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Stochastische Differentialgleichungen

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Homework 7

Problem 1. Fix $T > 0$. Let $\mathbf{W}$ be an $\mathbb{R}^m$-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{F} \equiv \mathbb{F}^{\mathbf{W}}$. Consider continuous maps

$$\mathbf{b} : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n \qquad \text{and} \qquad \boldsymbol{\sigma} : [0,T] \times \mathbb{R}^n \to \mathscr{L}(\mathbb{R}^m, \mathbb{R}^n)\,,$$

which satisfy the following properties for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all $t \in [0,T]$:

$$\begin{aligned}
&\text{(i)} \qquad \big|\mathbf{b}(t,\mathbf{x})\big|_{\mathbb{R}^n} + \|\boldsymbol{\sigma}(t,\mathbf{x})\|_{\mathscr{L}(\mathbb{R}^m,\mathbb{R}^n)} \leq C\big(1 + |\mathbf{x}|_{\mathbb{R}^n}\big)\,,\\
&\text{(ii)} \qquad \big|\mathbf{b}(t,\mathbf{x}) - \mathbf{b}(t,\mathbf{y})\big|_{\mathbb{R}^n} + \big\|\boldsymbol{\sigma}(t,\mathbf{x}) - \boldsymbol{\sigma}(t,\mathbf{y})\big\|_{\mathscr{L}(\mathbb{R}^m,\mathbb{R}^n)} \leq C\big|\mathbf{x} - \mathbf{y}\big|_{\mathbb{R}^n}\,.
\end{aligned}$$

Let $\boldsymbol{\xi} \in L^p(\Omega; \mathbb{R}^n)$ be given, for $p \geq 2$. Consider the SDE

$$\mathrm{d}\mathbf{X}_t = \mathbf{b}\big(t, \mathbf{X}_t\big)\mathrm{d}t + \boldsymbol{\sigma}\big(t, \mathbf{X}_t\big)\mathrm{d}\mathbf{W}_t \qquad (0 \leq t \leq T)\,, \qquad \mathbf{X}_0 = \boldsymbol{\xi}\,.$$

Show that there exist a constant $C > 0$, and a constant $C_p > 0$ which depends on p as well, such that

$$\mathbb{E}\big[\sup_{0\leq s\leq t} |\mathbf{X}_s|^p_{\mathbb{R}^n}\big] \leq C_p\big(\mathbb{E}[|\boldsymbol{\xi}|^p_{\mathbb{R}^n}] + 1\big)\exp\big(Ct\big)\,.$$

<u>**Hint**</u>: Use Ito's formula, and BDG-inequality.

Problem 2. Under the same conditions as in **Problem 1**, show that

$$\mathbb{E}\big[|\mathbf{X}_t - \mathbf{X}_s|^p_{\mathbb{R}^n}\big] \leq \widetilde{C}_p\big(\mathbb{E}\big[|\boldsymbol{\xi}|^p_{\mathbb{R}^n}\big] + 1\big)\big(t - s\big)^{\frac{p}{2}} \qquad (0 \leq s \leq t \leq T)\,,$$

with constants $\widetilde{C} > 0$ and $\widetilde{C}_p > 0$, where the latter are also depends on $p \geq 2$.

Problem 3. Assume the same conditions as in **Problem 1**. Let $\mathcal{I}_k = \{t_j\}_{j=0}^J$ be an equi-distant mesh of size $k \equiv k^J > 0$ that covers $[0,T]$. Consider iterates $\{\mathbf{Y}^j\}_{j=0}^J$ of the (explicit) Euler method

$$\mathbf{Y}^{j+1} - \mathbf{Y}^j = \mathbf{b}(\mathbf{Y}^j)k + \boldsymbol{\sigma}(\mathbf{Y}^j)\Delta_j\mathbf{W} \qquad (0 \leq j \leq J-1)\,, \qquad \mathbf{Y}^0 = \boldsymbol{\xi} \in L^4(\Omega; \mathbb{R}^n)\,,$$

where $\Delta_j\mathbf{W} := \mathbf{W}(t_{j+1}) - \mathbf{W}(t_j)$. Prove the existence of constants $C > 0$ and $C_p > 0$ such that

$$\sup_{k>0}\max_{0\leq j\leq J} \mathbb{E}\big[|\mathbf{Y}^j|^{2p}_{\mathbb{R}^n}\big] \leq C\big(\mathbb{E}[|\boldsymbol{\xi}|^{2p}_{\mathbb{R}^n}] + 1\big)\exp\big(CT\big)\,.$$

