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# Stochastische Differentialgleichungen

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## Homework 10

**Problem 1.** Fix $T > 0$. Let $W$ be an $\mathbb{R}$-valued Wiener process on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with is endowed with natural filtration $\mathbb{F} := \mathbb{F}^W$. Consider a map $f \in C^{1,2}([0,T] \times \mathbb{R})$, for which there exist numbers $K, \alpha > 0$, such that

$$\sup_{0 \le s \le T} \big| f(t,x) \big| \le K e^{\alpha |x|} .$$

Show that the process $\{f(t, W_t);\, 0 \le t \le T\}$ is an $\mathbb{F}$-martingale if and only if

$$\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} = 0 \qquad \forall\, (t,x) \in (0,T) \times \mathbb{R} .$$

<u>**Remark & Hint:**</u> **(a)** Note that $f(t,x) = t - x^2$ is admissible, which settles that $W^2 - t$ is $\mathbb{F}$-martingale.

**(b)** Use that $W$ is a Markov process, with given transition semigroup $\mathscr{S}^W$.

**Problem 2.** Let $p \in \mathbb{N}$. In **Problem 3** of **Homework 7**, you derived higher moment bounds for (explicit) Euler iterates $\{\mathbf{Y}^j\}_{j=0}^J \subset L^{2p}(\Omega; \mathbb{R}^n)$, which are meant to approximate the strong solution $\mathbf{X} \in L^2_{\mathbb{F}}\big(\Omega; C([0,T]; \mathbb{R}^n)\big)$ of the SDE given in **Problem 1** of **Homework 7**.

Now recall the notion of an equi-distant mesh $\{t_j\}_{j=0}^J$ covering $[0,T]$ of size $k = t_{j+1} - t_j$ from there: on every sub-interval $[t_j, t_{j+1}]$, we then define $\mathcal{Y}^{(k)}_{t_j} := \mathbf{Y}^j$, and refer to $\{\mathcal{Y}^{(k)}_t;\, t \in [t_j, t_{j+1})\}$ as strong solution of

$$\mathcal{Y}^{(k)}_t = \mathcal{Y}^{(k)}_{t_j} + \int_{t_j}^t \mathbf{b}\big(\mathcal{Y}^{(k)}_{t_j}\big)\, \mathrm{d}s + \int_{t_j}^t \boldsymbol{\sigma}\big(\mathcal{Y}^{(k)}_{t_j}\big)\, \mathrm{d}\mathbf{W}_s \qquad \forall\, t \in [t_j, t_{j+1}) . \tag{1}$$

1) Show that $\mathcal{Y}^{(k)} \equiv \{\mathcal{Y}^{(k)}_t;\, 0 \le t \le T\} \in L^2_{\mathbb{F}}\big(\Omega; C([0,T]; \mathbb{R}^n)\big)$ exists, and interpolates $\{\mathbf{Y}^j\}_{j=0}^J \subset L^2(\Omega; \mathbb{R}^n)$.

2) Show that there exists $C_T > 0$ such that

$$\max_{1 \le j \le J} \Big( \mathbb{E}\big[ \|\mathbf{X}_{t_j} - \mathbf{Y}^j\|^2_{\mathbb{R}^n} \big] \Big)^{1/2} \le C_T \sqrt{k} \tag{2}$$

3) Show that there exists $C_T > 0$ such that

$$\Big( \mathbb{E}\big[ \max_{1 \le j \le J} \|\mathbf{X}_{t_j} - \mathbf{Y}^j\|^2_{\mathbb{R}^n} \big] \Big)^{1/2} \le C_T \sqrt{k} \tag{3}$$

<u>**Hint**</u>: To show **2)** recall $\mathcal{Y}^{(k)}_{t_j} = \mathbf{Y}^j$. Now write an error identity on a fixed sub-interval $[t_j, t_{j+1}]$ first,

and use the properties given in **Problem 1** of **Homework 7** on data $(\mathbf{b}, \boldsymbol{\sigma})$ before summation over all sub-intervals.

To show **3)** use the result from **2)**, in combination with **BDG-inequality**.

**Problem 3**. Consider the following equation on $\mathcal{O} \subset \mathbb{R}^2$:

$$\begin{cases} -\Delta u = f & \text{on } \mathcal{O} := (0,1)^2\,, \\ \quad u = g & \text{on } \partial\mathcal{O}\,, \end{cases} \tag{4}$$

where $f : \mathcal{O} \to \mathbb{R}$ and $g : \partial\mathcal{O} \to \mathbb{R}$ be bounded measurable functions. Then the probabilistic representation of the solution of the above equation is given by the (generalized) Feynman-Kac formula as

$$u(\mathbf{x}) = \Big[g\big(\mathbf{X}^{\mathbf{x}}_{\tau_{\mathbf{x}}}\big) + Z^{\mathbf{x}}_{\tau_{\mathbf{x}}}\Big] \qquad \forall\, \mathbf{x} \equiv (x_1, x_2) \in \mathcal{O}\,,$$

where $\mathbf{X}^{\mathbf{x}} \equiv \big\{\mathbf{X}^{\mathbf{x}}_t; t \geq 0\big\}$ denotes the $\mathbb{R}^2$-valued solution of the following SDE

$$\mathrm{d}\mathbf{X}_t = \sqrt{2}\,\mathbf{1}_{\mathbb{R}^2}\,\mathrm{d}\mathbf{W}_t \quad \forall\, t > 0\,, \qquad \mathbf{X}_0 = \mathbf{x} \in \mathcal{O}\,, \tag{5}$$

with $\mathbf{1}_{\mathbb{R}^2} \in \mathbb{R}^{2\times 2}$ the identity matrix, with $\mathbf{W} \equiv \{\mathbf{W}_t; t \geq 0\}$ an $\mathbb{R}^2$-valued Wiener process on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and $\tau_{\mathbf{x}}$ the first exit time of $\mathbf{X}^{\mathbf{x}}$ from $\mathcal{O}$, *i.e.*,

$$\tau_{\mathbf{x}} := \inf\{t > 0 : \mathbf{X}^{\mathbf{x}}_t \notin \mathcal{O}\}\,,$$

and $Z^{\mathbf{x}} \equiv \{Z^{\mathbf{x}}_t; t \geq 0\}$ the $\mathbb{R}$-valued solution of the random ODE

$$\mathrm{d}Z_t = f(\mathbf{X}^{\mathbf{x}}_t)\,\mathrm{d}t \quad \forall t > 0, \qquad Z_0 = 0\,.$$

Let $f \equiv 1$ and $g = 0.5$ in (4). Partition the domain $\mathcal{O}$ by using the Matlab command `meshgrid` with grid size of $1/20$. For every spatial grid point in $\mathcal{O}$, compute (and plot) the approximate solution

$$u(\mathbf{x}) = 0.5 + \mathbb{E}\big[\tau_{\mathbf{x}}\big] \approx 0.5 + \mathbb{E}_{\mathtt{M}}\big[t_{j^*}\big]\,,$$

for $\mathtt{M} = 3000$, where $t_{j^*} := \min\{t_j : \mathbf{Y}^j_{\mathbf{x}} \notin \mathcal{O}, j \geq 0\}$, and $\{\mathbf{Y}^j_{\mathbf{x}}\}_{j\geq 0}$ are the explicit Euler iterates of the SDE (5).

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