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Stochastische Differentialgleichungen

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Homework 6

Problem 1. Let $\mathbf{W}$ be a $\mathbb{R}^d$-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$, and **B**, **S** $\in \mathbb{R}^{d\times d}$, as well as $\mathbf{b} \in \mathbb{R}^d$ be given. Consider the linear SDE

$$\mathrm{d}\mathbf{X}_t = \big(\boldsymbol{B}\mathbf{X}_t + \mathbf{b}\big)\mathrm{d}t + \boldsymbol{S}\mathrm{d}\mathbf{W}_t \qquad (0 \le t \le T)\,, \qquad \mathbf{X}_0 = \mathbf{x}_0 \in \mathbb{R}^d\,.$$

Show that there exists a unique strong solution; then derive a bound for $\frac{1}{2}\mathbb{E}\big[\|\mathbf{X}_t\|^2_{\mathbb{R}^d}\big]$, for $0 \le t \le T$.

Problem 2. Verify that the given processes solve the given corresponding stochastic SDEs:

(i) $X_t = e^{W_t}$ solves

$$\mathrm{d}X_t = \frac{1}{2}X_t\mathrm{d}t + X_t\mathrm{d}W_t \quad (t > 0)\,, \qquad X_0 = 0\,.$$

(ii) $X_t = \frac{W_t}{1+t}$ solves

$$\mathrm{d}W_t = -\frac{1}{1+t}X_t\mathrm{d}t + \frac{1}{1+t}\mathrm{d}W_t \quad (t > 0)\,, \qquad X_0 = 0\,.$$

Problem 3. Let $\mathbf{X}$ be the strong solution of the SDE considered in the lecture. Let $\mathcal{E} \subset \mathbb{R}^n$ be nonempty, and open or closed. Then the hitting time

$$\tau := \inf\big\{t \ge 0;\, \mathbf{X}_t \in \mathcal{E}\big\}$$

is an $\mathbb{F}$-stopping time.

Hint: consider the case where $\mathcal{E}$ is open independently from the case where $\mathcal{E}$ is closed.

Problem 4. Let $f \in M^2_T$, and τ be an $\mathbb{F}$-stopping time such that $0 \le \tau \le T$. Define

$$\int_0^\tau f_s\,\mathrm{d}W_s := \int_0^T \mathbb{1}_{\{s\le\tau\}} f_s\,\mathrm{d}W_s\,.$$

Show that

$$\text{i)} \qquad \mathbb{E}\Big[\int_0^\tau f_s\,\mathrm{d}W_s\Big] = 0\,.$$

$$\text{ii)} \qquad \mathbb{E}\Big[\Big(\int_0^\tau f_s\,\mathrm{d}W_s\Big)^2\Big] = \mathbb{E}\Big[\int_0^\tau f_s^2\,\mathrm{d}s\Big]\,.$$

Problem 5. We generalize **Problem 4** in **Homework 5** from $d = 1$ to general $d \in \mathbb{N}$, on using a $\mathbb{R}^d$-valued Wiener process $\mathbf{W} = [W_1, \ldots, W_d]^\top$, and considering the $\mathbb{R}^d$-valued process

$$\mathbf{Z} \equiv \{\mathbf{Z}(t);\, 0 \le t \le T\}, \qquad \text{where} \qquad \mathbf{Z}(t) = \begin{bmatrix} \mathbf{X}(t) \\ \mathbf{Y}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{Q}(t) \\ \mathbf{Q}'(t) \end{bmatrix},$$

which solves the following generalized problem instead of (2) from **Homework 5**,

$$\begin{cases} \mathrm{d}\mathbf{X}(t) & = \mathbf{Y}(t)\,\mathrm{d}t\,, \\ \mathrm{d}\mathbf{Y}(t) & = \big(-\mathbf{\Lambda}\mathbf{X}(t) - \mathbf{Y}(t) + \mathbf{F}(t)\big)\,\mathrm{d}t + \alpha\,\mathrm{d}\mathbf{W}(t)\,, \\ \mathbf{X}(0) & = \mathbf{x}_0 \in \mathbb{R}^d, \qquad \mathbf{Y}(0) = \mathbf{y}_0 \in \mathbb{R}^d\,, \end{cases} \tag{1}$$