Problem 4. Fix $T > 0$. Let $\mathbf{W}$ be an $\mathbb{R}^m$-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{F} \equiv \mathbb{F}^{\mathbf{W}}$. Let $\mathbf{b} : \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{\sigma} : \mathbb{R}^n \to \mathbb{R}]^{n\times m}$ be locally Lipschitz functions, *i.e.*, for every $N \geq 0$ there exists $C_N > 0$, *s.t.*

$$|\mathbf{b}(\mathbf{x}) - \mathbf{b}(\mathbf{y})|_{\mathbb{R}^n} + \|\boldsymbol{\sigma}(\mathbf{x}) - \boldsymbol{\sigma}(\mathbf{y})\|_{\mathscr{L}(\mathbb{R}^m,\mathbb{R}^n)} \leq C_N|\mathbf{x} - \mathbf{y}|_{\mathbb{R}^n} \qquad \forall\, \mathbf{x}, \mathbf{y} \in [-N, N]^n\,.$$

i) Show that for every $\mathbf{x} \in \mathbb{R}^n$, we can find a stopping time τ_x, almost surely positive, and a stochastic process $\{\mathbf{X}_t^{\mathbf{x}};\, 0 \leq t < \tau_{\mathbf{x}}\}$, *s.t.*

$$\mathbf{X}_t^{\mathbf{x}} = \mathbf{x} + \int_0^t \mathbf{b}(\mathbf{X}_s^{\mathbf{x}})\,\mathrm{d}s + \int_0^t \boldsymbol{\sigma}(\mathbf{X}_s^{\mathbf{x}})\,\mathrm{d}\mathbf{W}_s \qquad \forall\, t < \tau_x\,. \tag{1}$$

ii) Show that the process $\{\mathbf{X}_t^{\mathbf{x}};\, 0 \leq t < \tau_{\mathbf{x}}\}$ is unique in the sense that if $\widetilde{\tau}_{\mathbf{x}}$ is an almost surely positive stopping time and if $\{\mathbf{Y}_t^{\mathbf{x}};\, 0 \leq< \widetilde{\tau}_{\mathbf{x}}\}$ is a stochastic process such that

$$\mathbf{Y}_t^{\mathbf{x}} = \mathbf{x} + \int_0^t \mathbf{b}(\mathbf{Y}_s^{\mathbf{x}})\,\mathrm{d}s + \int_0^t \boldsymbol{\sigma}(\mathbf{Y}_s^{\mathbf{x}})\,\mathrm{d}\mathbf{W}_s \qquad \forall\, t < \widetilde{\tau}_x\,, \tag{2}$$

then $\widetilde{\tau}_{\mathbf{x}} \leq \tau_{\mathbf{x}}$, and $\mathbb{P}$-a.s.,

$$\mathbf{Y}_t^{\mathbf{x}}\mathbb{1}_{\{t<\widetilde{\tau}_{\mathbf{x}}\}} = \mathbf{X}_t^{\mathbf{x}}\mathbb{1}_{\{t<\widetilde{\tau}_{\mathbf{x}}\}} \qquad \forall\, t \geq 0\,.$$

The process $\{\mathbf{X}_t^{\mathbf{x}};\, 0 \leq t < \tau_{\mathbf{x}}\}$ is called the solution of the SDE (1) up to the explosion time $\tau_{\mathbf{x}}$.

