Exercise Sheet-2

Optimal control problem with ODE
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June 10, 2024

Exercise 1. Solve the optimal control problem

$$\text{minimize}_{u(\cdot)} \int_0^{t_f} (x(t) + u^2(t))\mathrm{d}t$$
$$\text{subject to} \begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} & = x(t) + u(t) + 1, \\ x(0) & = 0. \end{cases}$$

Exercise 2. Among all curves of length l in the upper half plane passing through the points $(-a, 0)$ and $(a, 0)$, find the one which encloses the largest area in the interval $[-a, a]$, i.e., solve

$$\text{maximize}_{x(\cdot)} \int_{-a}^{a} x(t)\mathrm{d}t$$
$$\text{subject to} \begin{cases} x(-a) & = 0, \\ x(a) & = 0, \\ K(x) & = \int_{-a}^{a} \sqrt{1 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2}\mathrm{d}t = l. \end{cases}$$

Exercise 3. Consider the optimal control problem

$$\text{minimize}_{u(\cdot)} \frac{\gamma}{2} x(t_f)^2 + \frac{1}{2} \int_{t_i}^{t_f} u^2(t)\mathrm{d}t$$
$$\text{subject to} \begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} & = u(t) \\ x(t_i) & = x_i. \end{cases}$$

Derive a optimal *feedback* policy using PMP.

Exercise 4. A producer with production rate $x(t)$ at time t may allocate a portion $u(t)$ of his/her production rate to reinvestments in a factory (thus increasing the production rate) and use the rest $(1 - u(t))$ to store goods in a warehouse. Thus $x(t)$ evolves according to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha u(t) x(t), \tag{1}$$

where α is given constant. The producer wants to maximize the total amount of goods stored summed with the capacity of the factory at final time. This gives us the following problem:

$$\begin{aligned} &\text{maximize}_{u(\cdot)} x(t_f) + \int_0^{t_f} (1 - u(t))x(t)\mathrm{d}t \\ &\text{subject to } \begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} &= \alpha u(t)x(t), \qquad 0 < \alpha < 1 \\ x(0) &= x_0 > 0, \\ 0 \leq &u(t) \leq 1, \qquad \forall\, t \in [0, t_f]. \end{cases} \end{aligned}$$

Find an analytical solution to the problem above using the PMP.

Deadline: 21th June 2024, 12:00.

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Note: Please meet on 14th June, 12:00, Room S 06 for the tutorial session to discuss this exercise sheet.