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Stochastische Differentialgleichungen

Sommer-Semester 2022 **Tübingen, 21.04.2022**

Homework 1

Problem 1. The Wiener process $W = \{W_t;\, t \geq 0\}$ (or: Brownian motion) on $(\Omega, \mathcal{F}, \mathbb{P})$ is an $\mathbb{R}$-valued process such that

i) $W_0 = 0$ $\mathbb{P}$-a.s.,

ii) W is continuous.

iii) For any finite sequence of times $0 < t_1 < t_2 < \ldots < t_n$ and Borel sets $\{\mathcal{A}_i\}_{i=1}^n \subset \mathscr{B}(\mathbb{R})$ there holds

$$\mathbb{P}\Big[W_{t_1} \in \mathcal{A}_1, \ldots, W_{t_n} \in \mathcal{A}_n\Big] = \int_{\mathcal{A}_1} \ldots \int_{\mathcal{A}_n} p(t_1, 0, x_1)p(t_2 - t_1, x_1, x_2) \ldots p(t_n - t_{n-1}, x_{n-1}, x_n)\, \mathrm{d}x_1 \ldots \mathrm{d}x_n\,,$$

where $p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{(x-y)^2}{2t}\big) \quad \forall\, x, y \in \mathbb{R},\ \forall\, t > 0.$

a) For all $t > 0$ show that $f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} \exp\big(-\frac{x^2}{2t}\big) \quad \forall\, x \in \mathbb{R}$ is the probability density of W_t. Moreover, compute the expectation, and the variance of W_t.

b) Show that $\mathbb{E}[W_s W_t] = \min\{s,t\}$.

<u>**Hint:**</u> Use the joint density of W_s and W_t of the form

$$f_{W_s, W_t}(x,y) = p(s, 0, x)p(t-s, x, y) \quad \forall\, x, y \in \mathbb{R} \quad \forall\, 0 < s < t\,.$$

c) Show that $\mathbb{E}\big[|W_t - W_s|^2\big] = |t-s| \quad \forall\, s, t \geq 0.$

d) Let $t > 0$. Compute the characteristic function

$$\phi_{W_t}(\lambda) := \mathbb{E}\big[\exp(i\lambda W_t)\big] \quad \forall\, \lambda \in \mathbb{R}\,.$$

<u>**Hint:**</u> Use the density of W_t in part $\mathrm{a)}$.

e) Compute $\mathbb{E}\big[|W_t|^4\big]$ with the help of part $\mathrm{d)}$.

Problem 2. We call $\mathbf{W} \equiv \{\mathbf{W}_t;\, t \geq 0\}$ an $\mathbb{R}^L$-valued Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ if its components $W^1, W^2, \ldots, W^L$ are independent $\mathbb{R}$-valued Wiener processes. Let $L = 2$. Compute the probability that $\|\mathbf{W}_t\|_{\mathbb{R}^2} < R$, where $R > 0$, and $\|\mathbf{x}\|_{\mathbb{R}^2}$ is the Euclidean norm of $\mathbf{x} = (x^1, x^2) \in \mathbb{R}^2$.

Date of Submission: 27.04.2022.