with initial data

$$\mathbf{x}_0^\top = \big(\sin(x_j)\big)_{j=1}^d\,, \qquad \mathbf{y}_0^\top = \big(x_j(1-x_j)\big)_{j=1}^d\,, \qquad \text{where} \qquad x_j = \frac{j}{d} \qquad (1 \le j \le d)\,,$$

and data for the SDE

$$\alpha \in [0,1]\,, \qquad \mathbf{\Lambda} = \begin{bmatrix} 2 & -1 & 0 & \ldots \\ -1 & 2 & -1 & \ddots \\ 0 & -1 & 2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \in \mathbb{R}^{d\times d}_{\mathtt{spd}}\,, \qquad \text{and} \qquad \mathbf{F}(t) \equiv t \cdot [1, \ldots, 1]^\top \in \mathbb{R}^d.$$

Let $\mathcal{I}_{k^J} = \big\{t_j^J\big\}_{j=0}^J \subset [0,1]$ be an equi-distant mesh of size k^J. For each j, let the $\mathbb{R}^{2d}$-valued tuple $\big(\mathbf{X}_j, \mathbf{Y}_j\big)$ denote the numerical approximation of $\big(\mathbf{X}(t_j^J), \mathbf{Y}(t_j^J)\big)$, which solves $(0 \le j \le J)$

$$\begin{cases} \mathbf{X}_{j+1} - \mathbf{X}_j & = k^J\,\mathbf{Y}_{j+1}\,, \\ \mathbf{Y}_{j+1} - \mathbf{Y}_j & = k^J\big(-\mathbf{\Lambda}\mathbf{X}_j - \mathbf{Y}_j + \mathbf{F}(t_j)\big) + \alpha\,\Delta_j\mathbf{W}\,, \end{cases} \tag{2}$$

where $\big(\Delta_j\mathbf{W}\big)_\ell := W_\ell(t_{j+1}^J) - W_\ell(t_j^J)$ for $1 \le \ell \le d$ denotes the ℓ-th component of the j-th $\mathbb{R}^d$-valued Wiener increment.

(a) Fix $d = 4$ and $T = 1$, and let $k^J = 10^{-3}$. Plot single trajectories of the 1st component each of the solution $\big(\{\mathbf{X}_j;\, 0 \le j \le J\}, \{\mathbf{Y}_j;\, 0 \le j \le J\}\big)$ for $\alpha \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$.

(b) Let $\mathbf{F} \equiv \mathbf{0}$. Consider the functional $\mathcal{E} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, via

$$\mathcal{E}\big(\mathbf{B}_1, \mathbf{B}_2\big) = \underbrace{\frac{1}{2}\|\mathbf{\Lambda}^{1/2}\mathbf{B}_1\|^2_{\mathbb{R}^d}}_{=:\mathcal{E}_1(\mathbf{B}_1)} + \underbrace{\frac{1}{2}\|\mathbf{B}_2\|^2_{\mathbb{R}^d}}_{=:\mathcal{E}_2(\mathbf{B}_2)}\,.$$

Consider $\alpha \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ below, and fix $\mathtt{M} = 10^3$. Plot the evolutions

$$[0,1] \ni t_j \mapsto \mathbb{E}_{\mathtt{M}}\big[\mathcal{E}_1(\mathbf{X}_j)\big]\,, \qquad [0,1] \ni t_j \mapsto \mathbb{E}_{\mathtt{M}}\big[\mathcal{E}_2(\mathbf{Y}_j)\big]\,, \qquad \text{and} \qquad [0,1] \ni t_j \mapsto \mathbb{E}_{\mathtt{M}}\big[\mathcal{E}(\mathbf{X}_j, \mathbf{Y}_j)\big]\,.$$

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