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Fachbereich Mathematik

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Stochastische Differentialgleichungen

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Homework 3

Problem 1. Let $T > 0$. For any $f, g \in M^2(T)$, there holds for $I(f) \equiv I_T(f) := \int_0^T f(s)\,\mathrm{d}W(s)$ that

$$\mathbb{E}\big[I(f)I(g)\big] = \mathbb{E}\Big[\int_0^T f(s)g(s)\,\mathrm{d}s\Big]\,.$$

<u>Hint:</u> Write the left-hand side in terms of $\mathbb{E}\big[\big|I(f)+I(g)\big|^2\big]$ and $\mathbb{E}\big[\big|I(f)-I(g)\big|^2\big]$, and the right-hand side in terms of $\mathbb{E}\big[\int_0^T |f(s)+g(s)|^2\,\mathrm{d}s\big]$ and $\mathbb{E}\big[\int_0^T |f(s)-g(s)|^2\,\mathrm{d}s\big]$.

Problem 2. Let $T > 0$. Show that $I : M^2_{\mathtt{step}}(T) \to L^2(\Omega)$ is a linear map, *i.e.*, for any $f, g \in M^2_{\mathtt{step}}(T)$ and any $\alpha, \beta \in \mathbb{R}$,

$$I\big(\alpha f + \beta g\big) = \alpha I(f) + \beta I(g)\,.$$

Generalize this property to Ito's integral $I : M^2(T) \to L^2(\Omega)$.

Problem 3. Let $T > 0$, and $f \in M^2_{\mathtt{step}}(T)$. Show that

$$I(f) := \int_0^T f(s)\,\mathrm{d}W(s)$$

is a martingale. Does this property remain valid for $f \in M^2(T)$?

Problem 4. Show that $W^2 = \{W_t^2;\, t \geq 0\}$ belongs to $M^2(T)$ for every $T > 0$, and verify the equality

$$\int_0^T W^2(s)\,\mathrm{d}W(s) = \frac{1}{3}W^3(T) - \int_0^T W(s)\,\mathrm{d}s\,,$$

where the integral on the right-hand side is a Riemann integral.

<u>Hint:</u> Use a partition of $[0, T]$ to define an approximating stochastic process of W^2. Then use the identity

$$a^2(b-a) = \frac{1}{3}(b^3 - a^3) - a(b-a)^2 - \frac{1}{3}(b-a)^3\,.$$

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