Universität Tübingen
Mathematisches Institut
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Exercise sheet no. 5 – Numerics for instationary differential equations

Exercise 13:

Show: If the bilinear form in the weak formulation of a initial boundary value problem satisfies just the Gårding inequality (see previous exercise) instead of being V-elliptic, all existence and uniqueness properties of the lecture are still true. The estimates for the solution still hold, with a factor e^{ct} on the right-hand side.

Hint: Formulate an equivalent problem for $w(x,t) = e^{-ct}u(x,t)$ and consider the corresponding weak formulation.

Exercise 14:

Show (with assumptions from the lecture): the solution $u(t) \in V$ of the homogeneous parabolic initial boundary value problem $u' + Au = 0$ in V', $u(0+) = u_0$ in H, satisfies for all $t > 0$

$$Au(t) \in H \quad \text{and} \quad |Au(t)| \leq \frac{C_1}{t}|u_0|$$

and thus

$$\|u(t)\| \leq \frac{C_2}{\sqrt{t}}|u_0|,$$

where the constants C_1, C_2 are independent of t and u_0.

Exercise 15:

Consider the Gelfand triple

$$V \overset{\iota}{\hookrightarrow} H \overset{r_H}{\cong} H' \overset{\iota'}{\hookrightarrow} V'$$

with Hilbert spaces $(V, \|\cdot\|)$, $(H, |\cdot|)$ and $(V', \|\cdot\|_{V'})$. Here ι is a dense continuous embedding and r_H the Riesz isomorphism. Derive ι' and show that it is also a dense continuous embedding.

Exercise 16:

Show that, under the assumptions of exercise 14

$$|u^{(k)}(t)| \leq \frac{C_k}{t^k}|u_0|, \qquad \text{for } t > 0 \text{ and } k \geq 1.$$

Solutions are discussed on 05.06.2024.

Contact person: Dominik Sulz - `dominik.sulz@uni-tuebingen.de`. Open door policy - just come to my office if you have any questions!