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Stochastische Differentialgleichungen

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Homework 2

Problem 1. Let $T > 0$, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. In the lecture, we introduced the Lagrange interpolation $\{\widetilde{Y}_{k,h}(t);\, 0 \le t \le T\}$ via

$$\widetilde{Y}_{k,h}(t) = \frac{t_{j+1} - t}{k} Y_{k,h}(t_j) + \frac{t - t_j}{k} Y_{k,h}(t_{j+1}) \qquad \forall\, t_j \le t < t_{j+1}\,,$$

where $Y_{k,h} = \sum_{l=1}^{j} \eta_l$ is the (scaled) random walk with $\mathcal{L}(\eta_\ell) = \frac{1}{2}\delta_{-h} + \frac{1}{2}\delta_h$, and $t_j = j\frac{T}{J} =: jk$ are the mesh points of the equi-distant mesh $\mathcal{I}_k := \{t_j\}_{j=0}^{J}$ covering $[0, T]$.

Let now $f \in C_{\mathtt{bdd}}(\mathbb{R})$[1]. Show that the iterates $\{u^j\}_{j=0}^{J} \subset C_{\mathtt{bdd}}(\mathbb{R})$ which are given by

$$u^j(x) = \mathbb{E}\big[f\big(x + \widetilde{Y}_{k,\sqrt{k}}(t_j)\big)\big] \quad \forall\, x \in \mathbb{R}$$

solves

$$\begin{cases} \frac{u^j(x) - u^{j-1}(x)}{k} = \frac{u^{j-1}(x+\sqrt{k}) - 2u^{j-1}(x) + u^{j-1}(x-\sqrt{k})}{2k} & (1 \le j \le N) \\ u^0(x) = f(x) \qquad \forall\, x \in \mathbb{R}\,, \end{cases} \tag{1}$$

where (1) is a finite difference discretization of the linear heat equation.

Problem 2. Let $t > 0$. In the lecture, we defined the quadratic variation of the Wiener process W as

$$Q^W(t) = L^2 - \lim_{J \uparrow \infty} Q^W_{k^J}(t)\,,$$

where $Q^W_{k^J}(t) = \sum_{j=1}^{J} |W(t^J_j) - W(t^J_{j-1})|^2$ on the equi-distant mesh $\mathcal{I}_{k^J} := \{0, t^J_1, \ldots, t^J_J\}$ of mesh size $k > 0$ covering $[0, t]$, such that $t^J_j = \frac{jt}{J} =: jk^J$. Let $\{\mathcal{I}_{k^l};\, l \in \mathbb{N}\}$ be a sequence of equi-distant meshes such that $\sum_{l=1}^{\infty} k^l < \infty$. Show that

$$Q^W_{k^l}(t) \to t \qquad \mathbb{P} - \text{a.s.} \qquad (l \uparrow \infty)\,.$$

Hint: Use Tschebycheff's inequality and the Borel-Cantelli lemma to use the corresponding L^2-convergence result from the lecture.

[1] I.e., the space that consists of bounded, continuous functions on $\mathbb{R}$.

Problem 3. Let $0 = t_0^J < t_1^J < \ldots < t_J^J = T$< where $t_j^J = \frac{jT}{J}$ be an equi-distant mesh of size $k^J = \frac{T}{J}$ covering $[0, T]$. Find the following limits

$$a) \qquad L^2 - \lim_{J\uparrow\infty} \sum_{j=0}^{J-1} W(t_j^J)\Big[W(t_{j+1}^J) - W(t_j^J)\Big],$$

$$b) \qquad L^2 - \lim_{J\uparrow\infty} \sum_{j=0}^{J-1} W(t_{j+1}^J)\Big[W(t_{j+1}^J) - W(t_j^J)\Big].$$

Problem 4. Let $f \in M^2_{\texttt{step}}$, *i.e.*, it is of the form $f(t) = \sum_{j=0}^{J-1} \eta_j 1_{[t_j, t_{j+1})}(t)$. Consider the stochastic integral

$$I(f) = \sum_{j=1}^{J-1} \eta_j \Big(W(t_{j+1}) - W(t_j)\Big).$$

Show that

$$\mathbb{E}\big[|I(f)|^2\big] = \mathbb{E}\Big[\int_0^\infty |f(s)|^2 \,\mathrm{d}s\Big].$$

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