EBERHARD KARLS
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Mathematisch-Naturwissenschaftliche Fakultät

Fachbereich Mathematik

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Statistical Learning 2

Summer-Semester 2023 **Tübingen, 24.10.2023**

Homework 1

Problem 1. Let $\{\mathscr{C}_k\}_{k=1}^K$ be a (non-overlapping) covering of $\mathbb{R}^2$, where each bounded $\mathscr{C}_k$ is a triangle. Fix $0 \leq B$. Define

$$\mathcal{G}_n = \Big\{ g \in C_0(\mathbb{R}^2; [0,B]); \quad g\big|_{\mathscr{C}_k} \in \mathcal{P}_1(\mathscr{C}_k) \qquad (1 \leq k \leq K) \Big\},$$

where $\mathcal{P}_1(\mathscr{C}_k)$ denotes the space of polynoms of degree ≤ 1 on $\mathscr{C}_k$.

(a) Fix $\varepsilon > 0$. Compute $\mathcal{N}_\infty(\varepsilon, \mathcal{G}_n)$.

(b) Give an upper bound for K in terms of n which still validates the uniform law of large numbers, *i.e.*,

$$\sup_{g \in \mathcal{G}_n} \Big| \frac{1}{n} \sum_{j=1}^n g(\mathbf{Z}_j) - \mathbb{E}\big[g(\mathbf{Z})\big] \Big| \overset{n\uparrow\infty}{\longrightarrow} 0 \quad \mathbb{P}\text{-a.s.} \tag{1}$$

Remark: Use the result from the lecture, according to which (1) for all $\varepsilon > 0$ in case

$$\sum_{n=1}^\infty \mathcal{N}_\infty\Big(\frac{\varepsilon}{3}, \mathcal{G}_n\Big) \exp\Big(-\frac{2n\varepsilon^2}{9B^2}\Big) < \infty .$$

Problem 2. We define the truncation operator $T_L : \mathbb{R} \to [-L, L]$ for given $L > 0$,

$$T_L u = \begin{cases} u & \text{if } |u| \leq L, \\ L\,\mathrm{sgn}(u) & \text{otherwise.} \end{cases}$$

For a given class of functions $\mathcal{G}_n$, we derive the following class of functions

$$T_L \mathcal{G}_n = \big\{ T_L g;\, g \in \mathcal{G}_n \big\} .$$

Now let $\{\phi_k\}_{k=1}^K : \mathbb{R}^d \to \mathbb{R}$ be given. Consider $\mathcal{G}_n = \big\{\sum_{k=1}^K \alpha_k \phi_k;\, \alpha_k \in \mathbb{R}\big\}$. For $L > 0$, we consider $T_L\mathcal{G}_n$. Then for any $0 < \varepsilon \leq \frac{L}{2}$ we have

$$\mathcal{N}\big(\varepsilon, T_L\mathcal{G}_n; \|\cdot\|_{\mathbb{L}^1(\nu)}\big) \leq \Big(\frac{6L}{\varepsilon}\Big)^{2(K+1)} .$$

Date of Submission: 12.00 on 02.05.2023.