Universität Tübingen
Mathematisches Institut
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Tübingen, July 6th, 2018

11th Exercise sheet – Numerics for instationary differential equations

Exercise 29:

Show that the Lax Wendroff method

$$\frac{u_j^{n+1} - u_j^n}{\tau} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h} + \frac{c^2\tau}{2}\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{n^2}$$

with the numerical boundary condition

$$\frac{u_0^{n+1} - u_0^{n-1}}{2\tau} = c\frac{u_1^n - u_0^n}{h}$$

is instable.

Exercise 30:

Compute the phase error of the Lax Wendroff method, i. e. determine $\gamma(\alpha)$, such that

$$G(\alpha) = |G(\alpha)| \exp(i\alpha\tau\gamma(\alpha)).$$

Hint: Show first that $\frac{\text{Im } G(\alpha)}{\text{Re } G(\alpha)} = \tan(\alpha\gamma(\alpha)\tau)$. Use a Taylor series expansion for the argument of arctan and for arctan. You finally arrive at $\gamma(\alpha) = c\left(1 - \frac{1}{6}(h\alpha)^2(1-r^2) + \mathcal{O}\left((h\alpha)^4\right)\right)$.

Exercise 31:

Show that the Leapfrog-method

$$\frac{u_j^{n+1} - u_j^{n-1}}{2\tau} = c\frac{u_{j+1}^n - u_{j-1}^n}{2h}$$

together with the boundary condition

$$\frac{u_0^{n+1} - u_0^n}{\tau} = c\frac{u_1^n - u_0^n}{h}$$

is stable. For this, consider the associated symbols

$$a(z,\xi) = z - z^{-1} - r(\xi - \xi^{-1}),$$
$$b(z,\xi) = z - 1 - r(\xi - 1)$$

(where $r = c\tau/h$) is the Courant number) and proceed the following:

(a) For $|z| > 1$, there exists exactly one zero $\xi_1(z)$ of $a(z,\xi) = 0$ with absolute value smaller than 1 and this zero satisfies $\lim_{z\to 1} \xi_1(z) = -1$.
Hint: Show that for real $z \in (1,\infty)$, there is one zero $\xi_1(z) \in (0,1)$ and one zero $\xi_2(z) \in (1,\infty)$. Then, consider $z = e^{\alpha}e^{i\varphi}$.

(b) The expansion

$$\frac{1}{b(z,\xi_1(z))} = \sum_{n=0}^{\infty} b_n z^{-n}, \quad |z| > 1$$

has a bounded coefficient sequence (b_n).
Hint: Show, that $b(z,\xi_1(z)) \neq 0$ for $|z| > 1$. Then, compute the Laurent series of $\frac{1}{b(z,\xi_1(z))}$, whose non-principal part vanishes. The method is therefore stable.

Solutions are discussed on July 13th.
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