_{n,r} = \begin{cases} T & \text{for } \|\mathbf{X}^n_t\|_{\mathbb{R}^d} < r \text{ for all } t \in [0,T]\,, \\ \inf\{t \in [0,T]; \|\mathbf{X}^n_t\|_{\mathbb{R}^d} \ge r\} & \text{else}\,. \end{cases} \tag{5}$$

Now generalize the proof in class for $q = 2$ of (4) to general $q \ge 2$. Then argue with the help of Fatou lemma that this implies the existence of $C_{q,T} > 0$, *s.t.*

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}\big[\|\mathbf{X}_t^n\|_{\mathbb{R}^n}^q\big] \leq C_{q,T}\,. \tag{6}$$

Problem 3. Based on **Assumption 1**, the *higher* moment bounds in (6) may now be used to verify:

$$\exists\, C > 0: \qquad \sup_{n \in \mathbb{N}} \|\mathbf{X}^n\|_{\mathbb{Y}} \leq C\,. \tag{7}$$

For this purpose, start again with

$$\mathbb{P}-\text{a.s.}: \qquad \mathbf{X}_t^n = \mathbf{x} + \int_0^t \mathbf{b}(\mathbf{X}_s^n)\,\mathrm{d}s + \int_0^t \boldsymbol{\sigma}_n(\mathbf{X}_s^n)\,\mathrm{d}\mathbf{W}_s \qquad \forall\, t \in [0,T]\,.$$

Now make squares, and use the **BDG-inequality**, in particular, to show (7).

Problem 4. The spread of an infection on a time interval $[0,T]$ subject to random disturbances is modeled via the stochastic SIR (Susceptible-Infectious-Recovered) equations, which are

$$\begin{cases} dS(t) &= \big[\alpha - \beta S(t)I(t) - \mu S(t)\big]\,\mathrm{d}t + \sigma_1\, S(t)\,\mathrm{d}W(t)\,, \\ dI(t) &= \big[\beta S(t)I(t) - (\mu+\rho+\gamma)I(t)\big]\,\mathrm{d}t + \sigma_2\, I(t)\,\mathrm{d}W(t)\,, \\ S(0) &= S_0, \qquad I(0) = I_0\,. \end{cases} \tag{8}$$

where

- $S(t)$ denotes the total *susceptible* population at time $t \in [0,T]$,
- $I(t)$ denotes the number of active *infections* at time $t \in [0,T]$,

and $\alpha, \mu, \rho, \beta, \gamma$ are given positive numbers. Here, $\sigma_1 > 0, \sigma_2 \neq 0$ are used to model the randomness in the evolution. Let $\{W(t)\}_{t \geq 0}$ be a Wiener process defined on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It can be shown that there exists a unique strong solution $\big\{(S(t), I(t));\, 0 \leq t \leq T\big\}$. Let $\mathcal{I}_{k^J} = \{t_j^J\}_{j=0}^J \subset [0,1]$ be an equi-distant mesh of size k^J. For each j, let the $\mathbb{R}^2$-valued tuple (S_j, I_j) denote the numerical approximation of $(S(t_j^J), I(t_j^J))$.

Use the Euler-Maruyama method to computationally study the system (8).

(a) Consider (8) with parameters $\alpha = 5, \beta = 5, \mu = 4, \rho = 1, \gamma = 1$. $\sigma_1 = \sigma_2 = 0$. For $(S_0, I_0) = (1,1)$, verify computationally that $I(t)$ converges a.s. to a non-zero number as $t \to \infty$. Then, consider $\sigma_1 = 2, \sigma_2 = -1$ and verify that $I(t) \to 0$ as $t \to \infty$, $\mathbb{P}$-a.s. (*i.e.,* the population will eventually be disease-free). To verify this, take $T = 30, k^J = 10^{-3}$ and plot the histogram of $I(t)$ for $t = 1, \cdots, 30$.

(b) For the same model parameters in (a) with non-zero σ_1 and σ_2, plot the histogram of $f^*(x)$ defined as

$$f^*(x) := \frac{125}{16} x^{-4} e^{-\frac{5}{2x}} \quad (x > 0)\,.$$

In the same snapshot, plot the empirical density of $S(t)$ at $t = 50$. Observe that $S(t) \to f^*$ as $t \to \infty$, $\mathbb{P}$-a.s.

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