Problem 5. Fix $T > 0$, as well as $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and a W on it. We wish to approximate the following so-called SPDE with solution $u \equiv \{u(t,x);\; 0 \leq t \leq T,\, 0 \leq x \leq 1\}$ via the spatial discretization below,

$$\begin{cases} \partial_t u(t,x) &= -\Delta u(t,x) + \alpha\, \partial_t W(t,x) \qquad \forall\, (t,x) \in (0,T) \times [0,1]\,, \\ u(0,x) &= u_0(x) \qquad \forall\, x \in [0,1]\,, \\ u(t,0) &= u(t,1) = 0 \qquad \forall\, t \in (0,T)\,, \end{cases} \tag{3}$$

for $\alpha \in \{0, 1\}$. Here, $W \equiv W(t,x)$ is of the form

$$W(t,x) := \sum_{i=1}^{N} \gamma_i^{1/2}\, \sqrt{2}\sin(i\pi x)\, W_i(t) \tag{4}$$

for some $N \in \mathbb{N}$, and $\{W_i(t);\, 0 \leq t \leq T\}$ are *i.i.d.*, real-valued Brownian motions, and $\{\gamma_i\}_{i\geq 1}^N \subset \mathbb{R}_{>0}$. For the numerical approximation of (3), we discretize[1] in both, space and time. For every $0 \leq j \leq J-1$, let $\mathbf{u}_j^h$ denote an approximation

$$\mathbf{u}_j^h \approx \left[u(t_j^J, x_1), u(t_j^J, x_2), \cdots, u(t_j^J, x_L)\right]^\top,$$

which solves the following (implicit) discretization

$$\begin{aligned} \mathbf{u}_{j+1}^h - \mathbf{u}_j^h &= -k^J \boldsymbol{\Lambda}^h\, \mathbf{u}_{j+1}^h + \alpha\, \Delta_j \mathbf{W} \qquad (0 \leq j \leq J-1)\,, \\ \mathbf{u}_0^h &= \left[u_0(x_1), u_0(x_2), \cdots, u_0(x_L)\right]^\top \end{aligned} \tag{5}$$

[1] For a spatial discretization, we divide the interval $[0,1]$ into subintervals $I_j := [x_{j-1}, x_j]$ ($1 \leq j \leq L$) of equi-distant mesh of size $h \equiv h^L$, where L is a positive integer, and $0 = x_0 < x_1 < \cdots < x_{L-1} < x_L = 1$. For the time discretization, let $\mathcal{I}_{k^J} = \{t_j^J\}_{j=0}^J \subset [0,T]$ be an equi-distant mesh of size $k \equiv k^J$.

where

$$\mathbf{\Lambda}^h := \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & \cdots \\ -1 & 2 & -1 & \ddots \\ 0 & -1 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{R}^{L\times L}, \qquad \Delta_j \mathbf{W} := \begin{bmatrix} \sum_{i=1}^N \gamma_i^{1/2}\sqrt{2}\sin(i\pi x_1)\left(W_i(t_{j+1}^J) - W_i(t_j^J)\right) \\ \sum_{i=1}^N \gamma_i^{1/2}\sqrt{2}\sin(i\pi x_2)\left(W_i(t_{j+1}^J) - W_i(t_j^J)\right) \\ \vdots \\ \sum_{i=1}^N \gamma_i^{1/2}\sqrt{2}\sin(i\pi x_L)\left(W_i(t_{j+1}^J) - W_i(t_j^J)\right) \end{bmatrix}.$$

Fix $N = 5$ in (4). Consider the following three different initial data for $x \in [0,1]$:

$$\text{(i)} \quad u_0(x) = \sin(\pi x)\,, \qquad \text{(ii)} \quad u_0(x) = \begin{cases} x, & \text{for } x \in [0, 0.5] \\ 1-x, & \text{otherwise} \end{cases}, \qquad \text{(iii)} \quad u_0(x) = \mathbb{1}_{\{x=0.5\}}\,.$$

Plot a single trajectory of the solution $\mathbf{u}^h \equiv \left[\mathbf{u}_0^h, \cdots, \mathbf{u}_L^h\right]^\top$ till $T = 1$ for the following cases with the above mentioned initial data $\text{(i)} - \text{(iii)}$:

(a) $\alpha = 0$ in (5).

(b) $\alpha = 1$ in (5), $\gamma_i = 1$ for $i = 1, \cdots, 5$ in (4).

(c) $\alpha = 1$ in (5), $\gamma_i = 1/i^2$ for $i = 1, \cdots, 5$ in (4).

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