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Stochastische Differentialgleichungen

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Homework 4

Problem 1. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space, where $\mathbb{F} = \{\mathcal{F}_t;\, t \geq 0\}$ is such that $\mathcal{F}_t = \sigma\big(\{W_s;\, 0 \leq s \leq t\}\big)$ is generated by the Wiener process W.

a) Show that $X = \{|W_t|^2 - t;\, t \geq 0\}$ is an $\mathbb{F}$-martingale.

b) Show that $X = \{X_t;\, t \geq 0\}$, with $X_t = \exp(W_t - \frac{t}{2})$ is an $\mathbb{F}$-martingale.

Problem 2. Show that the exponential martingale X from **Problem 1, b)** is an Ito process and verify that it satisfies the equation

$$\mathrm{d}X_t = X_t \mathrm{d}W_t\,.$$

<u>**Hint:**</u> Don't forget to show that $X \in M^2(T)$ for all $T > 0$ before applying Ito's formula.

Problem 3. Let $\alpha > 0$ and $\sigma \in \mathbb{R}$ be fixed. Define $Y = \{Y_t;\, t \geq 0\}$ via

$$Y_t = \sigma \exp\big(-\alpha t\big) \int_0^t \exp(\alpha s)\mathrm{d}W_s\,.$$

Show that Y satisfies

$$\mathrm{d}Y_t = -\alpha Y_t \mathrm{d}t + \sigma \mathrm{d}W_t \qquad (t \geq 0)\,.$$

The process Y is known as **Ornstein-Uhlenbeck process.**

Problem 4. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be the filtered probability space from **Problem 1**, and $X = \{X_t;\, 0 \leq t \leq T\} \subset M^2(T)$. Show that there exists a *continuous* modification of $Y = \{Y_t;\, 0 \leq t \leq T\}$, where $Y_t = \int_0^t X_s \mathrm{d}W_s = I_t(X)$.

<u>**Hint:**</u> Consider an approximating sequence $\{X_j;\, j \in \mathbb{N}\} \subset M^2_{\mathtt{step}}(T)$. Then

a) use Doob's inequality and Ito isometry to show that $\{I(X_j);\, j \in \mathbb{N}\} \subset L^2\big(\Omega; C([0,T];\mathbb{R})\big)$ is Cauchy, where $I(X_j) = \{I_t(X_j);\, 0 \leq t \leq T\}$. Recall that $L^2\big(\Omega; C([0,T];\mathbb{R})\big)$ is Banach space.

b) use Tschebycheff's inequality and the Borel-Cantelli lemma to step over to a subsequence $\{I(X_{j_\ell});\, \ell \in \mathbb{N}\}$ which already convergences $\mathbb{P}$-a.s. in $C([0,T];\mathbb{R})$.

Problem 5. Let $\mathcal{I}_{k^J} = \{t_j^J\}_{j=0}^J \subset [0,1]$ be an equi-distant mesh of size k^J. Starting with $W_0 = 0$, increments $\Delta_j^J W \sim \sqrt{k^J} N(0,1)$ are simulated in MATLAB via the random number generator `'randn'` to constitute an (approximate) Wiener process.

a) Let run the MATLAB codes `bpath1.m`, `bpath2.m`, which are available on the homepage:

`https://www.maths.ed.ac.uk/ dhigham/pubs.html`

to approximate the Wiener process W for $J = 100, 500, 1000$, and 2000. Discuss your observations.[1]

b) Simulate the process

$$t \mapsto F\big(W(t)\big) = \exp\big(t + \frac{1}{2}W(t)\big) \qquad (0 \le t \le 1)$$

for $J = 500, 1000, 2000$. Then plot

$$t \mapsto \mathbb{E}\big[F\big(W(t)\big)\big] \approx \frac{1}{M}\sum_{j=1}^{M} F\big(W_j(t)\big) \qquad \text{where } M \in \{10^\ell;\, 2 \le \ell \le 6\}$$

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[1] From the given homepage, we recommend the downloadable paper: *An Algorithmic Introduction to Numerical Simulation of Stochastic Differential equations'* for further reading to simulate SDEs.