



Statistisches Lernen

Winter-Semester 2022/23

Tübingen, 14.10.2026

Homework 1

Problem 1. For $(\Omega, \mathcal{F}, \mathbb{P})$ given, consider the disjoint decomposition

$$\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n, \quad \text{where} \quad \Omega_n \in \mathcal{F} \quad \text{and} \quad \mathbb{P}[\Omega_n] > 0 \quad (n \in \mathbb{N}). \quad (1)$$

Consider the sub- σ -Algebra $\mathcal{G} \subset \mathcal{F}$ which is generated from the family $\{\Omega_n\}_{n \in \mathbb{N}}$, i.e.,

$$\mathcal{G} = \left\{ \bigcup_{n \in \mathbb{N}} \Omega_n^{\varepsilon_n} : \varepsilon_1, \varepsilon_2, \dots \in \{0, 1\} \right\}, \quad \text{where} \quad \Omega_n^1 = \Omega_n \quad \text{and} \quad \Omega_n^0 = \emptyset.$$

Let now $\mathbf{X} : \Omega \rightarrow \mathbb{R}^d$ be a $\mathcal{F}$ -measurable random variable on Ω such that $\mathbb{E}[||\mathbf{X}||] < \infty$. Show that

$$\mathbb{E}[\mathbf{X}|\mathcal{G}](\omega) = \frac{\mathbb{E}[\mathbf{X}1_{\Omega_n}]}{\mathbb{P}[\Omega_n]} \quad \forall \omega \in \Omega_n.$$

Problem 2. For a given data set $D_n = \{(\mathbf{X}_i, Y_i)\}_{i=1}^n$, *linear* regression is an example for a parametric estimator m_n of regression function m . Here, the $\mathbb{R}^{1+d}$ -valued (random) parameter $\mathbf{a} \equiv (a_0, \underline{\mathbf{a}})^\top$ determines

$$m_n \equiv m_n(\cdot; \mathbf{a}, D_n) : \mathbf{x} \mapsto a_0 + \langle \underline{\mathbf{a}}, \mathbf{x} \rangle_{\mathbb{R}^d},$$

and the optimal parameter $\mathbf{a}^*$ that we look for minimizes the quadratic functional

$$\frac{1}{n} \sum_{i=1}^n |Y_i - (a_0 + \langle \underline{\mathbf{a}}, \mathbf{X}_i \rangle_{\mathbb{R}^d})|^2.$$

Discuss how the parameter $\mathbf{a}^*$ to set $m_n(\cdot; \mathbf{a}^*, D_n)$ is actually computed.

Problem 3. Let $D_n = \{(\mathbf{X}_j, Y_j)\}_{j=1}^n$ be a data set, and $m_n \equiv m_n(\cdot; D_n)$ be a regression estimator. Show that

$$\mathbb{E}[|m_n(\mathbf{X}) - Y|^2 | D_n] = \int_{\mathbb{R}^d} |m_n(\mathbf{x}) - m(\mathbf{x})|^2 \mu[d\mathbf{x}] + \mathbb{E}[|m(\mathbf{X}) - Y|^2].$$

Date of Submission: 12.00 on 21.10.2026.