Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

Tübingen, January 27, 2026

12. Sheet for Numerics of Stationary Differential Equations

Exercise 33

Let $A = M - N$ be a splitting of the symmetric, positive definite matrix A, and assume that N is also symmetric and positive definite. Show that the iteration

$$x_{k+1} = x_k + M^{-1}(b - Ax_k)$$

converges and that the eigenvalues of the iteration matrix are real and lie between 0 and 1.

Exercise 34

The iteration in the two-grid algorithm can be written in the form

$$u_h^{(k+1)} = Mu_h^{(k)} + v_h$$

with $v_h := (I - M)u_h$. Give the matrix M explicitly for the case that ν_1 smoothing steps are performed at the beginning of the iteration and ν_2 smoothing steps are performed during post-smoothing. Show that the spectral radius of M depends only on the sum $\nu_1 + \nu_2$ and not on how many smoothing steps are carried out a priori and a posteriori.

Exercise 35

Let A and C be symmetric positive definite matrices. Show that if $\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$ is invertible, then $\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}$ is also invertible.

Exercise 36

Let A be symmetric positive definite, and assume that

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

is invertible.

a) Show that $A + tB^TB$ is positive definite for every $t > 0$.

b) The solution of the minimization problem

$$(P_t) \qquad \frac{1}{2}u^T(A + tB^TB)u - u^Tf = \min!$$

under the constraint $Bu = 0$ does not depend on t. Now ignore the constraint, and instead introduce $\lambda = tBu$ as a new variable. Show that a problem with a matrix of the form from the previous exercise arises.

c) Show that the solution of (P_t) for $t \to +\infty$ (without constraint) converges to the solution of P_0 with the constraint $Bu = 0$.

Solutions are discussed on Tuesday February 3, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.