Universität Tübingen
Mathematisches Institut
Prof. Dr. Christian Lubich

Tübingen, January 13, 2026

10. Sheet for Numerics of Stationary Differential Equations

Exercise 27

a) Show for the reference triangle $\hat{K}$ with linear interpolation in the corners

$$\|v - \hat{\Pi} v\|_0 \leq C|v|_2 \qquad \text{for all } v \in H^2(\hat{K}).$$

Hint: Use

$$v(x) - v(0) = \int_0^1 \frac{d}{dt} v(tx)\, dt = Dv(x)\, x - \int_0^1 t\, \frac{d^2}{dt^2} v(tx)\, dt$$

and the same formula for $\hat{\Pi} v$.

b) With the help of part (a) show that for linear interpolation in the corners of an arbitrary triangle K with diameter h it holds

$$\|v - \Pi v\|_0 \leq Ch^2|v|_2 \qquad \text{for all } v \in H^2(K),$$

where C does not depend on K.

Exercise 28

a) Show for bilinear interpolation in the corners of the reference square $\hat{K}$

$$|v - \hat{\Pi} v|_1 \leq C|v|_2 \qquad \text{for all } v \in H^2(\hat{K}).$$

b) Let K be a finite element with diameter h and inner circle radius ρ, which is obtained via an affine transformation from $\hat{K}$. With the help of (a) show now for the interpolation error on K that it holds

$$|v - \Pi v|_1 \leq C\frac{h^2}{\rho}|v|_2 \qquad \text{for all } v \in H^2(K),$$

where C does not depend on K.

Exercise 29

Consider the elliptic variational problem $a(u, v) = l(v)\ \forall v \in V$ with $V \subset H^1(\Omega)$. It is approximated by a Galerkin method on the approximation space $V_h \leq V$, an approximated linear form $l_h : V_h \to \mathbb{R}$ and an approximated bilinear form $a_h : V_h \times V_h \to \mathbb{R}$. i.e.:

$$\text{Determine } u_h \in V_h \text{ such that } a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h.$$

Suppose further that the bilinear form a_h is uniformly elliptic, which means that for a $\alpha > 0$ (independent of h) it holds:

$$a_h(w_h, w_h) \geq \alpha \|w_h\|_1^2 v \quad \forall w_h \in V_h.$$

Show the error estimate (Lemma of Strang):

$$\|u - u_h\|_1 \leq c \left(\inf_{v_h \in V_h} (\|u - v_h\|_1 + \|a(v_h, \cdot) - a_h(v_h, \cdot)\|_*) + \|l - l_h\|_* \right),$$

where the dual norm $\|\cdot\|_*$ is defined by $\|F\|_* = \sup_{w_h \in V_h, w_h \neq 0} \frac{|F(w_h)|}{\|w_h\|_1}$.

Hint: Begin with the uniformly ellipticity of a_h for $w_h = u_h - v_h$.

Solutions are discussed on Tuesday January 20, 2026
Tutor: Georgios Vretinaris - if you have question just come to my office (C3P16) or write me an email.