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10. Exercise sheet for numerics of stationary differential equations

Exercise 27:

(a) Show for the reference triangle $\hat{K}$ with linear interpolation in the corners

$$\|v - \hat{\Pi}v\|_0 \leq C\,|v|_2 \qquad \text{for all } v \in H^2(\hat{K})\ .$$

Hint: Use

$$v(x) - v(0) = \int_0^1 \frac{d}{dt}v(tx)\,dt = Dv(x)x - \int_0^1 t\frac{d^2}{dt^2}v(tx)\,dt$$

and the same formula for $\hat{\Pi}v$.

(b) With the help of part (a) show that for linear interpolation in the corners of an arbitrary triangle K with diameter h it holds

$$\|v - \Pi v\|_0 \leq C\,h^2\,|v|_2 \qquad \text{for all } v \in H^2(K)\ ,$$

where C does not depend on K.

Exercise 28:

(a) Show for bilinear interpolation in the corners of the reference square $\hat{K}$

$$|v - \hat{\Pi}v|_1 \leq C\,|v|_2 \qquad \text{for all } v \in H^2(\hat{K})\ .$$

(b) Let K be a finite element with diameter h and inner circle radius ρ, which is obtained via an affine transformation from $\hat{K}$. With the help of (a) show now for the interpolation error of K that it holds

$$|v - \Pi v|_1 \leq C\,\frac{h^2}{\rho}\,|v|_2 \qquad \text{for all } v \in H^2(K)\ ,$$

where C does not depend on K.

Exercise 29:

Suppose the elliptic variational problem $a(u,v) = l(v) \quad \forall v \in V$ with $V \subset H^1(\Omega)$. It is approximated by a Galerkin method on the approximation space $V_h \leq V$, an approximated linear form $l_h : V_h \to \mathbb{R}$ and an approximated bilinear form $a_h : V_h \times V_h \to \mathbb{R}$. I.e.:

$$\text{Determine } u_h \in V_h \text{ such that } a_h(u_h, v_h) = l_h(v_h) \quad \forall v_h \in V_h\ .$$

Suppose further that the bilinear form a_h is uniformly elliptic, which means that for a $\alpha > 0$ (independent of h) it holds:

$$\alpha\|w_h\|_1^2 \leq a_h(w_h, w_h) \quad \forall w_h \in V_h\ .$$

Show the error estimate (*Lemma of Strang*):

$$\|u - u_h\|_1 \leq c\left(\inf_{v_h \in V_h} \left(\|u - v_h\|_1 + \|a(v_h,\cdot) - a_h(v_h,\cdot)\|_*\right) + \|l - l_h\|_*\right)\ ,$$

where the operator norm $\|\cdot\|_*$ is defined by $\|F\|_* = \sup_{0\neq w_h \in V_h} \frac{|F(w_h)|}{\|w_h\|_1}$.
Hint: Begin with the uniformly ellipticity of a_h for $w_h = u_h - v_h$.

Programming exercise 4 :

Solve with the finite elements method the problem

$$\begin{aligned} -d\Delta u + cu &= f \qquad \text{in} \qquad \Omega, \\ u &= 0 \qquad \text{auf} \qquad \partial\Omega, \end{aligned}$$

where the domain Ω and the parameters $d > 0, c \geq 0$ are given as the following:

- Ω: Direct as a triangulation through the matrices (elements, nodes);
- $\partial\Omega$: Through the list of boundary nodes (boundary)
- f: As a Matlab function (`func_f.m`).

Implement a function `[A,M]=matrix_assembly(Elements,Nodes)` for the computation of the stiffness- and mass-matrix. Use the exercises 23, 24 and 25. The vector b can be approximated by

$$\boldsymbol{b}|_j = \int_\Omega f\phi_j \approx \int_\Omega I_h f\phi_j = (\boldsymbol{M f})_j$$

An example code for handling the txt-files can be found under: `https://na.uni-tuebingen.de/ex/num3_ws2324/PA4_FEM_Bsp.zip`.

(a) Solve with the help of your `matrix_assembly` function the linear equation system, which corresponds to the problem above and Ω being the unit circle. A reference solution (func_solution) and the inhomogeneous function (func_f) are given as a m-file.

Compute the error of the numerical solution for different grids. The triangulations for the unit circle are encoded in the txt-files

`Elements_test_j.txt`
`Nodes_test_j.txt` $\qquad j = 1, 2, 3, 4.$

(b) Compute the error in the L^2 norm and the H^1 semi-norm through:

$$\begin{aligned} \|e_h\|^2_{L^2(\Omega)} &= \|\boldsymbol{e}\|^2_{\boldsymbol{M}} = \boldsymbol{e}^T \boldsymbol{M e}, \\ \|\nabla e_h\|^2_{L^2(\Omega)} &= \|\boldsymbol{e}\|^2_{\boldsymbol{A}} = \boldsymbol{e}^T \boldsymbol{A e} \end{aligned}$$

and plot the errors (loglog-plot!).

Discussion of the sheet on 08.01.2024.
Hand in the programming exercise by 15.01.2024.