Exercise Sheet-1

Optimal control problem with ODE
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Exercise 1. Let $\alpha(t), \beta(t)$, where $\beta(t) \geq 0$, and $x(t)$ be continuous scalar valued functions on $[a, b]$ and let $x(t)$ satisfy the inequality

$$x(t) \leq \alpha(t) + \int_a^t \beta(\tau)x(\tau)\mathrm{d}\tau, \ t \in [a, b].$$

Then for all $t \in [a, b]$, we have

$$x(t) \leq \alpha(t) + \int_a^t \beta(\tau)g(t, \tau)\mathrm{d}\tau,$$

where

$$g(t, \tau) = \alpha(\tau) \exp^{\int_\tau^t \beta(\sigma)\mathrm{d}\sigma}.$$

Exercise 2. Let us state the linear quadratic problem

$$\min \frac{1}{2} x(t_f)^T Q_0 x(t_f) + \frac{1}{2} \int_0^T \left[x(s)^T Q x(s) + u(s)^T R u(s)\right] \mathrm{d}t$$

$$\text{subj. to} \begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) & \forall t \in [0, t_f], \\ x(0) = x_0. \end{cases}$$

Where Q_0 and Q are symmetric positive semi-definite matrices and R is a symmetric positive definite matrix. As an application of Pontryagin's minimum principle prove that the optimal control u^* satisfies

$$u^*(t) = -R^{-1}B^T\lambda(t) \qquad \forall\, t \in [0, t_f],$$

where $\lambda(t)$ is the solution of the adjoint differential equation. Clearly write the adjoint system for this optimal control problem. Write also $u^*(t)$ in terms of given data x_0, Q, Q_0, R and A with help of the transition matrix as discussed in 3rd lecture, for details see Section 4.2 & 5.1 of lecture's notes.

Exercise 3. Consider the following problem

$$\min \|x(t_f)\|^2 + \int_0^{t_f} \|u(t)\|^2,$$

$$\text{subj to} \begin{cases} \frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t) & \forall t \in [0, t_f], \\ x(0) = \begin{bmatrix} z_0 \\ 0 \end{bmatrix}, \end{cases}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Show that optimal control u^* for the this problem is

$$u^*(t) = \frac{1}{(1 + t_f - t + (t_f - t)^2/3 + (t_f - t)^4/12)} \begin{bmatrix} t_f - t + (t_f - t)^2/2 \\ 1 + (t_f - t)^2 + \frac{1}{3}(t_f - t)^3 \end{bmatrix}.$$

Deadline: 16th May 2024, 12:00.

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Note: Please meet on 10th May, 12:00, Room S 06 for the tutorial session to discuss this exercise sheet, specially exercise 3.