Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

Tübingen, April 20th, 2018

1st Exercise sheet – Numerics for instationary differential equations

Exercise 1: Show that Runge–Kutta- and multistep methods are invariant under linear transformations $y = Tz$. That is, if the method is applied to $y' = f(t, y)$ and $z' = T^{-1}f(t, Tz)$ with initial values

$$y_0 = Tz_0 \quad \text{(RKM)}, \qquad\qquad y_j = Tz_j, \quad j = 0, \dots, k-1 \quad \text{(MSM)},$$

then we have $y_1 = Tz_1$ resp. $y_{n+k} = Tz_{n+k}$.

Exercise 2:

Show that an explicit s-step Runge–Kutta-Method of order $p = s$ has the stability function

$$R(z) = 1 + z + \frac{z^2}{2} + \cdots + \frac{z^s}{s!}.$$

Therefore, R is independent of the coefficients a_{ij}, b_j, c_j of the Runge–Kutta-Method.

Exercise 3:

Space discretization of the heat equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)\ ,\ u(0,t) = u(1,t) = 0\ ,\ u(x,0) = u_0(x) \quad (0 \le x \le 1\ ,\ t \ge 0)$$

using finite differences yields the following system of ordinary differential equations:

$$y' = Ay\ , \quad y(0) = y_0$$

where

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}, \qquad (N+1)h = 1\ .$$

(a) Compute eigenvalues and eigenvectors of A. What happens with the eigenvalues as $h \to 0$?

(b) How large can the time step size τ be chosen such that the explicit Euler method remains stable? What happens, if larger step sizes are chosen? Answer the same questions with regard to the implicit Euler method.

Hint for (a): An eigenvector $v = (v_1, \dots, v_N)^T$ corresponding to the eigenvalue λ of A satisfies

$$v_{n-1} - (2 + \lambda h^2)\cdot v_n + v_{n+1} = 0, \qquad (n = 1, \dots, N),$$

where $v_0 = v_{N+1} = 0$. Therefore (why?), v_n is a linear combination of the n-th powers of the zeros $z_{1,2}$ of the characteristic equation $z^2 - (2 + \lambda h^2)z + 1 = 0$. Remind Vieta's formulas.

(Result: $\lambda_k \cdot h^2 = -2 + 2\cos\frac{k\pi}{N+1}$, $k = 1, \dots, N$)

Programming exercise 1 :

Implement the explicit and implicit Euler method to solve the linear system of exercise 3. Test your implementation using

$$y_0 = \mathbb{1} \in \mathbb{R}^N, \qquad N = 100, \qquad \tau = 10^{-4}, \frac{1}{2} \cdot 10^{-4}, \frac{2}{5} \cdot 10^{-4},$$

Compare the approximations at $t = 0.1$ with a reference solution that is computed with the BDF2 method. Therefore, also implement the BDF2 method. Explain your observations for the different choices of τ and write a short explanation as a comment in one of your codes.

Hint: For the BDF2 method, a second initial value is needed. Note that this has to be done with a method of order 2 (or higher). Use for example the *classical Runge-Kutta method* RK4.

Solutions are discussed on April 27th.
The programming exercise needs to be handed in by May 4th, noon.
Contact person: Dominik Edelmann,
`edelmann@na.uni-tuebingen.de`, office hours Mo 14 - 16 and by arrangement per email.