EBERHARD KARLS
UNIVERSITÄT
TÜBINGEN

Mathematisch-Naturwissenschaftliche Fakultät

Fachbereich Mathematik

Prof. Dr. Andreas Prohl
Dr. Abhishek Chaudhary

Statistisches Lernen

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Homework 4

Problem 1. In the lecture, we defined the empirical measure μ_n for an i.i.d. sample set $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$. Let $\mathscr{P}_n$ be a partition of $\mathbb{R}^d$; we call a cell $\mathscr{A}_{n,j} \in \mathscr{P}_n$ to be empty, if $\mu_n[\mathscr{A}_{n,j}] = 0$. Let now M_n be the number of nonempty cells of $\mathscr{P}_n$. Prove that a.s.

$$\frac{1}{n} M_n \downarrow 0 \quad (n \uparrow \infty)\,,$$

provided that $\lim_{n\uparrow\infty} \frac{|\{j:\mathscr{A}_{n,j}\cap\mathscr{B}(\mathbf{0},R)\}|}{n} = 0$ for each $R > 0$.

Problem 2. Assume that the weights $\{\alpha_{n,i}\}_{i=1}^n$ are non-negative and that the corresponding local averaging estimator is weakly universally consistent. Prove that **assumption (iii)** in Stone's theorem is satisfied, *i.e.*,

$$\lim_{n\uparrow\infty} \mathbb{E}\Big[\sum_{i=1}^n |\alpha_{n,i}(\mathbf{X})| \mathbb{1}_{\{\|\mathbf{X}_i - \mathbf{X}\| > 0\}}\Big] = 0 \quad \forall\, a > 0\,.$$

Hint: For any fixed $\mathbf{x} \in \mathbb{R}^d$ and $a > 0$, let f be a non-negative continuous function which is 0 on $\mathscr{B}(\mathbf{x}; \frac{a}{3})$, and 1 on $\mathscr{B}(\mathbf{x}; \frac{2a}{3})$. Now choose $Y = f(\mathbf{X}) = m(\mathbf{X})$, then

$$\mathbb{1}_{\{\mathbf{X}\in\mathscr{B}(\mathbf{x};\frac{a}{3})\}} \sum_{i=1}^n \alpha_{n,i}(\mathbf{X}) f(\mathbf{X}_i) \geq \mathbb{1}_{\{\mathbf{X}\in\mathscr{B}(\mathbf{x};\frac{a}{3})\}} \sum_{i=1}^n \alpha_{n,i}(\mathbf{X}) \mathbb{1}_{\{\|\mathbf{X}_i - \mathbf{X}\| > 0\}} \downarrow 0 \quad (n \uparrow \infty)$$

in probability, therefore, for any compact set $\mathscr{A} \subset \mathbb{R}^d$,

$$\mathbb{1}_{\{\mathbf{X}\in\mathscr{A}\}} \sum_{i=1}^n \alpha_{n,i}(\mathbf{X}) \mathbb{1}_{\{\|\mathbf{X}_i - \mathbf{X}\| > a\}} \downarrow 0 \quad (n \uparrow \infty) \qquad \text{in probability.}$$

Date of Submission: 12.00 on 11.11.2026.