Universität Tübingen
Mathematisches Institut
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Exercise sheet no. 6 – Numerics for instationary differential equations

Exercise 17: (Characteristic equation of multi step methods)

(a) Show per induction for j, that the sequence $y_k = \zeta^k$, $k = 0, 1, \dots$ satisfies:

$$\nabla^j y_k = \zeta^k \left(1 - \frac{1}{\zeta}\right)^j ,$$

where $\nabla^0 y_k = y_k$, $\nabla^j y_k = \nabla^{j-1} y_k - \nabla^{j-1} y_{k-1}$ for $j \geq 1$.

(b) Using this, show that for BDF methods (given by $\sum_{j=1}^k j^{-1} \nabla^j y_{n+k} = h f_{n+k}$):

$$\alpha(\zeta) = \zeta^k \sum_{j=1}^k \frac{1}{j} \left(1 - \frac{1}{\zeta}\right)^j , \qquad \beta(\zeta) = \zeta^k.$$

Exercise 18: (Crank-Nicolson method)

Discretizing a parabolic problem using finite elements in space and the midpoint rule in time yields the following scheme:

For $n = 0, 1, 2, \dots$, find $u_{n+1} \in V_h$ such that

$$\Big((u_{n+1} - u_n)/\tau, v\Big) + a\Big((u_{n+1} + u_n)/2, v\Big) = \Big(f((t_{n+1} + t_n)/2), v\Big), \qquad \text{for all } \ v \in V_h \ .$$

(a) In each step, this is a linear equation system in $\mathbb{R}^N$. Derive this system.

(b) Derive a stability estimate using energy equations.

Exercise 19: (Crank-Nicolson method)

Show that, in the situation of the previous exercise and under suitable regularity assumptions, the following error estimates hold for $n\tau \leq T$

$$\begin{aligned} |u_n - u(t_n)| \qquad & \leq C\,(h^2 + \tau^2), \\ \left(\tau \sum_{j=0}^{n-1} \left\| \frac{u_{j+1} + u_j}{2} - u\Big(\frac{t_{j+1} + t_j}{2}\Big) \right\|^2 \right)^{1/2} & \leq C\,(h + \tau^2)\,. \end{aligned}$$

Solutions are discussed on 12.06.2024.

Contact person: Dominik Sulz - `dominik.sulz@uni-tuebingen.de`. Open door policy - just come to my office if you have any questions!