Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

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2. Exercise Sheet for Numerics of Stationary Differential Equations

Exercise 4:

Show that for the solution of the multi–shooting equation it holds:

$$\Delta x_j \;=\; \sum_{l=0}^{m-1} G_{jl} F_l \;-\; E_{m-j}^{-1} F_m,$$

with

$$E_{m-j} := A R_0^{-1} \cdots R_{j-1}^{-1} \;+\; B R_{m-1} \cdots R_j,$$

and

$$G_{jl} \;=\; \begin{cases} E_{m-j}^{-1} A R_0^{-1} \cdots R_l^{-1}, & l < j, \\ -\, E_{m-j}^{-1} B R_{m-1} \cdots R_{l+1}, & l \geq j, \end{cases}$$

where we define the empty product as I. The matrix (G_{jl}) can then be seen as a discrete analogue of the Green's function from Exercise 2.

Hint: Show and use

$$E_{m-(j+1)}\, R_j \;=\; E_{m-j}.$$

Exercise 5:

How does a Runge–Kutta method, which is equivalent to a collocation method with a (single) node $c_1 = 1/2$, look like?

Exercise 6:

Consider the linear boundary value problem

$$y' = C(t)\, y, \qquad Ay(a) + By(b) = r.$$

If you apply the collocation method from Exercise 4 as a multiple–shooting method to the boundary value problem, you obtain a linear system of equations. Provide the linear system.

Programming Exercise 1 :

Implement the single shooting method for the boundary value problem

$$u''(t) \;=\; \lambda \cdot \big(u(t)\big)^2, \qquad t \in [a,b],$$

$$u(a) = u_a, \quad u(b) = u_b,$$

with $\lambda \in \mathbb{R}$. Follow the steps:

(a) Implement a classical Runge–Kutta method to solve the corresponding initial value problem in each Newton step (as well as to solve the initial value problems to determine the resolvent needed in the Newton method) using the classical Runge–Kutta method.

(b) Calculate the matrices A^k, B^k, and C^k defined in the lecture by hand.

(c) To solve the linear equation system in each Newton step you can use the built-in `Julia/Matlab` operator \.

(d) Stop as soon as the boundary conditions are fulfilled up to a tolerance error $<$ TOL.

Test your program with $a = 0$, $b = 1$, $u_a = 0$, $u_b = 1$, $\lambda = \frac{1}{2}$ and TOL $=$ 1e$-$7. Use as initial value $u'(0) = -5$. Plot your trajectories of the found approximation to the solution against time t. Compare it with the trajectory of the corresponding initial value problem $u(0) = u_a,\ u'(0) = s$ with $s = -4, -12$.

Solutions are discussed on Tuesday 04.11.2025
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.