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Stochastische Differentialgleichungen

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Homework 5

Problem 1. Let $\mathbb{F} \equiv \mathbb{F}_W$, for a given Wiener process W on $(\Omega, \mathcal{F}, \mathbb{P})$. In the lecture, we construct a strong solution of the SDE as limit of a sequence of Picard iterates $\{X^J\}_{J\in\mathbb{N}_0} \subset \mathbb{Y} := L^2_{\mathbb{F}}\big(\Omega; C([0,T];\mathbb{R})\big)$. The latter denotes the space of continuous, $\mathbb{F}$-adapted processes, such that

$$\|X\|_{\mathbb{Y}} = \Big(\mathbb{E}\big[\sup_{0\le t\le T} |X_t|^2_{\mathbb{R}}\big]\Big)^{1/2} < \infty\,.$$

a) Show that $\mathbb{Y}$ is a Banach space.

b) Let $X^0 \equiv x_0$ on $[0,T]$. For given $X^{J-1} := \{X^{J-1}_t;\, 0\le t\le T\} \in \mathbb{Y}$, the subsequent Picard iterate $X^J := \{X^J_t;\, 0 \le t \le T\}$ is constructed as follows,

$$\mathbb{P}\text{-a.s.:} \qquad X^J_t := x_0 + \int_0^t b\big(s, X^{J-1}_s\big)\mathrm{d}s + \underbrace{\int_0^t \sigma\big(s, X^{J-1}_s\big)\,\mathrm{d}W_s}_{:=\mathtt{I}_t(\overline{\sigma})} \qquad \forall\, 0\le t\le T\,. \tag{1}$$

To show $X^J \in \mathbb{Y}$ — and herewith to proceed in the proof of existence of strong solution of the SDE —, required arguments were partly indicated in the lecture, mainly addressing $\{\mathtt{I}_t(\overline{\sigma});\, 0\le t\le T\}$. Please work out those again, and complement them with a discussion of the additional Riemann integral process in (1).

Problem 2. Let $\beta > 0$ and $f \in C([0,T];\mathbb{R})$ be given, and $\{\theta_n\}_{n=1}^\infty \subset L^1(0,T)$ be a sequence of functions such that ($n\in\mathbb{N}$)

$$\theta_{n+1}(t) \le f(t) + \beta \int_0^t \theta_n(s)\,\mathrm{d}s \qquad \forall\, t\in[0,T]\,.$$

Show that

$$\theta_{n+1}(t) \le f(t) + \beta\int_0^t f(\xi)\exp\big(\beta[t-\xi]\big)\,\mathrm{d}\xi + \beta^n \int_0^t \frac{(t-\xi)^{n-1}}{(n-1)!}\theta_1(\xi)\,\mathrm{d}\xi\,.$$

Remark: This result is needed to show that $\{X^J\}_{J\in\mathbb{N}_0} \subset \mathbb{Y}$ is Cauchy.

Problem 3. Fix $T>0$. Let W be an ($\mathbb{R}$-valued) Wiener process on $(\Omega,\mathcal{F},\mathbb{P})$. Consider the $\mathbb{R}^2$-valued stochastic process $\mathbf{X} := \{\mathbf{X}(t);\, 0 \le t \le T\}$, where

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \cos\big(W(t)\big) \\ \sin\big(W(t)\big) \end{bmatrix} .$$

This process is called **Wiener process on the unit circle**. Use Ito's formula to show that it satisfies

$$\mathrm{d}\mathbf{X}(t) = -\frac{1}{2}\mathbf{X}(t)\mathrm{d}t + \boldsymbol{K}\mathbf{X}(t)\,\mathrm{d}W(t) \qquad (0 \le t \le T)\,, \qquad \text{where} \qquad \boldsymbol{K} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} .$$

Discuss if the solvability criteria for this SDE from the lecture are valid.

Problem 4. In physics literature, the charge (process) $Q \equiv \{Q(t);\, 0 \le t \le T\}$ in an electrical circuit is explained to solve the 2nd order differential equation

$$Q^{''}(t) + Q'(t) + Q(t) = F(t) + \alpha W'(t) \qquad (0 \le t \le T)\,, \qquad \text{with} \quad W \text{ a Wiener process}\,, \tag{2}$$

where physical constants are all set equal to one, apart from the given potential source term $F \equiv \{F(t);\, 0 \le t \le T\}$, and $\alpha \in \mathbb{R}$. In this model, the term scaled by α is represents environmental noise in the circuit (*e.g.*, material impurities, etc.).

Give a proper mathematical reformulation of (2) which recasts it into a system of a first order SDE with initial condition, and discuss its solvability.

Problem 5. Fix $\mu, \sigma \in \mathbb{R}$. Consider the linear SDE[1]

$$dX(t) = \mu X(t)\,dt + \sigma X(t)\,dW(t) \quad (0 \le t \le 1), \quad X(0) = X_0. \tag{3}$$

The exact solution to this SDE is

$$X(t) = X(0)\exp\Big((\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\Big). \tag{4}$$

To apply a numerical method to (3) over the time interval $[0,1]$, we first discretize the interval. Let $\mathcal{I}_{k^J} = \{t_j^J\}_{j=0}^J \subset [0,1]$ be an equi-distant mesh of size k^J. The numerical approximation to $X(t_j^J)$ will be denoted by X_j. The Euler-Maruyama (EM) method then takes the form

$$X_j = X_{j-1} + \mu X_{j-1} k^J + \sigma X_{j-1}\big(W(t_j^J) - W(t_{j-1}^J)\big), \quad j = 1, \cdots, J\,. \tag{5}$$

Use the MATLAB code `em.m` from Highham's paper and

(a) Compute a discretize Brownian path over $[0,1]$ with $k^J = 2^{-8}$, and evaluate the solution in (4) (as true solution).

(b) Apply the EM method (5) to (3) using a stepsize $\Delta t = Rk^J$, with $R = 4$.

[1] This SDE arises, for example, as an asset price model in financial mathematics. The well-known Black-Scholes PDE can be derived from it.

(c) Change the mesh size k^J and value of R and discuss your observations.

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